

# Partial Differentiation of Vector-Valued Functions on $n$ -Dimensional Real Normed Linear Spaces

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**Summary.** In this article, we define and develop partial differentiation of vector-valued functions on  $n$ -dimensional real normed linear spaces (refer to [19] and [20]).

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The notation and terminology used in this paper have been introduced in the following papers: [7], [15], [2], [3], [24], [4], [5], [1], [11], [16], [6], [9], [12], [17], [18], [10], [8], [23], [14], [21], [13], and [22].

For simplicity, we use the following convention:  $n, m$  denote non empty elements of  $\mathbb{N}$ ,  $i, j$  denote elements of  $\mathbb{N}$ ,  $f$  denotes a partial function from  $\langle \mathcal{E}^m, \|\cdot\| \rangle$  to  $\langle \mathcal{E}^n, \|\cdot\| \rangle$ ,  $g$  denotes a partial function from  $\mathcal{R}^m$  to  $\mathcal{R}^n$ ,  $h$  denotes a partial function from  $\mathcal{R}^m$  to  $\mathbb{R}$ ,  $x$  denotes a point of  $\langle \mathcal{E}^m, \|\cdot\| \rangle$ ,  $y$  denotes an element of  $\mathcal{R}^m$ , and  $X$  denotes a set.

We now state a number of propositions:

- (1) If  $i \leq j$ , then  $\underbrace{\langle 0, \dots, 0 \rangle}_j \upharpoonright i = \underbrace{\langle 0, \dots, 0 \rangle}_i$ .
- (2) If  $i \leq j$ , then  $\underbrace{\langle 0, \dots, 0 \rangle}_j \upharpoonright (i - 1) = \underbrace{\langle 0, \dots, 0 \rangle}_{i-1}$ .
- (3)  $\underbrace{\langle 0, \dots, 0 \rangle}_j \upharpoonright i = \underbrace{\langle 0, \dots, 0 \rangle}_{j-i}$ .
- (4) If  $i \leq j$ , then  $\underbrace{\langle 0, \dots, 0 \rangle}_j \upharpoonright (i - 1) = \underbrace{\langle 0, \dots, 0 \rangle}_{i-1}$  and  $\underbrace{\langle 0, \dots, 0 \rangle}_j \upharpoonright i = \underbrace{\langle 0, \dots, 0 \rangle}_{j-i}$ .
- (5) For every element  $x_1$  of  $\langle \mathcal{E}^1, \|\cdot\| \rangle$  such that  $1 \leq i \leq j$  holds  $\|(\text{reproj}(i, 0_{\langle \mathcal{E}^j, \|\cdot\| \rangle}))(x_1)\| = \|x_1\|$ .
- (6) Let  $m, i$  be elements of  $\mathbb{N}$ ,  $x$  be an element of  $\mathcal{R}^m$ , and  $r$  be a real number. Then  $(\text{reproj}(i, x))(r) - x = (\text{reproj}(i, \underbrace{\langle 0, \dots, 0 \rangle}_m))(r - (\text{proj}(i, m))(x))$  and  $x - (\text{reproj}(i, x))(r) = (\text{reproj}(i, \underbrace{\langle 0, \dots, 0 \rangle}_m))((\text{proj}(i, m))(x) - r)$ .
