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The Mycielskian of a Graph[1]

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Summary. Let $\omega(G)$ and $\chi(G)$ be the clique number and the chromatic number of a graph G. Mycielski [11] presented a construction that for any n creates a graph M_n which is triangle-free ($\omega(G) = 2$) with $\chi(G) > n$. The starting point is the complete graph of two vertices (K_2). $M_{(n+1)}$ is obtained from M_n through the operation $\mu(G)$ called the Mycielskian of a graph G.

We first define the operation $\mu(G)$ and then show that $\omega(\mu(G)) = \omega(G)$ and $\chi(\mu(G)) = \chi(G) + 1$. This is done for arbitrary graph G, see also [10]. Then we define the sequence of graphs M_n each of exponential size in n and give their clique and chromatic numbers.

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The notation and terminology used here have been introduced in the following papers: [1], [15], [13], [8], [5], [2], [14], [9], [16], [3], [6], [18], [19], [12], [17], [4], and [7].

1. Preliminaries

One can prove the following propositions:

(1) For all real numbers x, y, z such that $0 \leq x$ holds $x \cdot (y -' z) = x \cdot y -' x \cdot z$.

(2) For all natural numbers x, y, z holds $x \in y \setminus z$ iff $z \leq x < y$.

(3) For all sets A, B, C, D, E, X such that $X \subseteq A$ or $X \subseteq B$ or $X \subseteq C$ or $X \subseteq D$ or $X \subseteq E$ holds $X \subseteq A \cup B \cup C \cup D \cup E$.

(4) For all sets A, B, C, D, E, x holds $x \in A \cup B \cup C \cup D \cup E$ iff $x \in A$ or $x \in B$ or $x \in C$ or $x \in D$ or $x \in E$.

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(5) Let R be a symmetric relational structure and x, y be sets. Suppose $x \in$ the carrier of R and $y \in$ the carrier of R and $\langle x, y\rangle \in$ the internal relation of R. Then $\langle y, x\rangle \in$ the internal relation of R.

(6) For every symmetric relational structure R and for all elements x, y of R such that $x \leq y$ holds $y \leq x$.

2. Partitions

One can prove the following proposition

(7) For every set X and for every partition P of X holds $\overline{\overline{P}} \subseteq \overline{\overline{X}}$.

Let X be a set, let P be a partition of X, and let S be a subset of X. The functor $P{\restriction}S$ yields a partition of S and is defined by:

(Def. 1) $P{\restriction}S = \{x \cap S; x$ ranges over elements of P: x meets $S\}$.

Let X be a set. Observe that there exists a partition of X which is finite.

Let X be a set, let P be a finite partition of X, and let S be a subset of X. Observe that $P{\restriction}S$ is finite.

One can prove the following propositions:

(8) For every set X and for every finite partition P of X and for every subset S of X holds $\overline{\overline{P{\restriction}S}} \leq \overline{\overline{P}}$.

(9) Let X be a set, P be a finite partition of X, and S be a subset of X. Then for every set p such that $p \in P$ holds p meets S if and only if $\overline{\overline{P{\restriction}S}} = \overline{\overline{P}}$.

(10) Let R be a relational structure, C be a coloring of R, and S be a subset of R. Then $C{\restriction}S$ is a coloring of $\mathrm{sub}(S)$.

3. Chromatic Number and Clique Cover Number

Let R be a relational structure. We say that R is finitely colorable if and only if:

(Def. 2) There exists a coloring of R which is finite.

One can check that there exists a relational structure which is finitely colorable.

Let us observe that every relational structure which is finite is also finitely colorable.

Let R be a finitely colorable relational structure. Observe that there exists a coloring of R which is finite.

Let R be a finitely colorable relational structure and let S be a subset of R. One can verify that $\mathrm{sub}(S)$ is finitely colorable.

Let R be a finitely colorable relational structure. The functor $\chi(R)$ yielding a natural number is defined by:

(Def. 3) There exists a finite coloring C of R such that $\overline{\overline{C}} = \chi(R)$ and for every finite coloring C of R holds $\chi(R) \leq \overline{\overline{C}}$.

Let R be an empty relational structure. Observe that $\chi(R)$ is empty.

Let R be a non empty finitely colorable relational structure. Observe that $\chi(R)$ is positive.

Let R be a relational structure. We say that R has finite clique cover if and only if:

(Def. 4) There exists a clique-partition of R which is finite.

One can verify that there exists a relational structure which has finite clique cover.

