

Partial Differentiation of Vector-Valued Functions on n -Dimensional Real Normed Linear Spaces

Takao Inoué
Inaba 2205, Wing-Minamikan
Nagano, Nagano, Japan

Adam Naumowicz
Institute of Computer Science
University of Białystok
Akademicka 2, 15-267 Białystok, Poland

Noboru Endou
Gifu National College of Technology
Japan

Yasunari Shidama
Shinshu University
Nagano, Japan

Summary. In this article, we define and develop partial differentiation of vector-valued functions on n -dimensional real normed linear spaces (refer to [19] and [20]).

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The notation and terminology used in this paper have been introduced in the following papers: [7], [15], [2], [3], [24], [4], [5], [1], [11], [16], [6], [9], [12], [17], [18], [10], [8], [23], [14], [21], [13], and [22].

For simplicity, we use the following convention: n, m denote non empty elements of \mathbb{N} , i, j denote elements of \mathbb{N} , f denotes a partial function from $\langle \mathcal{E}^m, \|\cdot\| \rangle$ to $\langle \mathcal{E}^n, \|\cdot\| \rangle$, g denotes a partial function from \mathcal{R}^m to \mathcal{R}^n , h denotes a partial function from \mathcal{R}^m to \mathbb{R} , x denotes a point of $\langle \mathcal{E}^m, \|\cdot\| \rangle$, y denotes an element of \mathcal{R}^m , and X denotes a set.

We now state a number of propositions:

- (1) If $i \leq j$, then $\underbrace{\langle 0, \dots, 0 \rangle}_j \upharpoonright i = \underbrace{\langle 0, \dots, 0 \rangle}_i$.
- (2) If $i \leq j$, then $\underbrace{\langle 0, \dots, 0 \rangle}_j \upharpoonright (i -' 1) = \underbrace{\langle 0, \dots, 0 \rangle}_{i -' 1}$.
- (3) $\underbrace{\langle 0, \dots, 0 \rangle}_j \upharpoonright i = \underbrace{\langle 0, \dots, 0 \rangle}_{j -' i}$.
- (4) If $i \leq j$, then $\underbrace{\langle 0, \dots, 0 \rangle}_j \upharpoonright (i -' 1) = \underbrace{\langle 0, \dots, 0 \rangle}_{i -' 1}$ and $\underbrace{\langle 0, \dots, 0 \rangle}_j \upharpoonright i = \underbrace{\langle 0, \dots, 0 \rangle}_{j -' i}$.
- (5) For every element x_1 of $\langle \mathcal{E}^1, \|\cdot\| \rangle$ such that $1 \leq i \leq j$ holds $\|(\text{reproj}(i, 0_{\langle \mathcal{E}^j, \|\cdot\| \rangle}))(x_1)\| = \|x_1\|$.