- (7) Let  $m, i$  be elements of  $\mathbb{N}$ ,  $x$  be a point of  $\langle \mathcal{E}^m, \|\cdot\| \rangle$ , and  $p$  be a point of  $\langle \mathcal{E}^1, \|\cdot\| \rangle$ . Then  $(\text{reproj}(i, x))(p) - x = (\text{reproj}(i, 0_{\langle \mathcal{E}^m, \|\cdot\| \rangle}))(p - (\text{Proj}(i, m))(x))$  and  $x - (\text{reproj}(i, x))(p) = (\text{reproj}(i, 0_{\langle \mathcal{E}^m, \|\cdot\| \rangle}))((\text{Proj}(i, m))(x) - p)$ .
- (8) Let  $m, n$  be non empty elements of  $\mathbb{N}$ ,  $i$  be an element of  $\mathbb{N}$ ,  $f$  be a partial function from  $\langle \mathcal{E}^m, \|\cdot\| \rangle$  to  $\langle \mathcal{E}^n, \|\cdot\| \rangle$ , and  $Z$  be a subset of  $\langle \mathcal{E}^m, \|\cdot\| \rangle$ . Suppose  $Z$  is open and  $1 \leq i \leq m$ . Then  $f$  is partially differentiable on  $Z$  w.r.t.  $i$  if and only if  $Z \subseteq \text{dom } f$  and for every point  $x$  of  $\langle \mathcal{E}^m, \|\cdot\| \rangle$  such that  $x \in Z$  holds  $f$  is partially differentiable in  $x$  w.r.t.  $i$ .
- (9) For all elements  $x, y$  of  $\mathbb{R}$  and for every element  $i$  of  $\mathbb{N}$  such that  $1 \leq i \leq m$  holds  $\text{Replace}(\underbrace{\langle 0, \dots, 0 \rangle}_m, i, x + y) = \text{Replace}(\underbrace{\langle 0, \dots, 0 \rangle}_m, i, x) + \text{Replace}(\underbrace{\langle 0, \dots, 0 \rangle}_m, i, y)$ .
- (10) For all elements  $x, a$  of  $\mathbb{R}$  and for every element  $i$  of  $\mathbb{N}$  such that  $1 \leq i \leq m$  holds  $\text{Replace}(\underbrace{\langle 0, \dots, 0 \rangle}_m, i, a \cdot x) = a \cdot \text{Replace}(\underbrace{\langle 0, \dots, 0 \rangle}_m, i, x)$ .
- (11) For every element  $x$  of  $\mathbb{R}$  and for every element  $i$  of  $\mathbb{N}$  such that  $1 \leq i \leq m$  and  $x \neq 0$  holds  $\text{Replace}(\underbrace{\langle 0, \dots, 0 \rangle}_m, i, x) \neq \underbrace{\langle 0, \dots, 0 \rangle}_m$ .
- (12) Let  $x, y$  be elements of  $\mathbb{R}$ ,  $z$  be an element of  $\mathcal{R}^m$ , and  $i$  be an element of  $\mathbb{N}$ . Suppose  $1 \leq i \leq m$  and  $y = (\text{proj}(i, m))(z)$ . Then  $\text{Replace}(z, i, x) - z = \text{Replace}(\underbrace{\langle 0, \dots, 0 \rangle}_m, i, x - y)$  and  $z - \text{Replace}(z, i, x) = \text{Replace}(\underbrace{\langle 0, \dots, 0 \rangle}_m, i, y - x)$ .

