

The Mycielskian of a Graph¹

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Summary. Let $\omega(G)$ and $\chi(G)$ be the clique number and the chromatic number of a graph G . Mycielski [11] presented a construction that for any n creates a graph M_n which is triangle-free ($\omega(G) = 2$) with $\chi(G) > n$. The starting point is the complete graph of two vertices (K_2). $M_{(n+1)}$ is obtained from M_n through the operation $\mu(G)$ called the Mycielskian of a graph G .

We first define the operation $\mu(G)$ and then show that $\omega(\mu(G)) = \omega(G)$ and $\chi(\mu(G)) = \chi(G) + 1$. This is done for arbitrary graph G , see also [10]. Then we define the sequence of graphs M_n each of exponential size in n and give their clique and chromatic numbers.

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The notation and terminology used here have been introduced in the following papers: [1], [15], [13], [8], [5], [2], [14], [9], [16], [3], [6], [18], [19], [12], [17], [4], and [7].

1. PRELIMINARIES

One can prove the following propositions:

- (1) For all real numbers x, y, z such that $0 \leq x$ holds $x \cdot (y -' z) = x \cdot y -' x \cdot z$.
- (2) For all natural numbers x, y, z holds $x \in y \setminus z$ iff $z \leq x < y$.
- (3) For all sets A, B, C, D, E, X such that $X \subseteq A$ or $X \subseteq B$ or $X \subseteq C$ or $X \subseteq D$ or $X \subseteq E$ holds $X \subseteq A \cup B \cup C \cup D \cup E$.
- (4) For all sets A, B, C, D, E, x holds $x \in A \cup B \cup C \cup D \cup E$ iff $x \in A$ or $x \in B$ or $x \in C$ or $x \in D$ or $x \in E$.

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- (5) Let R be a symmetric relational structure and x, y be sets. Suppose $x \in$ the carrier of R and $y \in$ the carrier of R and $\langle x, y \rangle \in$ the internal relation of R . Then $\langle y, x \rangle \in$ the internal relation of R .
- (6) For every symmetric relational structure R and for all elements x, y of R such that $x \leq y$ holds $y \leq x$.

2. PARTITIONS

One can prove the following proposition

- (7) For every set X and for every partition P of X holds $\overline{P} \subseteq \overline{X}$.

Let X be a set, let P be a partition of X , and let S be a subset of X . The functor $P \upharpoonright S$ yields a partition of S and is defined by:

(Def. 1) $P \upharpoonright S = \{x \cap S; x \text{ ranges over elements of } P: x \text{ meets } S\}$.

Let X be a set. Observe that there exists a partition of X which is finite.

Let X be a set, let P be a finite partition of X , and let S be a subset of X . Observe that $P \upharpoonright S$ is finite.

One can prove the following propositions:

- (8) For every set X and for every finite partition P of X and for every subset S of X holds $\overline{P \upharpoonright S} \leq \overline{P}$.
- (9) Let X be a set, P be a finite partition of X , and S be a subset of X . Then for every set p such that $p \in P$ holds p meets S if and only if $\overline{P \upharpoonright S} = \overline{P}$.
- (10) Let R be a relational structure, C be a coloring of R , and S be a subset of R . Then $C \upharpoonright S$ is a coloring of $\text{sub}(S)$.

3. CHROMATIC NUMBER AND CLIQUE COVER NUMBER

Let R be a relational structure. We say that R is finitely colorable if and only if:

(Def. 2) There exists a coloring of R which is finite.

One can check that there exists a relational structure which is finitely colorable.

Let us observe that every relational structure which is finite is also finitely colorable.

Let R be a finitely colorable relational structure. Observe that there exists a coloring of R which is finite.

Let R be a finitely colorable relational structure and let S be a subset of R . One can verify that $\text{sub}(S)$ is finitely colorable.