- (6) Let m, i be elements of \mathbb{N} , x be an element of \mathcal{R}^m , and r be a real number. Then $(\text{reproj}(i, x))(r) - x = (\text{reproj}(i, \underbrace{\langle 0, \dots, 0 \rangle}_m))(r - (\text{proj}(i, m))(x))$ and $x - (\text{reproj}(i, x))(r) = (\text{reproj}(i, \underbrace{\langle 0, \dots, 0 \rangle}_m))((\text{proj}(i, m))(x) - r)$.
- (7) Let m, i be elements of \mathbb{N} , x be a point of $\langle \mathcal{E}^m, \|\cdot\| \rangle$, and p be a point of $\langle \mathcal{E}^1, \|\cdot\| \rangle$. Then $(\text{reproj}(i, x))(p) - x = (\text{reproj}(i, 0_{\langle \mathcal{E}^m, \|\cdot\| \rangle}))(p - (\text{Proj}(i, m))(x))$ and $x - (\text{reproj}(i, x))(p) = (\text{reproj}(i, 0_{\langle \mathcal{E}^m, \|\cdot\| \rangle}))((\text{Proj}(i, m))(x) - p)$.
- (8) Let m, n be non empty elements of \mathbb{N} , i be an element of \mathbb{N} , f be a partial function from $\langle \mathcal{E}^m, \|\cdot\| \rangle$ to $\langle \mathcal{E}^n, \|\cdot\| \rangle$, and Z be a subset of $\langle \mathcal{E}^m, \|\cdot\| \rangle$. Suppose Z is open and $1 \leq i \leq m$. Then f is partially differentiable on Z w.r.t. i if and only if $Z \subseteq \text{dom } f$ and for every point x of $\langle \mathcal{E}^m, \|\cdot\| \rangle$ such that $x \in Z$ holds f is partially differentiable in x w.r.t. i .
- (9) For all elements x, y of \mathbb{R} and for every element i of \mathbb{N} such that $1 \leq i \leq m$ holds $\text{Replace}(\underbrace{\langle 0, \dots, 0 \rangle}_m, i, x + y) = \text{Replace}(\underbrace{\langle 0, \dots, 0 \rangle}_m, i, x) + \text{Replace}(\underbrace{\langle 0, \dots, 0 \rangle}_m, i, y)$.
- (10) For all elements x, a of \mathbb{R} and for every element i of \mathbb{N} such that $1 \leq i \leq m$ holds $\text{Replace}(\underbrace{\langle 0, \dots, 0 \rangle}_m, i, a \cdot x) = a \cdot \text{Replace}(\underbrace{\langle 0, \dots, 0 \rangle}_m, i, x)$.
- (11) For every element x of \mathbb{R} and for every element i of \mathbb{N} such that $1 \leq i \leq m$ and $x \neq 0$ holds $\text{Replace}(\underbrace{\langle 0, \dots, 0 \rangle}_m, i, x) \neq \underbrace{\langle 0, \dots, 0 \rangle}_m$.
- (12) Let x, y be elements of \mathbb{R} , z be an element of \mathcal{R}^m , and i be an element of \mathbb{N} . Suppose $1 \leq i \leq m$ and $y = (\text{proj}(i, m))(z)$. Then $\text{Replace}(z, i, x) - z = \text{Replace}(\underbrace{\langle 0, \dots, 0 \rangle}_m, i, x - y)$ and $z - \text{Replace}(z, i, x) = \text{Replace}(\underbrace{\langle 0, \dots, 0 \rangle}_m, i, y - x)$.