- (13) For all elements  $x, y$  of  $\mathbb{R}$  and for every element  $i$  of  $\mathbb{N}$  such that  $1 \leq i \leq m$  holds  $(\text{reproj}(i, \underbrace{\langle 0, \dots, 0 \rangle}_m))(x + y) = (\text{reproj}(i, \underbrace{\langle 0, \dots, 0 \rangle}_m))(x) + (\text{reproj}(i, \underbrace{\langle 0, \dots, 0 \rangle}_m))(y)$ .
- (14) For all points  $x, y$  of  $\langle \mathcal{E}^1, \|\cdot\| \rangle$  and for every element  $i$  of  $\mathbb{N}$  such that  $1 \leq i \leq m$  holds  $(\text{reproj}(i, 0_{\langle \mathcal{E}^m, \|\cdot\| \rangle}))(x + y) = (\text{reproj}(i, 0_{\langle \mathcal{E}^m, \|\cdot\| \rangle}))(x) + (\text{reproj}(i, 0_{\langle \mathcal{E}^m, \|\cdot\| \rangle}))(y)$ .
- (15) For all elements  $x, a$  of  $\mathbb{R}$  and for every element  $i$  of  $\mathbb{N}$  such that  $1 \leq i \leq m$  holds  $(\text{reproj}(i, \underbrace{\langle 0, \dots, 0 \rangle}_m))(a \cdot x) = a \cdot (\text{reproj}(i, \underbrace{\langle 0, \dots, 0 \rangle}_m))(x)$ .
- (16) Let  $x$  be a point of  $\langle \mathcal{E}^1, \|\cdot\| \rangle$ ,  $a$  be an element of  $\mathbb{R}$ , and  $i$  be an element of  $\mathbb{N}$ . If  $1 \leq i \leq m$ , then  $(\text{reproj}(i, 0_{\langle \mathcal{E}^m, \|\cdot\| \rangle}))(a \cdot x) = a \cdot (\text{reproj}(i, 0_{\langle \mathcal{E}^m, \|\cdot\| \rangle}))(x)$ .
- (17) For every element  $x$  of  $\mathbb{R}$  and for every element  $i$  of  $\mathbb{N}$  such that  $1 \leq i \leq m$  and  $x \neq 0$  holds  $(\text{reproj}(i, \underbrace{\langle 0, \dots, 0 \rangle}_m))(x) \neq \underbrace{\langle 0, \dots, 0 \rangle}_m$ .
- (18) For every point  $x$  of  $\langle \mathcal{E}^1, \|\cdot\| \rangle$  and for every element  $i$  of  $\mathbb{N}$  such that  $1 \leq i \leq m$  and  $x \neq 0_{\langle \mathcal{E}^1, \|\cdot\| \rangle}$  holds  $(\text{reproj}(i, 0_{\langle \mathcal{E}^m, \|\cdot\| \rangle}))(x) \neq 0_{\langle \mathcal{E}^m, \|\cdot\| \rangle}$ .
- (19) Let  $x, y$  be elements of  $\mathbb{R}$ ,  $z$  be an element of  $\mathcal{R}^m$ , and  $i$  be an element of  $\mathbb{N}$ . Suppose  $1 \leq i \leq m$  and  $y = (\text{proj}(i, m))(z)$ . Then  $(\text{reproj}(i, z))(x) - z = (\text{reproj}(i, \underbrace{\langle 0, \dots, 0 \rangle}_m))(x - y)$  and  $z - (\text{reproj}(i, z))(x) = (\text{reproj}(i, \underbrace{\langle 0, \dots, 0 \rangle}_m))(y - x)$ .
- (20) Let  $x, y$  be points of  $\langle \mathcal{E}^1, \|\cdot\| \rangle$ ,  $i$  be an element of  $\mathbb{N}$ , and  $z$  be a point of  $\langle \mathcal{E}^m, \|\cdot\| \rangle$ . Suppose  $1 \leq i \leq m$  and  $y = (\text{Proj}(i, m))(z)$ . Then  $(\text{reproj}(i, z))(x) - z = (\text{reproj}(i, 0_{\langle \mathcal{E}^m, \|\cdot\| \rangle}))(x - y)$  and  $z - (\text{reproj}(i, z))(x) = (\text{reproj}(i, 0_{\langle \mathcal{E}^m, \|\cdot\| \rangle}))(y - x)$ .
- (21) Suppose  $f$  is differentiable in  $x$  and  $1 \leq i \leq m$ . Then  $f$  is partially differentiable in  $x$  w.r.t.  $i$  and  $\text{partdiff}(f, x, i) = f'(x) \cdot \text{reproj}(i, 0_{\langle \mathcal{E}^m, \|\cdot\| \rangle})$ .
- (22) Suppose  $g$  is differentiable in  $y$  and  $1 \leq i \leq m$ . Then  $g$  is partially differentiable in  $y$  w.r.t.  $i$  and  $\text{partdiff}(g, y, i) = (g'(y) \cdot \text{reproj}(i, 0_{\langle \mathcal{E}^m, \|\cdot\| \rangle}))(\langle 1 \rangle)$ .

