

More on Continuous Functions on Normed Linear Spaces

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Summary. In this article we formalize the definition and some facts about continuous functions from \mathbb{R} into normed linear spaces [14].

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The terminology and notation used in this paper have been introduced in the following papers: [2], [12], [3], [4], [10], [11], [1], [5], [13], [7], [17], [18], [15], [9], [8], [16], [19], and [6].

1. PRELIMINARIES

For simplicity, we adopt the following rules: n denotes an element of \mathbb{N} , X , X_1 denote sets, r , p denote real numbers, s , x_0 , x_1 , x_2 denote real numbers, S , T denote real normed spaces, f , f_1 , f_2 denote partial functions from \mathbb{R} to the carrier of S , s_1 denotes a sequence of real numbers, and Y denotes a subset of \mathbb{R} .

The following propositions are true:

- (1) Let s_2 be a sequence of real numbers and h be a partial function from \mathbb{R} to the carrier of S . If $\text{rng } s_2 \subseteq \text{dom } h$, then $s_2(n) \in \text{dom } h$.
- (2) Let h_1 , h_2 be partial functions from \mathbb{R} to the carrier of S and s_2 be a sequence of real numbers. If $\text{rng } s_2 \subseteq \text{dom } h_1 \cap \text{dom } h_2$, then $(h_1 + h_2)_* s_2 = (h_1 * s_2) + (h_2 * s_2)$ and $(h_1 - h_2)_* s_2 = (h_1 * s_2) - (h_2 * s_2)$.

- (3) For every sequence h of S and for every real number r holds $rh = r \cdot h$.
- (4) Let h be a partial function from \mathbb{R} to the carrier of S , s_2 be a sequence of real numbers, and r be a real number. If $\text{rng } s_2 \subseteq \text{dom } h$, then $rh_*s_2 = r \cdot (h_*s_2)$.
- (5) Let h be a partial function from \mathbb{R} to the carrier of S and s_2 be a sequence of real numbers. If $\text{rng } s_2 \subseteq \text{dom } h$, then $\|h_*s_2\| = \|h\|_*s_2$ and $-(h_*s_2) = -h_*s_2$.

2. CONTINUOUS REAL FUNCTIONS INTO NORMED LINEAR SPACES

Let us consider S, f, x_0 . We say that f is continuous in x_0 if and only if:

- (Def. 1) $x_0 \in \text{dom } f$ and for every s_1 such that $\text{rng } s_1 \subseteq \text{dom } f$ and s_1 is convergent and $\lim s_1 = x_0$ holds f_*s_1 is convergent and $f_{x_0} = \lim(f_*s_1)$.

Next we state a number of propositions:

- (6) If $x_0 \in X$ and f is continuous in x_0 , then $f|X$ is continuous in x_0 .
- (7) f is continuous in x_0 if and only if the following conditions are satisfied:
 - (i) $x_0 \in \text{dom } f$, and
 - (ii) for every s_1 such that $\text{rng } s_1 \subseteq \text{dom } f$ and s_1 is convergent and $\lim s_1 = x_0$ and for every n holds $s_1(n) \neq x_0$ holds f_*s_1 is convergent and $f_{x_0} = \lim(f_*s_1)$.
- (8) f is continuous in x_0 if and only if the following conditions are satisfied:
 - (i) $x_0 \in \text{dom } f$, and
 - (ii) for every r such that $0 < r$ there exists s such that $0 < s$ and for every x_1 such that $x_1 \in \text{dom } f$ and $|x_1 - x_0| < s$ holds $\|f_{x_1} - f_{x_0}\| < r$.
- (9) Let given S, f, x_0 . Then f is continuous in x_0 if and only if the following conditions are satisfied:
 - (i) $x_0 \in \text{dom } f$, and
 - (ii) for every neighbourhood N_1 of f_{x_0} there exists a neighbourhood N of x_0 such that for every x_1 such that $x_1 \in \text{dom } f$ and $x_1 \in N$ holds $f_{x_1} \in N_1$.
- (10) Let given S, f, x_0 . Then f is continuous in x_0 if and only if the following conditions are satisfied:
 - (i) $x_0 \in \text{dom } f$, and
 - (ii) for every neighbourhood N_1 of f_{x_0} there exists a neighbourhood N of x_0 such that $f^\circ N \subseteq N_1$.
- (11) If there exists a neighbourhood N of x_0 such that $\text{dom } f \cap N = \{x_0\}$, then f is continuous in x_0 .
- (12) If $x_0 \in \text{dom } f_1 \cap \text{dom } f_2$ and f_1 is continuous in x_0 and f_2 is continuous in x_0 , then $f_1 + f_2$ is continuous in x_0 and $f_1 - f_2$ is continuous in x_0 .
- (13) If f is continuous in x_0 , then rf is continuous in x_0 .