- (13) For all elements x, y of \mathbb{R} and for every element i of \mathbb{N} such that $1 \leq i \leq m$ holds $(\text{reproj}(i, \underbrace{\langle 0, \dots, 0 \rangle}_m))(x + y) = (\text{reproj}(i, \underbrace{\langle 0, \dots, 0 \rangle}_m))(x) + (\text{reproj}(i, \underbrace{\langle 0, \dots, 0 \rangle}_m))(y)$.
- (14) For all points x, y of $\langle \mathcal{E}^1, \|\cdot\| \rangle$ and for every element i of \mathbb{N} such that $1 \leq i \leq m$ holds $(\text{reproj}(i, 0_{\langle \mathcal{E}^m, \|\cdot\| \rangle}))(x + y) = (\text{reproj}(i, 0_{\langle \mathcal{E}^m, \|\cdot\| \rangle}))(x) + (\text{reproj}(i, 0_{\langle \mathcal{E}^m, \|\cdot\| \rangle}))(y)$.
- (15) For all elements x, a of \mathbb{R} and for every element i of \mathbb{N} such that $1 \leq i \leq m$ holds $(\text{reproj}(i, \underbrace{\langle 0, \dots, 0 \rangle}_m))(a \cdot x) = a \cdot (\text{reproj}(i, \underbrace{\langle 0, \dots, 0 \rangle}_m))(x)$.
- (16) Let x be a point of $\langle \mathcal{E}^1, \|\cdot\| \rangle$, a be an element of \mathbb{R} , and i be an element of \mathbb{N} . If $1 \leq i \leq m$, then $(\text{reproj}(i, 0_{\langle \mathcal{E}^m, \|\cdot\| \rangle}))(a \cdot x) = a \cdot (\text{reproj}(i, 0_{\langle \mathcal{E}^m, \|\cdot\| \rangle}))(x)$.
- (17) For every element x of \mathbb{R} and for every element i of \mathbb{N} such that $1 \leq i \leq m$ and $x \neq 0$ holds $(\text{reproj}(i, \underbrace{\langle 0, \dots, 0 \rangle}_m))(x) \neq \underbrace{\langle 0, \dots, 0 \rangle}_m$.
- (18) For every point x of $\langle \mathcal{E}^1, \|\cdot\| \rangle$ and for every element i of \mathbb{N} such that $1 \leq i \leq m$ and $x \neq 0_{\langle \mathcal{E}^1, \|\cdot\| \rangle}$ holds $(\text{reproj}(i, 0_{\langle \mathcal{E}^m, \|\cdot\| \rangle}))(x) \neq 0_{\langle \mathcal{E}^m, \|\cdot\| \rangle}$.
- (19) Let x, y be elements of \mathbb{R} , z be an element of \mathcal{R}^m , and i be an element of \mathbb{N} . Suppose $1 \leq i \leq m$ and $y = (\text{proj}(i, m))(z)$. Then $(\text{reproj}(i, z))(x) - z = (\text{reproj}(i, \underbrace{\langle 0, \dots, 0 \rangle}_m))(x - y)$ and $z - (\text{reproj}(i, z))(x) = (\text{reproj}(i, \underbrace{\langle 0, \dots, 0 \rangle}_m))(y - x)$.
- (20) Let x, y be points of $\langle \mathcal{E}^1, \|\cdot\| \rangle$, i be an element of \mathbb{N} , and z be a point of $\langle \mathcal{E}^m, \|\cdot\| \rangle$. Suppose $1 \leq i \leq m$ and $y = (\text{Proj}(i, m))(z)$. Then $(\text{reproj}(i, z))(x) - z = (\text{reproj}(i, 0_{\langle \mathcal{E}^m, \|\cdot\| \rangle}))(x - y)$ and $z - (\text{reproj}(i, z))(x) = (\text{reproj}(i, 0_{\langle \mathcal{E}^m, \|\cdot\| \rangle}))(y - x)$.
- (21) Suppose f is differentiable in x and $1 \leq i \leq m$. Then f is partially differentiable in x w.r.t. i and $\text{partdiff}(f, x, i) = f'(x) \cdot \text{reproj}(i, 0_{\langle \mathcal{E}^m, \|\cdot\| \rangle})$.
- (22) Suppose g is differentiable in y and $1 \leq i \leq m$. Then g is partially differentiable in y w.r.t. i and $\text{partdiff}(g, y, i) = (g'(y) \cdot \text{reproj}(i, 0_{\langle \mathcal{E}^m, \|\cdot\| \rangle}))(\langle 1 \rangle)$.