Let  $n$  be a non empty element of  $\mathbb{N}$ , let  $f$  be a partial function from  $\mathcal{R}^n$  to  $\mathbb{R}$ , and let  $x$  be an element of  $\mathcal{R}^n$ . We say that  $f$  is differentiable in  $x$  if and only if:

(Def. 1)  $\langle f \rangle$  is differentiable in  $x$ .

Let  $n$  be a non empty element of  $\mathbb{N}$ , let  $f$  be a partial function from  $\mathcal{R}^n$  to  $\mathbb{R}$ , and let  $x$  be an element of  $\mathcal{R}^n$ . The functor  $f'(x)$  yielding a function from  $\mathcal{R}^n$  into  $\mathbb{R}$  is defined as follows:

(Def. 2)  $f'(x) = \text{proj}(1, 1) \cdot \langle f \rangle'(x)$ .

Next we state several propositions:

- (23) Suppose  $h$  is differentiable in  $y$  and  $1 \leq i \leq m$ . Then  $h$  is partially differentiable in  $y$  w.r.t.  $i$  and  
 $\text{partdiff}(h, y, i) = (h \cdot \text{reproj}(i, y))'((\text{proj}(i, m))(y))$  and  
 $\text{partdiff}(h, y, i) = h'(y)((\text{reproj}(i, \underbrace{\langle 0, \dots, 0 \rangle}_m))(1))$ .
- (24) Let  $m$  be a non empty element of  $\mathbb{N}$  and  $v, w, u$  be finite sequences of elements of  $\mathcal{R}^m$ . If  $\text{dom } v = \text{dom } w$  and  $u = v + w$ , then  $\sum u = \sum v + \sum w$ .
- (25) Let  $m$  be a non empty element of  $\mathbb{N}$ ,  $r$  be a real number, and  $w, u$  be finite sequences of elements of  $\mathcal{R}^m$ . If  $u = r w$ , then  $\sum u = r \cdot \sum w$ .
- (26) Let  $n$  be a non empty element of  $\mathbb{N}$  and  $h, g$  be finite sequences of elements of  $\mathcal{R}^n$ . Suppose  $\text{len } h = \text{len } g + 1$  and for every natural number  $i$  such that  $i \in \text{dom } g$  holds  $g_i = h_i - h_{i+1}$ . Then  $h_1 - h_{\text{len } h} = \sum g$ .
- (27) Let  $n$  be a non empty element of  $\mathbb{N}$  and  $h, g, j$  be finite sequences of elements of  $\mathcal{R}^n$ . Suppose  $\text{len } h = \text{len } j$  and  $\text{len } g = \text{len } j$  and for every natural number  $i$  such that  $i \in \text{dom } j$  holds  $j_i = h_i - g_i$ . Then  $\sum j = \sum h - \sum g$ .
- (28) Let  $m, n$  be non empty elements of  $\mathbb{N}$ ,  $f$  be a partial function from  $\mathcal{R}^m$  to  $\mathcal{R}^n$ , and  $x, y$  be elements of  $\mathcal{R}^m$ . Then there exists a finite sequence  $h$  of elements of  $\mathcal{R}^m$  and there exists a finite sequence  $g$  of elements of  $\mathcal{R}^n$  such that
- (i)  $\text{len } h = m + 1$ ,
  - (ii)  $\text{len } g = m$ ,
  - (iii) for every natural number  $i$  such that  $i \in \text{dom } h$  holds  $h_i = (y \upharpoonright ((m + 1) - i)) \frown \underbrace{\langle 0, \dots, 0 \rangle}_{i-1}$ ,
  - (iv) for every natural number  $i$  such that  $i \in \text{dom } g$  holds  $g_i = f_{x+h_i} - f_{x+h_{i+1}}$ ,
  - (v) for every natural number  $i$  and for every element  $h_1$  of  $\mathcal{R}^m$  such that  $i \in \text{dom } h$  and  $h_i = h_1$  holds  $|h_1| \leq |y|$ , and
  - (vi)  $f_{x+y} - f_x = \sum g$ .
- (29) Let  $m$  be a non empty element of  $\mathbb{N}$  and  $f$  be a partial function from  $\mathcal{R}^m$  to  $\mathcal{R}^1$ . Then there exists a partial function  $f_0$  from  $\mathcal{R}^m$  to  $\mathbb{R}$  such that  $f = \langle f_0 \rangle$ .
- (30) Let  $m, n$  be non empty elements of  $\mathbb{N}$ ,  $f$  be a partial function from  $\mathcal{R}^m$  to  $\mathcal{R}^n$ ,  $f_0$  be a partial function from  $\langle \mathcal{E}^m, \|\cdot\| \rangle$  to  $\langle \mathcal{E}^n, \|\cdot\| \rangle$ ,  $x$  be an element of  $\mathcal{R}^m$ , and  $x_0$  be an element of  $\langle \mathcal{E}^m, \|\cdot\| \rangle$ . If  $x \in \text{dom } f$  and  $x = x_0$  and  $f = f_0$ , then  $f_x = (f_0)_{x_0}$ .

Let  $m$  be a non empty element of  $\mathbb{N}$  and let  $X$  be a subset of  $\mathcal{R}^m$ . We say that  $X$  is open if and only if:

(Def. 3) There exists a subset  $X_0$  of  $\langle \mathcal{E}^m, \|\cdot\| \rangle$  such that  $X_0 = X$  and  $X_0$  is open.