Let R be a finitely colorable relational structure. The functor $\chi(R)$ yielding a natural number is defined by:

(Def. 3) There exists a finite coloring C of R such that $\overline{\overline{C}} = \chi(R)$ and for every finite coloring C of R holds $\chi(R) \leq \overline{\overline{C}}$.

Let R be an empty relational structure. Observe that $\chi(R)$ is empty.

Let R be a non empty finitely colorable relational structure. Observe that $\chi(R)$ is positive.

Let R be a relational structure. We say that R has finite clique cover if and only if:

(Def. 4) There exists a clique-partition of R which is finite.

One can verify that there exists a relational structure which has finite clique cover.

One can verify that every relational structure which is finite has also finite clique cover.

Let R be a relational structure with finite clique cover. Observe that there exists a clique-partition of R which is finite.

Let R be a relational structure with finite clique cover and let S be a subset of R . Observe that $\text{sub}(S)$ has finite clique cover.

Let R be a relational structure with finite clique cover. The functor $\kappa(R)$ yielding a natural number is defined by:

(Def. 5) There exists a finite clique-partition C of R such that $\overline{\overline{C}} = \kappa(R)$ and for every finite clique-partition C of R holds $\kappa(R) \leq \overline{\overline{C}}$.

Let R be an empty relational structure. One can check that $\kappa(R)$ is empty.

Let R be a non empty relational structure with finite clique cover. One can verify that $\kappa(R)$ is positive.

We now state several propositions:

- (11) For every finite relational structure R holds $\omega(R) \leq \overline{\overline{\text{the carrier of } R}}$.
- (12) For every finite relational structure R holds $\alpha(R) \leq \overline{\overline{\text{the carrier of } R}}$.
- (13) For every finite relational structure R holds $\chi(R) \leq \overline{\overline{\text{the carrier of } R}}$.
- (14) For every finite relational structure R holds $\kappa(R) \leq \overline{\overline{\text{the carrier of } R}}$.
- (15) For every finitely colorable relational structure R with finite clique number holds $\omega(R) \leq \chi(R)$.
- (16) For every relational structure R with finite stability number and finite clique cover holds $\alpha(R) \leq \kappa(R)$.

4. COMPLEMENT

The following two propositions are true:

- (17) Let R be a relational structure, x, y be elements of R , and a, b be elements of $\text{CompRelStr } R$. If $x = a$ and $y = b$ and $x \leq y$, then $a \not\leq b$.

- (18) Let R be a relational structure, x, y be elements of R , and a, b be elements of $\text{ComplRelStr } R$. If $x = a$ and $y = b$ and $x \neq y$ and $x \in$ the carrier of R and $a \not\leq b$, then $x \leq y$.

Let R be a finite relational structure. Note that $\text{ComplRelStr } R$ is finite.

Next we state four propositions:

- (19) For every symmetric relational structure R holds every clique of R is a stable set of $\text{ComplRelStr } R$.
- (20) For every symmetric relational structure R holds every clique of $\text{ComplRelStr } R$ is a stable set of R .
- (21) For every relational structure R holds every stable set of R is a clique of $\text{ComplRelStr } R$.
- (22) For every relational structure R holds every stable set of $\text{ComplRelStr } R$ is a clique of R .

Let R be a relational structure with finite clique number.

One can verify that $\text{ComplRelStr } R$ has finite stability number.

Let R be a symmetric relational structure with finite stability number. Observe that $\text{ComplRelStr } R$ has finite clique number.

The following propositions are true:

- (23) For every symmetric relational structure R with finite clique number holds $\omega(R) = \alpha(\text{ComplRelStr } R)$.
- (24) For every symmetric relational structure R with finite stability number holds $\alpha(R) = \omega(\text{ComplRelStr } R)$.
- (25) For every relational structure R holds every coloring of R is a clique-partition of $\text{ComplRelStr } R$.
- (26) For every symmetric relational structure R holds every clique-partition of $\text{ComplRelStr } R$ is a coloring of R .
- (27) For every symmetric relational structure R holds every clique-partition of R is a coloring of $\text{ComplRelStr } R$.
- (28) For every relational structure R holds every coloring of $\text{ComplRelStr } R$ is a clique-partition of R .