Let n be a non empty element of \mathbb{N} , let f be a partial function from \mathcal{R}^n to \mathbb{R} , and let x be an element of \mathcal{R}^n . We say that f is differentiable in x if and only if:

(Def. 1) $\langle f \rangle$ is differentiable in x .

Let n be a non empty element of \mathbb{N} , let f be a partial function from \mathcal{R}^n to \mathbb{R} , and let x be an element of \mathcal{R}^n . The functor $f'(x)$ yielding a function from \mathcal{R}^n into \mathbb{R} is defined as follows:

(Def. 2) $f'(x) = \text{proj}(1, 1) \cdot \langle f \rangle'(x)$.

Next we state several propositions:

- (23) Suppose h is differentiable in y and $1 \leq i \leq m$. Then h is partially differentiable in y w.r.t. i and
 $\text{partdiff}(h, y, i) = (h \cdot \text{reproj}(i, y))'((\text{proj}(i, m))(y))$ and
 $\text{partdiff}(h, y, i) = h'(y)((\text{reproj}(i, \underbrace{\langle 0, \dots, 0 \rangle}_m))(1))$.
- (24) Let m be a non empty element of \mathbb{N} and v, w, u be finite sequences of elements of \mathcal{R}^m . If $\text{dom } v = \text{dom } w$ and $u = v + w$, then $\sum u = \sum v + \sum w$.
- (25) Let m be a non empty element of \mathbb{N} , r be a real number, and w, u be finite sequences of elements of \mathcal{R}^m . If $u = r w$, then $\sum u = r \cdot \sum w$.
- (26) Let n be a non empty element of \mathbb{N} and h, g be finite sequences of elements of \mathcal{R}^n . Suppose $\text{len } h = \text{len } g + 1$ and for every natural number i such that $i \in \text{dom } g$ holds $g_i = h_i - h_{i+1}$. Then $h_1 - h_{\text{len } h} = \sum g$.
- (27) Let n be a non empty element of \mathbb{N} and h, g, j be finite sequences of elements of \mathcal{R}^n . Suppose $\text{len } h = \text{len } j$ and $\text{len } g = \text{len } j$ and for every natural number i such that $i \in \text{dom } j$ holds $j_i = h_i - g_i$. Then $\sum j = \sum h - \sum g$.
- (28) Let m, n be non empty elements of \mathbb{N} , f be a partial function from \mathcal{R}^m to \mathcal{R}^n , and x, y be elements of \mathcal{R}^m . Then there exists a finite sequence h of elements of \mathcal{R}^m and there exists a finite sequence g of elements of \mathcal{R}^n such that
- (i) $\text{len } h = m + 1$,
 - (ii) $\text{len } g = m$,
 - (iii) for every natural number i such that $i \in \text{dom } h$ holds $h_i = (y \upharpoonright ((m + 1) - i)) \frown \underbrace{\langle 0, \dots, 0 \rangle}_{i-1}$,
 - (iv) for every natural number i such that $i \in \text{dom } g$ holds $g_i = f_{x+h_i} - f_{x+h_{i+1}}$,
 - (v) for every natural number i and for every element h_1 of \mathcal{R}^m such that $i \in \text{dom } h$ and $h_i = h_1$ holds $|h_1| \leq |y|$, and
 - (vi) $f_{x+y} - f_x = \sum g$.
- (29) Let m be a non empty element of \mathbb{N} and f be a partial function from \mathcal{R}^m to \mathcal{R}^1 . Then there exists a partial function f_0 from \mathcal{R}^m to \mathbb{R} such that $f = \langle f_0 \rangle$.
- (30) Let m, n be non empty elements of \mathbb{N} , f be a partial function from \mathcal{R}^m to \mathcal{R}^n , f_0 be a partial function from $\langle \mathcal{E}^m, \|\cdot\| \rangle$ to $\langle \mathcal{E}^n, \|\cdot\| \rangle$, x be an element of \mathcal{R}^m , and x_0 be an element of $\langle \mathcal{E}^m, \|\cdot\| \rangle$. If $x \in \text{dom } f$ and $x = x_0$ and $f = f_0$, then $f_x = (f_0)_{x_0}$.

Let m be a non empty element of \mathbb{N} and let X be a subset of \mathcal{R}^m . We say that X is open if and only if:

(Def. 3) There exists a subset X_0 of $\langle \mathcal{E}^m, \|\cdot\| \rangle$ such that $X_0 = X$ and X_0 is open.

The following proposition is true

- (31) Let m be a non empty element of \mathbb{N} and X be a subset of \mathcal{R}^m . Then X is open if and only if for every element x of \mathcal{R}^m such that $x \in X$ there exists a real number r such that $r > 0$ and $\{y \in \mathcal{R}^m: |y - x| < r\} \subseteq X$.

Let m, n be non empty elements of \mathbb{N} , let i be an element of \mathbb{N} , let f be a partial function from \mathcal{R}^m to \mathcal{R}^n , and let X be a set. We say that f is partially differentiable on X w.r.t. i if and only if:

- (Def. 4) $X \subseteq \text{dom } f$ and for every element x of \mathcal{R}^m such that $x \in X$ holds $f|_X$ is partially differentiable in x w.r.t. i .