The following proposition is true

- (31) Let  $m$  be a non empty element of  $\mathbb{N}$  and  $X$  be a subset of  $\mathcal{R}^m$ . Then  $X$  is open if and only if for every element  $x$  of  $\mathcal{R}^m$  such that  $x \in X$  there exists a real number  $r$  such that  $r > 0$  and  $\{y \in \mathcal{R}^m: |y - x| < r\} \subseteq X$ .

Let  $m, n$  be non empty elements of  $\mathbb{N}$ , let  $i$  be an element of  $\mathbb{N}$ , let  $f$  be a partial function from  $\mathcal{R}^m$  to  $\mathcal{R}^n$ , and let  $X$  be a set. We say that  $f$  is partially differentiable on  $X$  w.r.t.  $i$  if and only if:

- (Def. 4)  $X \subseteq \text{dom } f$  and for every element  $x$  of  $\mathcal{R}^m$  such that  $x \in X$  holds  $f|_X$  is partially differentiable in  $x$  w.r.t.  $i$ .

One can prove the following propositions:

- (32) Let  $m, n$  be non empty elements of  $\mathbb{N}$  and  $f$  be a partial function from  $\mathcal{R}^m$  to  $\mathcal{R}^n$ . Suppose  $f$  is partially differentiable on  $X$  w.r.t.  $i$ . Then  $X$  is a subset of  $\mathcal{R}^m$ .
- (33) Let  $m, n$  be non empty elements of  $\mathbb{N}$ ,  $i$  be an element of  $\mathbb{N}$ ,  $f$  be a partial function from  $\mathcal{R}^m$  to  $\mathcal{R}^n$ ,  $g$  be a partial function from  $\langle \mathcal{E}^m, \|\cdot\| \rangle$  to  $\langle \mathcal{E}^n, \|\cdot\| \rangle$ , and  $Z$  be a set. Suppose  $f = g$ . Then  $f$  is partially differentiable on  $Z$  w.r.t.  $i$  if and only if  $g$  is partially differentiable on  $Z$  w.r.t.  $i$ .
- (34) Let  $m, n$  be non empty elements of  $\mathbb{N}$ ,  $i$  be an element of  $\mathbb{N}$ ,  $f$  be a partial function from  $\mathcal{R}^m$  to  $\mathcal{R}^n$ , and  $Z$  be a subset of  $\mathcal{R}^m$ . Suppose  $Z$  is open and  $1 \leq i \leq m$ . Then  $f$  is partially differentiable on  $Z$  w.r.t.  $i$  if and only if  $Z \subseteq \text{dom } f$  and for every element  $x$  of  $\mathcal{R}^m$  such that  $x \in Z$  holds  $f$  is partially differentiable in  $x$  w.r.t.  $i$ .

Let  $m, n$  be non empty elements of  $\mathbb{N}$ , let  $i$  be an element of  $\mathbb{N}$ , let  $f$  be a partial function from  $\mathcal{R}^m$  to  $\mathcal{R}^n$ , and let us consider  $X$ . Let us assume that  $f$  is partially differentiable on  $X$  w.r.t.  $i$ . The functor  $f|_X^i$  yielding a partial function from  $\mathcal{R}^m$  to  $\mathcal{R}^n$  is defined as follows:

- (Def. 5)  $\text{dom}(f|_X^i) = X$  and for every element  $x$  of  $\mathcal{R}^m$  such that  $x \in X$  holds  $(f|_X^i)_x = \text{partdiff}(f, x, i)$ .

Let  $m, n$  be non empty elements of  $\mathbb{N}$ , let  $f$  be a partial function from  $\mathcal{R}^m$  to  $\mathcal{R}^n$ , and let  $x_0$  be an element of  $\mathcal{R}^m$ . We say that  $f$  is continuous in  $x_0$  if and only if:

- (Def. 6) There exists a point  $y_0$  of  $\langle \mathcal{E}^m, \|\cdot\| \rangle$  and there exists a partial function  $g$  from  $\langle \mathcal{E}^m, \|\cdot\| \rangle$  to  $\langle \mathcal{E}^n, \|\cdot\| \rangle$  such that  $x_0 = y_0$  and  $f = g$  and  $g$  is continuous in  $y_0$ .