- (14) If $x_0 \in \text{dom } f$ and f is continuous in x_0 , then $\|f\|$ is continuous in x_0 and $-f$ is continuous in x_0 .
- (15) Let f_1 be a partial function from \mathbb{R} to the carrier of S and f_2 be a partial function from the carrier of S to the carrier of T . Suppose $x_0 \in \text{dom}(f_2 \cdot f_1)$ and f_1 is continuous in x_0 and f_2 is continuous in $(f_1)_{x_0}$. Then $f_2 \cdot f_1$ is continuous in x_0 .

Let us consider S, f . We say that f is continuous if and only if:

(Def. 2) For every x_0 such that $x_0 \in \text{dom } f$ holds f is continuous in x_0 .

Next we state two propositions:

- (16) Let given X, f . Suppose $X \subseteq \text{dom } f$. Then $f \upharpoonright X$ is continuous if and only if for every s_1 such that $\text{rng } s_1 \subseteq X$ and s_1 is convergent and $\lim s_1 \in X$ holds f_*s_1 is convergent and $f_{\lim s_1} = \lim(f_*s_1)$.
- (17) Suppose $X \subseteq \text{dom } f$. Then $f \upharpoonright X$ is continuous if and only if for all x_0, r such that $x_0 \in X$ and $0 < r$ there exists s such that $0 < s$ and for every x_1 such that $x_1 \in X$ and $|x_1 - x_0| < s$ holds $\|f_{x_1} - f_{x_0}\| < r$.

Let us consider S . One can check that every partial function from \mathbb{R} to the carrier of S which is constant is also continuous.

Let us consider S . Note that there exists a partial function from \mathbb{R} to the carrier of S which is continuous.

Let us consider S , let f be a continuous partial function from \mathbb{R} to the carrier of S , and let X be a set. Observe that $f \upharpoonright X$ is continuous.

Next we state the proposition

- (18) If $f \upharpoonright X$ is continuous and $X_1 \subseteq X$, then $f \upharpoonright X_1$ is continuous.

Let us consider S . Observe that every partial function from \mathbb{R} to the carrier of S which is empty is also continuous.

Let us consider S, f and let X be a trivial set. Observe that $f \upharpoonright X$ is continuous.

Let us consider S and let f_1, f_2 be continuous partial functions from \mathbb{R} to the carrier of S . Observe that $f_1 + f_2$ is continuous and $f_1 - f_2$ is continuous.

The following two propositions are true:

- (19) Let given X, f_1, f_2 . Suppose $X \subseteq \text{dom } f_1 \cap \text{dom } f_2$ and $f_1 \upharpoonright X$ is continuous and $f_2 \upharpoonright X$ is continuous. Then $(f_1 + f_2) \upharpoonright X$ is continuous and $(f_1 - f_2) \upharpoonright X$ is continuous.
- (20) Let given X, X_1, f_1, f_2 . Suppose $X \subseteq \text{dom } f_1$ and $X_1 \subseteq \text{dom } f_2$ and $f_1 \upharpoonright X$ is continuous and $f_2 \upharpoonright X_1$ is continuous. Then $(f_1 + f_2) \upharpoonright (X \cap X_1)$ is continuous and $(f_1 - f_2) \upharpoonright (X \cap X_1)$ is continuous.

Let us consider S , let f be a continuous partial function from \mathbb{R} to the carrier of S , and let us consider r . One can check that $r f$ is continuous.

We now state several propositions:

- (21) If $X \subseteq \text{dom } f$ and $f \upharpoonright X$ is continuous, then $(r f) \upharpoonright X$ is continuous.

- (22) If $X \subseteq \text{dom } f$ and $f \upharpoonright X$ is continuous, then $\|f\| \upharpoonright X$ is continuous and $(-f) \upharpoonright X$ is continuous.
- (23) If f is total and for all x_1, x_2 holds $f_{x_1+x_2} = f_{x_1} + f_{x_2}$ and there exists x_0 such that f is continuous in x_0 , then $f \upharpoonright \mathbb{R}$ is continuous.
- (24) If $\text{dom } f$ is compact and $f \upharpoonright \text{dom } f$ is continuous, then $\text{rng } f$ is compact.
- (25) If $Y \subseteq \text{dom } f$ and Y is compact and $f \upharpoonright Y$ is continuous, then $f^\circ Y$ is compact.

3. LIPSCHITZ CONTINUITY

Let us consider S, f . We say that f is Lipschitzian if and only if:

- (Def. 3) There exists a real number r such that $0 < r$ and for all x_1, x_2 such that $x_1, x_2 \in \text{dom } f$ holds $\|f_{x_1} - f_{x_2}\| \leq r \cdot |x_1 - x_2|$.

The following proposition is true

- (26) $f \upharpoonright X$ is Lipschitzian if and only if there exists a real number r such that $0 < r$ and for all x_1, x_2 such that $x_1, x_2 \in \text{dom}(f \upharpoonright X)$ holds $\|f_{x_1} - f_{x_2}\| \leq r \cdot |x_1 - x_2|$.