One can verify that every relational structure which is finite has also finite clique cover.

Let R be a relational structure with finite clique cover. Observe that there exists a clique-partition of R which is finite.

Let R be a relational structure with finite clique cover and let S be a subset of R. Observe that $\mathrm{sub}(S)$ has finite clique cover.

Let R be a relational structure with finite clique cover. The functor $\kappa(R)$ yielding a natural number is defined by:

(Def. 5) There exists a finite clique-partition C of R such that $\overline{\overline{C}} = \kappa(R)$ and for every finite clique-partition C of R holds $\kappa(R) \leq \overline{\overline{C}}$.

Let R be an empty relational structure. One can check that $\kappa(R)$ is empty.

Let R be a non empty relational structure with finite clique cover. One can verify that $\kappa(R)$ is positive.

We now state several propositions:

(11) For every finite relational structure R holds $\omega(R) \leq \overline{\overline{\text{the carrier of } R}}$.

(12) For every finite relational structure R holds $\alpha(R) \leq \overline{\overline{\text{the carrier of } R}}$.

(13) For every finite relational structure R holds $\chi(R) \leq \overline{\overline{\text{the carrier of } R}}$.

(14) For every finite relational structure R holds $\kappa(R) \leq \overline{\overline{\text{the carrier of } R}}$.

(15) For every finitely colorable relational structure R with finite clique number holds $\omega(R) \leq \chi(R)$.

(16) For every relational structure R with finite stability number and finite clique cover holds $\alpha(R) \leq \kappa(R)$.

4. COMPLEMENT

The following two propositions are true:

(17) Let R be a relational structure, x, y be elements of R, and a, b be elements of ComplRelStr R. If $x = a$ and $y = b$ and $x \leq y$, then $a \not\leq b$.

(18) Let R be a relational structure, x, y be elements of R, and a, b be elements of ComplRelStr R. If $x = a$ and $y = b$ and $x \neq y$ and $x \in$ the carrier of R and $a \not\leq b$, then $x \leq y$.

Let R be a finite relational structure. Note that ComplRelStr R is finite.

Next we state four propositions:

(19) For every symmetric relational structure R holds every clique of R is a stable set of ComplRelStr R.

(20) For every symmetric relational structure R holds every clique of ComplRelStr R is a stable set of R.

(21) For every relational structure R holds every stable set of R is a clique of ComplRelStr R.

(22) For every relational structure R holds every stable set of ComplRelStr R is a clique of R.

Let R be a relational structure with finite clique number.

One can verify that ComplRelStr R has finite stability number.

Let R be a symmetric relational structure with finite stability number. Observe that ComplRelStr R has finite clique number.

The following propositions are true:

(23) For every symmetric relational structure R with finite clique number holds $\omega(R) = \alpha(\text{ComplRelStr}\, R)$.

(24) For every symmetric relational structure R with finite stability number holds $\alpha(R) = \omega(\text{ComplRelStr}\, R)$.

(25) For every relational structure R holds every coloring of R is a clique-partition of ComplRelStr R.

(26) For every symmetric relational structure R holds every clique-partition of ComplRelStr R is a coloring of R.

(27) For every symmetric relational structure R holds every clique-partition of R is a coloring of ComplRelStr R.

(28) For every relational structure R holds every coloring of ComplRelStr R is a clique-partition of R.

Let R be a finitely colorable relational structure.

Observe that ComplRelStr R has finite clique cover.

Let R be a symmetric relational structure with finite clique cover. One can check that ComplRelStr R is finitely colorable.

The following propositions are true:

(29) For every finitely colorable symmetric relational structure R holds $\chi(R) = \kappa(\text{ComplRelStr}\, R)$.

(30) For every symmetric relational structure R with finite clique cover holds $\kappa(R) = \chi(\text{ComplRelStr}\, R)$.

5. Adjacent Set

Let R be a relational structure and let v be an element of R. The functor Adjacent(v) yields a subset of R and is defined as follows:

(Def. 6) For every element x of R holds $x \in \mathrm{Adjacent}(v)$ iff $x < v$ or $v < x$.

The following proposition is true

(31) Let R be a finitely colorable relational structure, C be a finite coloring of R, and c be a set. Suppose $c \in C$ and $\overline{\overline{C}} = \chi(R)$. Then there exists an element v of R such that $v \in c$ and for every element d of C such that $d \neq c$ there exists an element w of R such that $w \in \mathrm{Adjacent}(v)$ and $w \in d$.