The following propositions are true:

- (35) Let  $m, n$  be non empty elements of  $\mathbb{N}$ ,  $f$  be a partial function from  $\mathcal{R}^m$  to  $\mathcal{R}^n$ ,  $g$  be a partial function from  $\langle \mathcal{E}^m, \|\cdot\| \rangle$  to  $\langle \mathcal{E}^n, \|\cdot\| \rangle$ ,  $x$  be an element of  $\mathcal{R}^m$ , and  $y$  be a point of  $\langle \mathcal{E}^m, \|\cdot\| \rangle$ . Suppose  $f = g$  and  $x = y$ . Then  $f$  is continuous in  $x$  if and only if  $g$  is continuous in  $y$ .
- (36) Let  $m, n$  be non empty elements of  $\mathbb{N}$ ,  $f$  be a partial function from  $\mathcal{R}^m$  to  $\mathcal{R}^n$ , and  $x_0$  be an element of  $\mathcal{R}^m$ . Then  $f$  is continuous in  $x_0$  if and

only if the following conditions are satisfied:

- (i)  $x_0 \in \text{dom } f$ , and
- (ii) for every real number  $r$  such that  $0 < r$  there exists a real number  $s$  such that  $0 < s$  and for every element  $x_2$  of  $\mathcal{R}^m$  such that  $x_2 \in \text{dom } f$  and  $|x_2 - x_0| < s$  holds  $|f_{x_2} - f_{x_0}| < r$ .

Let  $m, n$  be non empty elements of  $\mathbb{N}$ , let  $f$  be a partial function from  $\mathcal{R}^m$  to  $\mathcal{R}^n$ , and let us consider  $X$ . We say that  $f$  is continuous on  $X$  if and only if:

(Def. 7)  $X \subseteq \text{dom } f$  and for every element  $x_0$  of  $\mathcal{R}^m$  such that  $x_0 \in X$  holds  $f|_X$  is continuous in  $x_0$ .

Next we state a number of propositions:

- (37) Let  $m, n$  be non empty elements of  $\mathbb{N}$ ,  $f$  be a partial function from  $\mathcal{R}^m$  to  $\mathcal{R}^n$ ,  $g$  be a partial function from  $\langle \mathcal{E}^m, \|\cdot\| \rangle$  to  $\langle \mathcal{E}^n, \|\cdot\| \rangle$ , and  $X$  be a set. If  $f = g$ , then  $f$  is continuous on  $X$  iff  $g$  is continuous on  $X$ .
- (38) Let  $m, n$  be non empty elements of  $\mathbb{N}$ ,  $f$  be a partial function from  $\mathcal{R}^m$  to  $\mathcal{R}^n$ , and  $X$  be a set. Then  $f$  is continuous on  $X$  if and only if the following conditions are satisfied:
  - (i)  $X \subseteq \text{dom } f$ , and
  - (ii) for every element  $x_0$  of  $\mathcal{R}^m$  and for every real number  $r$  such that  $x_0 \in X$  and  $0 < r$  there exists a real number  $s$  such that  $0 < s$  and for every element  $x_2$  of  $\mathcal{R}^m$  such that  $x_2 \in X$  and  $|x_2 - x_0| < s$  holds  $|f_{x_2} - f_{x_0}| < r$ .
- (39) Let  $m$  be a non empty element of  $\mathbb{N}$ ,  $x, y$  be elements of  $\mathcal{R}^m$ ,  $i$  be an element of  $\mathbb{N}$ , and  $x_1$  be a real number. If  $1 \leq i \leq m$  and  $y = (\text{reproj}(i, x))(x_1)$ , then  $(\text{proj}(i, m))(y) = x_1$ .
- (40) Let  $m$  be a non empty element of  $\mathbb{N}$ ,  $f$  be a partial function from  $\mathcal{R}^m$  to  $\mathbb{R}$ ,  $x, y$  be elements of  $\mathcal{R}^m$ ,  $i$  be an element of  $\mathbb{N}$ , and  $x_1$  be a real number. If  $1 \leq i \leq m$  and  $y = (\text{reproj}(i, x))(x_1)$ , then  $\text{reproj}(i, x) = \text{reproj}(i, y)$ .
- (41) Let  $m$  be a non empty element of  $\mathbb{N}$ ,  $f$  be a partial function from  $\mathcal{R}^m$  to  $\mathbb{R}$ ,  $g$  be a partial function from  $\mathbb{R}$  to  $\mathbb{R}$ ,  $x, y$  be elements of  $\mathcal{R}^m$ ,  $i$  be an element of  $\mathbb{N}$ , and  $x_1$  be a real number. If  $1 \leq i \leq m$  and  $y = (\text{reproj}(i, x))(x_1)$  and  $g = f \cdot \text{reproj}(i, x)$ , then  $g'(x_1) = \text{partdiff}(f, y, i)$ .
- (42) Let  $m$  be a non empty element of  $\mathbb{N}$ ,  $f$  be a partial function from  $\mathcal{R}^m$  to  $\mathbb{R}$ ,  $p, q$  be real numbers,  $x$  be an element of  $\mathcal{R}^m$ , and  $i$  be an element of  $\mathbb{N}$ . Suppose that
  - (i)  $1 \leq i$ ,
  - (ii)  $i \leq m$ ,
  - (iii)  $p < q$ ,
  - (iv) for every real number  $h$  such that  $h \in [p, q]$  holds  $(\text{reproj}(i, x))(h) \in \text{dom } f$ , and