Let R be a finitely colorable relational structure.

Observe that $\text{ComplRelStr } R$ has finite clique cover.

Let R be a symmetric relational structure with finite clique cover. One can check that $\text{ComplRelStr } R$ is finitely colorable.

The following propositions are true:

- (29) For every finitely colorable symmetric relational structure R holds $\chi(R) = \kappa(\text{ComplRelStr } R)$.
- (30) For every symmetric relational structure R with finite clique cover holds $\kappa(R) = \chi(\text{ComplRelStr } R)$.

5. ADJACENT SET

Let R be a relational structure and let v be an element of R . The functor $\text{Adjacent}(v)$ yields a subset of R and is defined as follows:

(Def. 6) For every element x of R holds $x \in \text{Adjacent}(v)$ iff $x < v$ or $v < x$.

The following proposition is true

(31) Let R be a finitely colorable relational structure, C be a finite coloring of R , and c be a set. Suppose $c \in C$ and $\overline{C} = \chi(R)$. Then there exists an element v of R such that $v \in c$ and for every element d of C such that $d \neq c$ there exists an element w of R such that $w \in \text{Adjacent}(v)$ and $w \in d$.

6. NATURAL NUMBERS AS VERTICES

Let n be a natural number. A strict relational structure is said to be a relational structure of n if:

(Def. 7) The carrier of it = n .

Let us observe that every relational structure of 0 is empty.

Let n be a non empty natural number. Note that every relational structure of n is non empty.

Let n be a natural number. Note that every relational structure of n is finite and there exists a relational structure of n which is irreflexive.

Let n be a natural number. The functor $K(n)$ yields a relational structure of n and is defined as follows:

(Def. 8) The internal relation of $K(n) = n \times n \setminus \text{id}_n$.

The following proposition is true

(32) Let n be a natural number and x, y be sets. Suppose $x, y \in n$. Then $\langle x, y \rangle \in$ the internal relation of $K(n)$ if and only if $x \neq y$.

Let n be a natural number. Note that $K(n)$ is irreflexive and symmetric.

Let n be a natural number. Observe that $\Omega_{K(n)}$ is a clique.

The following propositions are true:

(33) For every natural number n holds $\omega(K(n)) = n$.

(34) For every non empty natural number n holds $\alpha(K(n)) = 1$.

(35) For every natural number n holds $\chi(K(n)) = n$.

(36) For every non empty natural number n holds $\kappa(K(n)) = 1$.

7. MYCIELSKIAN OF A GRAPH

Let n be a natural number and let R be a relational structure of n . The functor Mycielskian R yields a relational structure of $2 \cdot n + 1$ and is defined by the condition (Def. 9).

(Def. 9) The internal relation of Mycielskian $R = (\text{the internal relation of } R) \cup \{\langle x, y + n \rangle; x \text{ ranges over elements of } \mathbb{N}, y \text{ ranges over elements of } \mathbb{N}: \langle x, y \rangle \in \text{the internal relation of } R\} \cup \{\langle x + n, y \rangle; x \text{ ranges over elements of } \mathbb{N}, y \text{ ranges over elements of } \mathbb{N}: \langle x, y \rangle \in \text{the internal relation of } R\} \cup \{2 \cdot n\} \times (2 \cdot n \setminus n) \cup (2 \cdot n \setminus n) \times \{2 \cdot n\}$.