6. Natural Numbers as Vertices

Let n be a natural number. A strict relational structure is said to be a relational structure of n if:

(Def. 7) The carrier of it $= n$.

Let us observe that every relational structure of 0 is empty.

Let n be a non empty natural number. Note that every relational structure of n is non empty.

Let n be a natural number. Note that every relational structure of n is finite and there exists a relational structure of n which is irreflexive.

Let n be a natural number. The functor $K(n)$ yields a relational structure of n and is defined as follows:

(Def. 8) The internal relation of $K(n) = n \times n \setminus \mathrm{id}_n$.

The following proposition is true

(32) Let n be a natural number and x, y be sets. Suppose $x, y \in n$. Then $\langle x, y \rangle \in$ the internal relation of $K(n)$ if and only if $x \neq y$.

Let n be a natural number. Note that $K(n)$ is irreflexive and symmetric.

Let n be a natural number. Observe that $\Omega_{K(n)}$ is a clique.

The following propositions are true:

(33) For every natural number n holds $\omega(K(n)) = n$.

(34) For every non empty natural number n holds $\alpha(K(n)) = 1$.

(35) For every natural number n holds $\chi(K(n)) = n$.

(36) For every non empty natural number n holds $\kappa(K(n)) = 1$.

7. Mycielskian of a Graph

Let n be a natural number and let R be a relational structure of n. The functor Mycielskian R yields a relational structure of $2 \cdot n + 1$ and is defined by the condition (Def. 9).

(Def. 9) The internal relation of Mycielskian R = (the internal relation of R) $\cup$ $\{\langle x, y+n \rangle; x$ ranges over elements of $\mathbb{N}$, y ranges over elements of $\mathbb{N}$: $\langle x, y \rangle \in$ the internal relation of $R\} \cup \{\langle x+n, y \rangle; x$ ranges over elements of $\mathbb{N}$, y ranges over elements of $\mathbb{N}$: $\langle x, y \rangle \in$ the internal relation of $R\} \cup \{2 \cdot n\} \times (2 \cdot n \setminus n) \cup (2 \cdot n \setminus n) \times \{2 \cdot n\}$.

One can prove the following propositions:

(37) Let n be a natural number and R be a relational structure of n. Then the carrier of $R \subseteq$ the carrier of Mycielskian R.

(38) Let n be a natural number, R be a relational structure of n, and x, y be natural numbers. Suppose $\langle x, y \rangle \in$ the internal relation of Mycielskian R. Then

- (i) $x < n$ and $y < n$, or
- (ii) $x < n \leq y < 2 \cdot n$, or
- (iii) $n \leq x < 2 \cdot n$ and $y < n$, or
- (iv) $x = 2 \cdot n$ and $n \leq y < 2 \cdot n$, or
- (v) $n \leq x < 2 \cdot n$ and $y = 2 \cdot n$.

(39) Let n be a natural number and R be a relational structure of n. Then the internal relation of $R \subseteq$ the internal relation of Mycielskian R.

(40) Let n be a natural number, R be a relational structure of n, and x, y be sets. Suppose $x, y \in n$ and $\langle x, y \rangle \in$ the internal relation of Mycielskian R. Then $\langle x, y \rangle \in$ the internal relation of R.

(41) Let n be a natural number, R be a relational structure of n, and x, y be natural numbers. Suppose $\langle x, y \rangle \in$ the internal relation of R. Then $\langle x, y+n \rangle \in$ the internal relation of Mycielskian R and $\langle x+n, y \rangle \in$ the internal relation of Mycielskian R.

(42) Let n be a natural number, R be a relational structure of n, and x, y be natural numbers. Suppose $x \in n$ and $\langle x, y+n \rangle \in$ the internal relation of Mycielskian R. Then $\langle x, y \rangle \in$ the internal relation of R.

(43) Let n be a natural number, R be a relational structure of n, and x, y be natural numbers. Suppose $y \in n$ and $\langle x+n, y \rangle \in$ the internal relation of Mycielskian R. Then $\langle x, y \rangle \in$ the internal relation of R.

(44) Let n be a natural number, R be a relational structure of n, and m be a natural number. Suppose $n \leq m < 2 \cdot n$. Then $\langle m, 2 \cdot n \rangle \in$ the internal relation of Mycielskian R and $\langle 2 \cdot n, m \rangle \in$ the internal relation of Mycielskian R.