- (v) for every real number  $h$  such that  $h \in [p, q]$  holds  $f$  is partially differentiable in  $(\text{reproj}(i, x))(h)$  w.r.t.  $i$ .

Then there exists a real number  $r$  and there exists an element  $y$  of  $\mathcal{R}^m$  such that  $r \in ]p, q[$  and  $y = (\text{reproj}(i, x))(r)$  and  $f_{(\text{reproj}(i, x))(q)} - f_{(\text{reproj}(i, x))(p)} = (q - p) \cdot \text{partdiff}(f, y, i)$ .

- (43) Let  $m$  be a non empty element of  $\mathbb{N}$ ,  $f$  be a partial function from  $\mathcal{R}^m$  to  $\mathbb{R}$ ,  $p, q$  be real numbers,  $x$  be an element of  $\mathcal{R}^m$ , and  $i$  be an element of  $\mathbb{N}$ . Suppose that

- (i)  $1 \leq i$ ,

- (ii)  $i \leq m$ ,

- (iii)  $p \leq q$ ,

- (iv) for every real number  $h$  such that  $h \in [p, q]$  holds  $(\text{reproj}(i, x))(h) \in \text{dom } f$ , and

- (v) for every real number  $h$  such that  $h \in [p, q]$  holds  $f$  is partially differentiable in  $(\text{reproj}(i, x))(h)$  w.r.t.  $i$ .

Then there exists a real number  $r$  and there exists an element  $y$  of  $\mathcal{R}^m$  such that  $r \in [p, q]$  and  $y = (\text{reproj}(i, x))(r)$  and  $f_{(\text{reproj}(i, x))(q)} - f_{(\text{reproj}(i, x))(p)} = (q - p) \cdot \text{partdiff}(f, y, i)$ .

- (44) Let  $m$  be a non empty element of  $\mathbb{N}$ ,  $x, y, z, w$  be elements of  $\mathcal{R}^m$ ,  $i$  be an element of  $\mathbb{N}$ , and  $d, p, q, r$  be real numbers. Suppose  $1 \leq i \leq m$  and  $|y - x| < d$  and  $|z - x| < d$  and  $p = (\text{proj}(i, m))(y)$  and  $z = (\text{reproj}(i, y))(q)$  and  $r \in [p, q]$  and  $w = (\text{reproj}(i, y))(r)$ . Then  $|w - x| < d$ .

- (45) Let  $m$  be a non empty element of  $\mathbb{N}$ ,  $f$  be a partial function from  $\mathcal{R}^m$  to  $\mathbb{R}$ ,  $X$  be a subset of  $\mathcal{R}^m$ ,  $x, y, z$  be elements of  $\mathcal{R}^m$ ,  $i$  be an element of  $\mathbb{N}$ , and  $d, p, q$  be real numbers. Suppose that  $1 \leq i \leq m$  and  $X$  is open and  $x \in X$  and  $|y - x| < d$  and  $|z - x| < d$  and  $X \subseteq \text{dom } f$  and for every element  $x$  of  $\mathcal{R}^m$  such that  $x \in X$  holds  $f$  is partially differentiable in  $x$  w.r.t.  $i$  and  $0 < d$  and for every element  $z$  of  $\mathcal{R}^m$  such that  $|z - x| < d$  holds  $z \in X$  and  $z = (\text{reproj}(i, y))(p)$  and  $q = (\text{proj}(i, m))(y)$ . Then there exists an element  $w$  of  $\mathcal{R}^m$  such that  $|w - x| < d$  and  $f$  is partially differentiable in  $w$  w.r.t.  $i$  and  $f_z - f_y = (p - q) \cdot \text{partdiff}(f, w, i)$ .

