

Normal Subgroup of Product of Groups

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Summary. In [6] it was formalized that the direct product of a family of groups gives a new group. In this article, we formalize that for all $j \in I$, the group $G = \prod_{i \in I} G_i$ has a normal subgroup isomorphic to G_j . Moreover, we show some relations between a family of groups and its direct product.

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The papers [2], [4], [5], [3], [8], [9], [7], [10], [11], [6], [1], [13], and [12] provide the terminology and notation for this paper.

1. NORMAL SUBGROUP OF PRODUCT OF GROUPS

Let I be a non empty set, let F be a group-like multiplicative magma family of I , and let i be an element of I . Note that $F(i)$ is group-like.

Let I be a non empty set, let F be an associative multiplicative magma family of I , and let i be an element of I . Observe that $F(i)$ is associative.

Let I be a non empty set, let F be a commutative multiplicative magma family of I , and let i be an element of I . Note that $F(i)$ is commutative.

In the sequel I is a non empty set, F is an associative group-like multiplicative magma family of I , and i, j are elements of I .

We now state the proposition

- (1) Let x be a function and g be an element of $F(i)$. Then $\text{dom } x = I$ and $x(i) = g$ and for every element j of I such that $j \neq i$ holds $x(j) = \mathbf{1}_{F(j)}$ if and only if $x = \mathbf{1}_{\prod F} + \cdot (i, g)$.

Let I be a non empty set, let F be an associative group-like multiplicative magma family of I , and let i be an element of I . The functor $\text{ProjSet}(F, i)$ yields a subset of $\prod F$ and is defined by:

(Def. 1) For every set x holds $x \in \text{ProjSet}(F, i)$ iff there exists an element g of $F(i)$ such that $x = \mathbf{1}_{\prod F} + \cdot (i, g)$.

Let I be a non empty set, let F be an associative group-like multiplicative magma family of I , and let i be an element of I . Observe that $\text{ProjSet}(F, i)$ is non empty.

Next we state several propositions:

- (2) Let x_0 be a set. Then $x_0 \in \text{ProjSet}(F, i)$ if and only if there exists a function x and there exists an element g of $F(i)$ such that $x = x_0$ and $\text{dom } x = I$ and $x(i) = g$ and for every element j of I such that $j \neq i$ holds $x(j) = \mathbf{1}_{F(j)}$.
- (3) Let g_1, g_2 be elements of $\prod F$ and z_1, z_2 be elements of $F(i)$. If $g_1 = \mathbf{1}_{\prod F} + \cdot (i, z_1)$ and $g_2 = \mathbf{1}_{\prod F} + \cdot (i, z_2)$, then $g_1 \cdot g_2 = \mathbf{1}_{\prod F} + \cdot (i, z_1 \cdot z_2)$.
- (4) For every element g_1 of $\prod F$ and for every element z_1 of $F(i)$ such that $g_1 = \mathbf{1}_{\prod F} + \cdot (i, z_1)$ holds $g_1^{-1} = \mathbf{1}_{\prod F} + \cdot (i, z_1^{-1})$.
- (5) For all elements g_1, g_2 of $\prod F$ such that $g_1, g_2 \in \text{ProjSet}(F, i)$ holds $g_1 \cdot g_2 \in \text{ProjSet}(F, i)$.
- (6) For every element g of $\prod F$ such that $g \in \text{ProjSet}(F, i)$ holds $g^{-1} \in \text{ProjSet}(F, i)$.

Let I be a non empty set, let F be an associative group-like multiplicative magma family of I , and let i be an element of I . The functor $\text{ProjGroup}(F, i)$ yields a strict subgroup of $\prod F$ and is defined as follows:

(Def. 2) The carrier of $\text{ProjGroup}(F, i) = \text{ProjSet}(F, i)$.

Let us consider I, F, i . The functor $1\text{ProdHom}(F, i)$ yielding a homomorphism from $F(i)$ to $\text{ProjGroup}(F, i)$ is defined as follows:

(Def. 3) For every element x of $F(i)$ holds $(1\text{ProdHom}(F, i))(x) = \mathbf{1}_{\prod F} + \cdot (i, x)$.

Let us consider I, F, i . Note that $1\text{ProdHom}(F, i)$ is bijective.