(45) Let n be a natural number, R be a relational structure of n, and S be a subset of Mycielskian R. If $S = n$, then $R = \mathrm{sub}(S)$.

(46) For every natural number n and for every irreflexive relational structure R of n such that $2 \leq \omega(R)$ holds $\omega(R) = \omega(\mathrm{Mycielskian}\, R)$.

(47) For every finitely colorable relational structure R and for every subset S of R holds $\chi(R) \geq \chi(\mathrm{sub}(S))$.

(48) For every natural number n and for every irreflexive relational structure R of n holds $\chi(\mathrm{Mycielskian}\, R) = 1 + \chi(R)$.

Let n be a natural number. The functor Mycielskian n yielding a relational structure of $3 \cdot 2^n -' 1$ is defined by the condition (Def. 10).

(Def. 10) There exists a function m_1 such that

(i) Mycielskian $n = m_1(n)$,

(ii) $\mathrm{dom}\, m_1 = \mathbb{N}$,

(iii) $m_1(0) = K(2)$, and

(iv) for every natural number k and for every relational structure R of $3 \cdot 2^k -' 1$ such that $R = m_1(k)$ holds $m_1(k+1) = \mathrm{Mycielskian}\, R$.

The following proposition is true

(49) Mycielskian $0 = K(2)$ and for every natural number k holds $\mathrm{Mycielskian}(k+1) = \mathrm{Mycielskian}\, \mathrm{Mycielskian}\, k$.

Let n be a natural number. One can verify that Mycielskian n is irreflexive.

Let n be a natural number. Observe that Mycielskian n is symmetric.

We now state three propositions:

(50) For every natural number n holds $\omega(\mathrm{Mycielskian}\, n) = 2$ and $\chi(\mathrm{Mycielskian}\, n) = n + 2$.

(51) For every natural number n there exists a finite relational structure R such that $\omega(R) = 2$ and $\chi(R) > n$.

(52) For every natural number n there exists a finite relational structure R such that $\alpha(R) = 2$ and $\kappa(R) > n$.

References

[1] Grzegorz Bancerek. Cardinal numbers. *Formalized Mathematics*, 1(**2**):377–382, 1990.
[2] Grzegorz Bancerek. The fundamental properties of natural numbers. *Formalized Mathematics*, 1(**1**):41–46, 1990.
[3] Grzegorz Bancerek. The ordinal numbers. *Formalized Mathematics*, 1(**1**):91–96, 1990.
[4] Grzegorz Bancerek. Bounds in posets and relational substructures. *Formalized Mathematics*, 6(**1**):81–91, 1997.
[5] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.
[6] Czesław Byliński. Partial functions. *Formalized Mathematics*, 1(**2**):357–367, 1990.
[7] Czesław Byliński. Some basic properties of sets. *Formalized Mathematics*, 1(**1**):47–53, 1990.
[8] Agata Darmochwał. Finite sets. *Formalized Mathematics*, 1(**1**):165–167, 1990.
[9] Rafał Kwiatek. Factorial and Newton coefficients. *Formalized Mathematics*, 1(**5**):887–890, 1990.

[10] M. Larsen, J. Propp, and D. Ullman. The fractional chromatic number of Mycielski's graphs. *Journal of Graph Theory*, 19:411–416, 1995.

[11] J. Mycielski. Sur le coloriage des graphes. *Colloquium Mathematicum*, 3:161–162, 1955.

[12] Beata Padlewska. Families of sets. *Formalized Mathematics*, 1(**1**):147–152, 1990.

[13] Konrad Raczkowski and Paweł Sadowski. Equivalence relations and classes of abstraction. *Formalized Mathematics*, 1(**3**):441–444, 1990.

[14] Krzysztof Retel. The class of series – parallel graphs. Part I. *Formalized Mathematics*, 11(**1**):99–103, 2003.

[15] Piotr Rudnicki. Dilworth's decomposition theorem for posets. *Formalized Mathematics*, 17(**4**):223–232, 2009, doi: 10.2478/v10037-009-0028-4.

[16] Wojciech A. Trybulec and Grzegorz Bancerek. Kuratowski – Zorn lemma. *Formalized Mathematics*, 1(**2**):387–393, 1990.

[17] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(**1**):67–71, 1990.

[18] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1(**1**):73–83, 1990.

[19] Edmund Woronowicz. Relations defined on sets. *Formalized Mathematics*, 1(**1**):181–186, 1990.

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