- (46) Let  $m$  be a non empty element of  $\mathbb{N}$ ,  $h$  be a finite sequence of elements of  $\mathcal{R}^m$ ,  $y, x$  be elements of  $\mathcal{R}^m$ , and  $j$  be an element of  $\mathbb{N}$ . Suppose  $\text{len } h = m + 1$  and  $1 \leq j \leq m$  and for every natural number  $i$  such that  $i \in \text{dom } h$  holds  $h_i = (y \upharpoonright ((m + 1) -' i)) \frown \underbrace{\langle 0, \dots, 0 \rangle}_{i-1}$ . Then  $x + h_j = (\text{reproj}((m + 1) -' j, x + h_{j+1}))((\text{proj}((m + 1) -' j, m))(x + y))$ .

- (47) Let  $m$  be a non empty element of  $\mathbb{N}$ ,  $f$  be a partial function from  $\mathcal{R}^m$  to  $\mathcal{R}^1$ ,  $X$  be a subset of  $\mathcal{R}^m$ , and  $x$  be an element of  $\mathcal{R}^m$ . Suppose that

- (i)  $X$  is open,

- (ii)  $x \in X$ , and

- (iii) for every element  $i$  of  $\mathbb{N}$  such that  $1 \leq i \leq m$  holds  $f$  is partially differentiable on  $X$  w.r.t.  $i$  and  $f|_X^i$  is continuous on  $X$ .

Then

- (iv)  $f$  is differentiable in  $x$ , and  
 (v) for every element  $h$  of  $\mathcal{R}^m$  there exists a finite sequence  $w$  of elements of  $\mathcal{R}^1$  such that  $\text{dom } w = \text{Seg } m$  and for every element  $i$  of  $\mathbb{N}$  such that  $i \in \text{Seg } m$  holds  $w(i) = (\text{proj}(i, m))(h) \cdot \text{partdiff}(f, x, i)$  and  $f'(x)(h) = \sum w$ .

- (48) Let  $m$  be a non empty element of  $\mathbb{N}$ ,  $f$  be a partial function from  $\langle \mathcal{E}^m, \|\cdot\| \rangle$  to  $\langle \mathcal{E}^1, \|\cdot\| \rangle$ ,  $X$  be a subset of  $\langle \mathcal{E}^m, \|\cdot\| \rangle$ , and  $x$  be a point of  $\langle \mathcal{E}^m, \|\cdot\| \rangle$ . Suppose that

- (i)  $X$  is open,  
 (ii)  $x \in X$ , and  
 (iii) for every element  $i$  of  $\mathbb{N}$  such that  $1 \leq i \leq m$  holds  $f$  is partially differentiable on  $X$  w.r.t.  $i$  and  $f|_X^i$  is continuous on  $X$ .

Then

- (iv)  $f$  is differentiable in  $x$ , and  
 (v) for every point  $h$  of  $\langle \mathcal{E}^m, \|\cdot\| \rangle$  there exists a finite sequence  $w$  of elements of  $\mathcal{R}^1$  such that  $\text{dom } w = \text{Seg } m$  and for every element  $i$  of  $\mathbb{N}$  such that  $i \in \text{Seg } m$  holds  $w(i) = (\text{partdiff}(f, x, i))(\langle (\text{proj}(i, m))(h) \rangle)$  and  $f'(x)(h) = \sum w$ .

- (49) Let  $m$  be a non empty element of  $\mathbb{N}$ ,  $f$  be a partial function from  $\langle \mathcal{E}^m, \|\cdot\| \rangle$  to  $\langle \mathcal{E}^1, \|\cdot\| \rangle$ , and  $X$  be a subset of  $\langle \mathcal{E}^m, \|\cdot\| \rangle$ . Suppose  $X$  is open. Then for every element  $i$  of  $\mathbb{N}$  such that  $1 \leq i \leq m$  holds  $f$  is partially differentiable on  $X$  w.r.t.  $i$  and  $f|_X^i$  is continuous on  $X$  if and only if  $f$  is differentiable on  $X$  and  $f|_X$  is continuous on  $X$ .

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