One can prove the following propositions:

- (32) Let m, n be non empty elements of \mathbb{N} and f be a partial function from \mathcal{R}^m to \mathcal{R}^n . Suppose f is partially differentiable on X w.r.t. i . Then X is a subset of \mathcal{R}^m .
- (33) Let m, n be non empty elements of \mathbb{N} , i be an element of \mathbb{N} , f be a partial function from \mathcal{R}^m to \mathcal{R}^n , g be a partial function from $\langle \mathcal{E}^m, \|\cdot\| \rangle$ to $\langle \mathcal{E}^n, \|\cdot\| \rangle$, and Z be a set. Suppose $f = g$. Then f is partially differentiable on Z w.r.t. i if and only if g is partially differentiable on Z w.r.t. i .
- (34) Let m, n be non empty elements of \mathbb{N} , i be an element of \mathbb{N} , f be a partial function from \mathcal{R}^m to \mathcal{R}^n , and Z be a subset of \mathcal{R}^m . Suppose Z is open and $1 \leq i \leq m$. Then f is partially differentiable on Z w.r.t. i if and only if $Z \subseteq \text{dom } f$ and for every element x of \mathcal{R}^m such that $x \in Z$ holds f is partially differentiable in x w.r.t. i .

Let m, n be non empty elements of \mathbb{N} , let i be an element of \mathbb{N} , let f be a partial function from \mathcal{R}^m to \mathcal{R}^n , and let us consider X . Let us assume that f is partially differentiable on X w.r.t. i . The functor $f|_X^i$ yielding a partial function from \mathcal{R}^m to \mathcal{R}^n is defined as follows:

- (Def. 5) $\text{dom}(f|_X^i) = X$ and for every element x of \mathcal{R}^m such that $x \in X$ holds $(f|_X^i)_x = \text{partdiff}(f, x, i)$.

Let m, n be non empty elements of \mathbb{N} , let f be a partial function from \mathcal{R}^m to \mathcal{R}^n , and let x_0 be an element of \mathcal{R}^m . We say that f is continuous in x_0 if and only if:

- (Def. 6) There exists a point y_0 of $\langle \mathcal{E}^m, \|\cdot\| \rangle$ and there exists a partial function g from $\langle \mathcal{E}^m, \|\cdot\| \rangle$ to $\langle \mathcal{E}^n, \|\cdot\| \rangle$ such that $x_0 = y_0$ and $f = g$ and g is continuous in y_0 .

The following propositions are true:

- (35) Let m, n be non empty elements of \mathbb{N} , f be a partial function from \mathcal{R}^m to \mathcal{R}^n , g be a partial function from $\langle \mathcal{E}^m, \|\cdot\| \rangle$ to $\langle \mathcal{E}^n, \|\cdot\| \rangle$, x be an element of \mathcal{R}^m , and y be a point of $\langle \mathcal{E}^m, \|\cdot\| \rangle$. Suppose $f = g$ and $x = y$. Then f is continuous in x if and only if g is continuous in y .
- (36) Let m, n be non empty elements of \mathbb{N} , f be a partial function from \mathcal{R}^m to \mathcal{R}^n , and x_0 be an element of \mathcal{R}^m . Then f is continuous in x_0 if and

only if the following conditions are satisfied:

- (i) $x_0 \in \text{dom } f$, and
- (ii) for every real number r such that $0 < r$ there exists a real number s such that $0 < s$ and for every element x_2 of \mathcal{R}^m such that $x_2 \in \text{dom } f$ and $|x_2 - x_0| < s$ holds $|f_{x_2} - f_{x_0}| < r$.

Let m, n be non empty elements of \mathbb{N} , let f be a partial function from \mathcal{R}^m to \mathcal{R}^n , and let us consider X . We say that f is continuous on X if and only if:

(Def. 7) $X \subseteq \text{dom } f$ and for every element x_0 of \mathcal{R}^m such that $x_0 \in X$ holds $f|_X$ is continuous in x_0 .

Next we state a number of propositions:

- (37) Let m, n be non empty elements of \mathbb{N} , f be a partial function from \mathcal{R}^m to \mathcal{R}^n , g be a partial function from $\langle \mathcal{E}^m, \|\cdot\| \rangle$ to $\langle \mathcal{E}^n, \|\cdot\| \rangle$, and X be a set. If $f = g$, then f is continuous on X iff g is continuous on X .
- (38) Let m, n be non empty elements of \mathbb{N} , f be a partial function from \mathcal{R}^m to \mathcal{R}^n , and X be a set. Then f is continuous on X if and only if the following conditions are satisfied:
 - (i) $X \subseteq \text{dom } f$, and
 - (ii) for every element x_0 of \mathcal{R}^m and for every real number r such that $x_0 \in X$ and $0 < r$ there exists a real number s such that $0 < s$ and for every element x_2 of \mathcal{R}^m such that $x_2 \in X$ and $|x_2 - x_0| < s$ holds $|f_{x_2} - f_{x_0}| < r$.
- (39) Let m be a non empty element of \mathbb{N} , x, y be elements of \mathcal{R}^m , i be an element of \mathbb{N} , and x_1 be a real number. If $1 \leq i \leq m$ and $y = (\text{reproj}(i, x))(x_1)$, then $(\text{proj}(i, m))(y) = x_1$.
- (40) Let m be a non empty element of \mathbb{N} , f be a partial function from \mathcal{R}^m to \mathbb{R} , x, y be elements of \mathcal{R}^m , i be an element of \mathbb{N} , and x_1 be a real number. If $1 \leq i \leq m$ and $y = (\text{reproj}(i, x))(x_1)$, then $\text{reproj}(i, x) = \text{reproj}(i, y)$.
- (41) Let m be a non empty element of \mathbb{N} , f be a partial function from \mathcal{R}^m to \mathbb{R} , g be a partial function from \mathbb{R} to \mathbb{R} , x, y be elements of \mathcal{R}^m , i be an element of \mathbb{N} , and x_1 be a real number. If $1 \leq i \leq m$ and $y = (\text{reproj}(i, x))(x_1)$ and $g = f \cdot \text{reproj}(i, x)$, then $g'(x_1) = \text{partdiff}(f, y, i)$.
- (42) Let m be a non empty element of \mathbb{N} , f be a partial function from \mathcal{R}^m to \mathbb{R} , p, q be real numbers, x be an element of \mathcal{R}^m , and i be an element of \mathbb{N} . Suppose that
 - (i) $1 \leq i$,
 - (ii) $i \leq m$,
 - (iii) $p < q$,
 - (iv) for every real number h such that $h \in [p, q]$ holds $(\text{reproj}(i, x))(h) \in \text{dom } f$, and

- (v) for every real number h such that $h \in [p, q]$ holds f is partially differentiable in $(\text{reproj}(i, x))(h)$ w.r.t. i .

Then there exists a real number r and there exists an element y of \mathcal{R}^m such that $r \in]p, q[$ and $y = (\text{reproj}(i, x))(r)$ and $f_{(\text{reproj}(i, x))(q)} - f_{(\text{reproj}(i, x))(p)} = (q - p) \cdot \text{partdiff}(f, y, i)$.

- (43) Let m be a non empty element of \mathbb{N} , f be a partial function from \mathcal{R}^m to \mathbb{R} , p, q be real numbers, x be an element of \mathcal{R}^m , and i be an element of \mathbb{N} . Suppose that

- (i) $1 \leq i$,
- (ii) $i \leq m$,
- (iii) $p \leq q$,
- (iv) for every real number h such that $h \in [p, q]$ holds $(\text{reproj}(i, x))(h) \in \text{dom } f$, and
- (v) for every real number h such that $h \in [p, q]$ holds f is partially differentiable in $(\text{reproj}(i, x))(h)$ w.r.t. i .

Then there exists a real number r and there exists an element y of \mathcal{R}^m such that $r \in [p, q]$ and $y = (\text{reproj}(i, x))(r)$ and $f_{(\text{reproj}(i, x))(q)} - f_{(\text{reproj}(i, x))(p)} = (q - p) \cdot \text{partdiff}(f, y, i)$.

- (44) Let m be a non empty element of \mathbb{N} , x, y, z, w be elements of \mathcal{R}^m , i be an element of \mathbb{N} , and d, p, q, r be real numbers. Suppose $1 \leq i \leq m$ and $|y - x| < d$ and $|z - x| < d$ and $p = (\text{proj}(i, m))(y)$ and $z = (\text{reproj}(i, y))(q)$ and $r \in [p, q]$ and $w = (\text{reproj}(i, y))(r)$. Then $|w - x| < d$.