One can prove the following propositions:

- (37) Let n be a natural number and R be a relational structure of n . Then the carrier of $R \subseteq$ the carrier of Mycielskian R .
- (38) Let n be a natural number, R be a relational structure of n , and x, y be natural numbers. Suppose $\langle x, y \rangle \in$ the internal relation of Mycielskian R . Then
- (i) $x < n$ and $y < n$, or
 - (ii) $x < n \leq y < 2 \cdot n$, or
 - (iii) $n \leq x < 2 \cdot n$ and $y < n$, or
 - (iv) $x = 2 \cdot n$ and $n \leq y < 2 \cdot n$, or
 - (v) $n \leq x < 2 \cdot n$ and $y = 2 \cdot n$.
- (39) Let n be a natural number and R be a relational structure of n . Then the internal relation of $R \subseteq$ the internal relation of Mycielskian R .
- (40) Let n be a natural number, R be a relational structure of n , and x, y be sets. Suppose $x, y \in n$ and $\langle x, y \rangle \in$ the internal relation of Mycielskian R . Then $\langle x, y \rangle \in$ the internal relation of R .
- (41) Let n be a natural number, R be a relational structure of n , and x, y be natural numbers. Suppose $\langle x, y \rangle \in$ the internal relation of R . Then $\langle x, y + n \rangle \in$ the internal relation of Mycielskian R and $\langle x + n, y \rangle \in$ the internal relation of Mycielskian R .
- (42) Let n be a natural number, R be a relational structure of n , and x, y be natural numbers. Suppose $x \in n$ and $\langle x, y + n \rangle \in$ the internal relation of Mycielskian R . Then $\langle x, y \rangle \in$ the internal relation of R .
- (43) Let n be a natural number, R be a relational structure of n , and x, y be natural numbers. Suppose $y \in n$ and $\langle x + n, y \rangle \in$ the internal relation of Mycielskian R . Then $\langle x, y \rangle \in$ the internal relation of R .
- (44) Let n be a natural number, R be a relational structure of n , and m be a natural number. Suppose $n \leq m < 2 \cdot n$. Then $\langle m, 2 \cdot n \rangle \in$ the internal relation of Mycielskian R and $\langle 2 \cdot n, m \rangle \in$ the internal relation of Mycielskian R .

- (45) Let n be a natural number, R be a relational structure of n , and S be a subset of Mycielskian R . If $S = n$, then $R = \text{sub}(S)$.
- (46) For every natural number n and for every irreflexive relational structure R of n such that $2 \leq \omega(R)$ holds $\omega(R) = \omega(\text{Mycielskian } R)$.
- (47) For every finitely colorable relational structure R and for every subset S of R holds $\chi(R) \geq \chi(\text{sub}(S))$.
- (48) For every natural number n and for every irreflexive relational structure R of n holds $\chi(\text{Mycielskian } R) = 1 + \chi(R)$.

Let n be a natural number. The functor Mycielskian n yielding a relational structure of $3 \cdot 2^n - 1$ is defined by the condition (Def. 10).

(Def. 10) There exists a function m_1 such that

- (i) Mycielskian $n = m_1(n)$,
- (ii) $\text{dom } m_1 = \mathbb{N}$,
- (iii) $m_1(0) = K(2)$, and
- (iv) for every natural number k and for every relational structure R of $3 \cdot 2^k - 1$ such that $R = m_1(k)$ holds $m_1(k + 1) = \text{Mycielskian } R$.

The following proposition is true

- (49) Mycielskian $0 = K(2)$ and for every natural number k holds Mycielskian $(k + 1) = \text{Mycielskian Mycielskian } k$.

Let n be a natural number. One can verify that Mycielskian n is irreflexive.

Let n be a natural number. Observe that Mycielskian n is symmetric.

We now state three propositions:

- (50) For every natural number n holds $\omega(\text{Mycielskian } n) = 2$ and $\chi(\text{Mycielskian } n) = n + 2$.
- (51) For every natural number n there exists a finite relational structure R such that $\omega(R) = 2$ and $\chi(R) > n$.
- (52) For every natural number n there exists a finite relational structure R such that $\alpha(R) = 2$ and $\kappa(R) > n$.

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