Let us consider I, F, i . One can check that $\text{ProjGroup}(F, i)$ is normal.

One can prove the following proposition

- (7) For all elements x, y of $\prod F$ such that $i \neq j$ and $x \in \text{ProjGroup}(F, i)$ and $y \in \text{ProjGroup}(F, j)$ holds $x \cdot y = y \cdot x$.

2. PRODUCT OF SUBGROUPS OF A GROUP

In the sequel n denotes a non empty natural number.

One can prove the following propositions:

- (8) Let F be an associative group-like multiplicative magma family of $\text{Seg } n$, J be a natural number, and G_1 be a group. Suppose $1 \leq J \leq n$ and $G_1 = F(J)$. Let x be an element of $\prod F$ and s be a finite sequence of elements of $\prod F$. Suppose $\text{len } s < J$ and for every element k of $\text{Seg } n$

such that $k \in \text{dom } s$ holds $s(k) \in \text{ProjGroup}(F, k)$ and $x = \prod s$. Then $x(J) = \mathbf{1}_{(G_1)}$.

- (9) Let F be an associative group-like multiplicative magma family of $\text{Seg } n$, x be an element of $\prod F$, and s be a finite sequence of elements of $\prod F$. Suppose $\text{len } s = n$ and for every element k of $\text{Seg } n$ holds $s(k) \in \text{ProjGroup}(F, k)$ and $x = \prod s$. Let i be a natural number. Suppose $1 \leq i \leq n$. Then there exists an element s_1 of $\prod F$ such that $s_1 = s(i)$ and $x(i) = s_1(i)$.
- (10) Let F be an associative group-like multiplicative magma family of $\text{Seg } n$, x be an element of $\prod F$, and s, t be finite sequences of elements of $\prod F$. Suppose that
 - (i) $\text{len } s = n$,
 - (ii) for every element k of $\text{Seg } n$ holds $s(k) \in \text{ProjGroup}(F, k)$,
 - (iii) $x = \prod s$,
 - (iv) $\text{len } t = n$,
 - (v) for every element k of $\text{Seg } n$ holds $t(k) \in \text{ProjGroup}(F, k)$, and
 - (vi) $x = \prod t$.
 Then $s = t$.
- (11) Let F be an associative group-like multiplicative magma family of $\text{Seg } n$ and x be an element of $\prod F$. Then there exists a finite sequence s of elements of $\prod F$ such that $\text{len } s = n$ and for every element k of $\text{Seg } n$ holds $s(k) \in \text{ProjGroup}(F, k)$ and $x = \prod s$.
- (12) Let G be a commutative group and F be an associative group-like multiplicative magma family of $\text{Seg } n$. Suppose that
 - (i) for every element i of $\text{Seg } n$ holds $F(i)$ is a subgroup of G ,
 - (ii) for every element x of G there exists a finite sequence s of elements of G such that $\text{len } s = n$ and for every element k of $\text{Seg } n$ holds $s(k) \in F(k)$ and $x = \prod s$, and
 - (iii) for all finite sequences s, t of elements of G such that $\text{len } s = n$ and for every element k of $\text{Seg } n$ holds $s(k) \in F(k)$ and $\text{len } t = n$ and for every element k of $\text{Seg } n$ holds $t(k) \in F(k)$ and $\prod s = \prod t$ holds $s = t$.
 Then there exists a homomorphism f from $\prod F$ to G such that
 - (iv) f is bijective, and
 - (v) for every element x of $\prod F$ there exists a finite sequence s of elements of G such that $\text{len } s = n$ and for every element k of $\text{Seg } n$ holds $s(k) \in F(k)$ and $s = x$ and $f(x) = \prod s$.
- (13) Let G, F be associative commutative group-like multiplicative magma families of $\text{Seg } n$. Suppose that for every element k of $\text{Seg } n$ holds $F(k) = \text{ProjGroup}(G, k)$. Then there exists a homomorphism f from $\prod F$ to $\prod G$ such that
 - (i) f is bijective, and

- (ii) for every element x of $\prod F$ there exists a finite sequence s of elements of $\prod G$ such that $\text{len } s = n$ and for every element k of $\text{Seg } n$ holds $s(k) \in F(k)$ and $s = x$ and $f(x) = \prod s$.

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