- (45) Let m be a non empty element of \mathbb{N} , f be a partial function from \mathcal{R}^m to \mathbb{R} , X be a subset of \mathcal{R}^m , x, y, z be elements of \mathcal{R}^m , i be an element of \mathbb{N} , and d, p, q be real numbers. Suppose that $1 \leq i \leq m$ and X is open and $x \in X$ and $|y - x| < d$ and $|z - x| < d$ and $X \subseteq \text{dom } f$ and for every element x of \mathcal{R}^m such that $x \in X$ holds f is partially differentiable in x w.r.t. i and $0 < d$ and for every element z of \mathcal{R}^m such that $|z - x| < d$ holds $z \in X$ and $z = (\text{reproj}(i, y))(p)$ and $q = (\text{proj}(i, m))(y)$. Then there exists an element w of \mathcal{R}^m such that $|w - x| < d$ and f is partially differentiable in w w.r.t. i and $f_z - f_y = (p - q) \cdot \text{partdiff}(f, w, i)$.

- (46) Let m be a non empty element of \mathbb{N} , h be a finite sequence of elements of \mathcal{R}^m , y, x be elements of \mathcal{R}^m , and j be an element of \mathbb{N} . Suppose $\text{len } h = m + 1$ and $1 \leq j \leq m$ and for every natural number i such that $i \in \text{dom } h$ holds $h_i = (y \upharpoonright ((m + 1) -' i)) \wedge \underbrace{\langle 0, \dots, 0 \rangle}_{i-1}$. Then $x + h_j = (\text{reproj}((m + 1) -' j, x + h_{j+1}))((\text{proj}((m + 1) -' j, m))(x + y))$.

- (47) Let m be a non empty element of \mathbb{N} , f be a partial function from \mathcal{R}^m to \mathcal{R}^1 , X be a subset of \mathcal{R}^m , and x be an element of \mathcal{R}^m . Suppose that

- (i) X is open,
- (ii) $x \in X$, and

- (iii) for every element i of \mathbb{N} such that $1 \leq i \leq m$ holds f is partially differentiable on X w.r.t. i and $f|_X^i$ is continuous on X .

Then

- (iv) f is differentiable in x , and
 (v) for every element h of \mathcal{R}^m there exists a finite sequence w of elements of \mathcal{R}^1 such that $\text{dom } w = \text{Seg } m$ and for every element i of \mathbb{N} such that $i \in \text{Seg } m$ holds $w(i) = (\text{proj}(i, m))(h) \cdot \text{partdiff}(f, x, i)$ and $f'(x)(h) = \sum w$.

- (48) Let m be a non empty element of \mathbb{N} , f be a partial function from $\langle \mathcal{E}^m, \|\cdot\| \rangle$ to $\langle \mathcal{E}^1, \|\cdot\| \rangle$, X be a subset of $\langle \mathcal{E}^m, \|\cdot\| \rangle$, and x be a point of $\langle \mathcal{E}^m, \|\cdot\| \rangle$. Suppose that

- (i) X is open,
 (ii) $x \in X$, and
 (iii) for every element i of \mathbb{N} such that $1 \leq i \leq m$ holds f is partially differentiable on X w.r.t. i and $f|_X^i$ is continuous on X .

Then

- (iv) f is differentiable in x , and
 (v) for every point h of $\langle \mathcal{E}^m, \|\cdot\| \rangle$ there exists a finite sequence w of elements of \mathcal{R}^1 such that $\text{dom } w = \text{Seg } m$ and for every element i of \mathbb{N} such that $i \in \text{Seg } m$ holds $w(i) = (\text{partdiff}(f, x, i))(\langle (\text{proj}(i, m))(h) \rangle)$ and $f'(x)(h) = \sum w$.

- (49) Let m be a non empty element of \mathbb{N} , f be a partial function from $\langle \mathcal{E}^m, \|\cdot\| \rangle$ to $\langle \mathcal{E}^1, \|\cdot\| \rangle$, and X be a subset of $\langle \mathcal{E}^m, \|\cdot\| \rangle$. Suppose X is open. Then for every element i of \mathbb{N} such that $1 \leq i \leq m$ holds f is partially differentiable on X w.r.t. i and $f|_X^i$ is continuous on X if and only if f is differentiable on X and $f|_X$ is continuous on X .

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