Let us consider S . Observe that every partial function from \mathbb{R} to the carrier of S which is empty is also Lipschitzian.

Let us consider S . One can verify that there exists a partial function from \mathbb{R} to the carrier of S which is empty.

Let us consider S , let f be a Lipschitzian partial function from \mathbb{R} to the carrier of S , and let X be a set. One can check that $f \upharpoonright X$ is Lipschitzian.

The following proposition is true

- (27) If $f \upharpoonright X$ is Lipschitzian and $X_1 \subseteq X$, then $f \upharpoonright X_1$ is Lipschitzian.

Let us consider S and let f_1, f_2 be Lipschitzian partial functions from \mathbb{R} to the carrier of S . One can check that $f_1 + f_2$ is Lipschitzian and $f_1 - f_2$ is Lipschitzian.

One can prove the following propositions:

- (28) If $f_1 \upharpoonright X$ is Lipschitzian and $f_2 \upharpoonright X_1$ is Lipschitzian, then $(f_1 + f_2) \upharpoonright (X \cap X_1)$ is Lipschitzian.
- (29) If $f_1 \upharpoonright X$ is Lipschitzian and $f_2 \upharpoonright X_1$ is Lipschitzian, then $(f_1 - f_2) \upharpoonright (X \cap X_1)$ is Lipschitzian.

Let us consider S , let f be a Lipschitzian partial function from \mathbb{R} to the carrier of S , and let us consider p . Note that $p f$ is Lipschitzian.

Next we state the proposition

- (30) If $f \upharpoonright X$ is Lipschitzian and $X \subseteq \text{dom } f$, then $(p f) \upharpoonright X$ is Lipschitzian.

Let us consider S and let f be a Lipschitzian partial function from \mathbb{R} to the carrier of S . Note that $\|f\|$ is Lipschitzian.

One can prove the following proposition

- (31) If $f|X$ is Lipschitzian, then $-f|X$ is Lipschitzian and $(-f)|X$ is Lipschitzian and $\|f\||X$ is Lipschitzian.

Let us consider S . One can verify that every partial function from \mathbb{R} to the carrier of S which is constant is also Lipschitzian.

Let us consider S . Observe that every partial function from \mathbb{R} to the carrier of S which is Lipschitzian is also continuous.

Next we state two propositions:

- (32) If there exists a point r of S such that $\text{rng } f = \{r\}$, then f is continuous.
 (33) For all points r, p of S such that for every x_0 such that $x_0 \in X$ holds $f_{x_0} = x_0 \cdot r + p$ holds $f|X$ is continuous.

REFERENCES

- [1] Grzegorz Bancerek. The ordinal numbers. *Formalized Mathematics*, 1(1):91–96, 1990.
- [2] Czesław Byliński. The complex numbers. *Formalized Mathematics*, 1(3):507–513, 1990.
- [3] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(1):55–65, 1990.
- [4] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(1):153–164, 1990.
- [5] Czesław Byliński. Partial functions. *Formalized Mathematics*, 1(2):357–367, 1990.
- [6] Czesław Byliński. Some basic properties of sets. *Formalized Mathematics*, 1(1):47–53, 1990.
- [7] Krzysztof Hryniewiecki. Basic properties of real numbers. *Formalized Mathematics*, 1(1):35–40, 1990.
- [8] Jarosław Kotowicz. Convergent sequences and the limit of sequences. *Formalized Mathematics*, 1(2):273–275, 1990.
- [9] Jarosław Kotowicz. Real sequences and basic operations on them. *Formalized Mathematics*, 1(2):269–272, 1990.
- [10] Takaya Nishiyama, Keiji Ohkubo, and Yasunari Shidama. The continuous functions on normed linear spaces. *Formalized Mathematics*, 12(3):269–275, 2004.
- [11] Jan Popiołek. Real normed space. *Formalized Mathematics*, 2(1):111–115, 1991.
- [12] Konrad Raczkowski and Paweł Sadowski. Real function continuity. *Formalized Mathematics*, 1(4):787–791, 1990.
- [13] Konrad Raczkowski and Paweł Sadowski. Topological properties of subsets in real numbers. *Formalized Mathematics*, 1(4):777–780, 1990.
- [14] Laurent Schwartz. Cours d’analyse, vol. 1. *Hermann Paris*, 1967.
- [15] Wojciech A. Trybulec. Vectors in real linear space. *Formalized Mathematics*, 1(2):291–296, 1990.
- [16] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(1):67–71, 1990.
- [17] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1(1):73–83, 1990.
- [18] Edmund Woronowicz. Relations defined on sets. *Formalized Mathematics*, 1(1):181–186, 1990.
- [19] Hiroshi Yamazaki and Yasunari Shidama. Algebra of vector functions. *Formalized Mathematics*, 3(2):171–175, 1992.

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