

Cartesian Products of Family of Real Linear Spaces

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Summary. In this article we introduced the isomorphism mapping between cartesian products of family of linear spaces [4]. Those products had been formalized by two different ways, i.e., the way using the functor $[:X,Y:]$ and ones using the functor “product”. By the same way, the isomorphism mapping was defined between Cartesian products of family of linear normed spaces also.

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The notation and terminology used in this paper are introduced in the following articles: [5], [1], [16], [11], [3], [6], [17], [7], [8], [15], [14], [2], [13], [12], [20], [18], [10], [19], and [9].

1. PRELIMINARIES

One can prove the following propositions:

- (1) Let D, E, F, G be non empty sets. Then there exists a function I from $D \times E \times (F \times G)$ into $D \times F \times (E \times G)$ such that
 - (i) I is one-to-one and onto, and
 - (ii) for all sets d, e, f, g such that $d \in D$ and $e \in E$ and $f \in F$ and $g \in G$ holds $I(\langle d, e \rangle, \langle f, g \rangle) = \langle \langle d, f \rangle, \langle e, g \rangle \rangle$.

- (2) Let X be a non empty set and D be a function. Suppose $\text{dom } D = \{1\}$ and $D(1) = X$. Then there exists a function I from X into $\prod D$ such that I is one-to-one and onto and for every set x such that $x \in X$ holds $I(x) = \langle x \rangle$.
- (3) Let X, Y be non empty sets and D be a function. Suppose $\text{dom } D = \{1, 2\}$ and $D(1) = X$ and $D(2) = Y$. Then there exists a function I from $X \times Y$ into $\prod D$ such that I is one-to-one and onto and for all sets x, y such that $x \in X$ and $y \in Y$ holds $I(x, y) = \langle x, y \rangle$.
- (4) Let X be a non empty set. Then there exists a function I from X into $\prod \langle X \rangle$ such that I is one-to-one and onto and for every set x such that $x \in X$ holds $I(x) = \langle x \rangle$.

Let X, Y be non-empty non empty finite sequences. Observe that $X \cap Y$ is non-empty.

We now state two propositions:

- (5) Let X, Y be non empty sets. Then there exists a function I from $X \times Y$ into $\prod \langle X, Y \rangle$ such that I is one-to-one and onto and for all sets x, y such that $x \in X$ and $y \in Y$ holds $I(x, y) = \langle x, y \rangle$.
- (6) Let X, Y be non-empty non empty finite sequences. Then there exists a function I from $\prod X \times \prod Y$ into $\prod (X \cap Y)$ such that I is one-to-one and onto and for all finite sequences x, y such that $x \in \prod X$ and $y \in \prod Y$ holds $I(x, y) = x \cap y$.

Let G, F be non empty additive loop structures. The functor $\text{prodadd}(G, F)$ yielding a binary operation on $(\text{the carrier of } G) \times (\text{the carrier of } F)$ is defined by:

- (Def. 1) For all points g_1, g_2 of G and for all points f_1, f_2 of F holds $(\text{prodadd}(G, F))(\langle g_1, f_1 \rangle, \langle g_2, f_2 \rangle) = \langle g_1 + g_2, f_1 + f_2 \rangle$.

Let G, F be non empty RLS structures. The functor $\text{prodmlt}(G, F)$ yielding a function from $\mathbb{R} \times ((\text{the carrier of } G) \times (\text{the carrier of } F))$ into $(\text{the carrier of } G) \times (\text{the carrier of } F)$ is defined by:

- (Def. 2) For every element r of \mathbb{R} and for every point g of G and for every point f of F holds $(\text{prodmlt}(G, F))(r, \langle g, f \rangle) = \langle r \cdot g, r \cdot f \rangle$.

Let G, F be non empty additive loop structures. The functor $\text{prodzero}(G, F)$ yields an element of $(\text{the carrier of } G) \times (\text{the carrier of } F)$ and is defined by:

- (Def. 3) $\text{prodzero}(G, F) = \langle 0_G, 0_F \rangle$.

Let G, F be non empty additive loop structures. The functor $G \times F$ yielding a strict non empty additive loop structure is defined by:

- (Def. 4) $G \times F = \langle (\text{the carrier of } G) \times (\text{the carrier of } F), \text{prodadd}(G, F), \text{prodzero}(G, F) \rangle$.

Let G, F be Abelian non empty additive loop structures. Observe that $G \times F$ is Abelian.

Let G, F be add-associative non empty additive loop structures. Note that $G \times F$ is add-associative.

Let G, F be right zeroed non empty additive loop structures. Note that $G \times F$ is right zeroed.

Let G, F be right complementable non empty additive loop structures. Note that $G \times F$ is right complementable.

Next we state two propositions:

- (7) Let G, F be non empty additive loop structures. Then
 - (i) for every set x holds x is a point of $G \times F$ iff there exists a point x_1 of G and there exists a point x_2 of F such that $x = \langle x_1, x_2 \rangle$,
 - (ii) for all points x, y of $G \times F$ and for all points x_1, y_1 of G and for all points x_2, y_2 of F such that $x = \langle x_1, x_2 \rangle$ and $y = \langle y_1, y_2 \rangle$ holds $x + y = \langle x_1 + y_1, x_2 + y_2 \rangle$, and
 - (iii) $0_{G \times F} = \langle 0_G, 0_F \rangle$.
- (8) Let G, F be add-associative right zeroed right complementable non empty additive loop structures, x be a point of $G \times F$, x_1 be a point of G , and x_2 be a point of F . If $x = \langle x_1, x_2 \rangle$, then $-x = \langle -x_1, -x_2 \rangle$.

Let G, F be Abelian add-associative right zeroed right complementable strict non empty additive loop structures. One can check that $G \times F$ is strict, Abelian, add-associative, right zeroed, and right complementable.

Let G, F be non empty RLS structures. The functor $G \times F$ yields a strict non empty RLS structure and is defined by:

(Def. 5) $G \times F = \langle (\text{the carrier of } G) \times (\text{the carrier of } F), \text{prodzero}(G, F), \text{prodadd}(G, F), \text{prodmult}(G, F) \rangle$.

Let G, F be Abelian non empty RLS structures. Observe that $G \times F$ is Abelian.

Let G, F be add-associative non empty RLS structures. Note that $G \times F$ is add-associative.

Let G, F be right zeroed non empty RLS structures. Note that $G \times F$ is right zeroed.

Let G, F be right complementable non empty RLS structures. One can check that $G \times F$ is right complementable.

Next we state two propositions:

- (9) Let G, F be non empty RLS structures. Then
 - (i) for every set x holds x is a point of $G \times F$ iff there exists a point x_1 of G and there exists a point x_2 of F such that $x = \langle x_1, x_2 \rangle$,
 - (ii) for all points x, y of $G \times F$ and for all points x_1, y_1 of G and for all points x_2, y_2 of F such that $x = \langle x_1, x_2 \rangle$ and $y = \langle y_1, y_2 \rangle$ holds $x + y = \langle x_1 + y_1, x_2 + y_2 \rangle$,
 - (iii) $0_{G \times F} = \langle 0_G, 0_F \rangle$, and

- (iv) for every point x of $G \times F$ and for every point x_1 of G and for every point x_2 of F and for every real number a such that $x = \langle x_1, x_2 \rangle$ holds $a \cdot x = \langle a \cdot x_1, a \cdot x_2 \rangle$.
- (10) Let G, F be add-associative right zeroed right complementable non empty RLS structures, x be a point of $G \times F$, x_1 be a point of G , and x_2 be a point of F . If $x = \langle x_1, x_2 \rangle$, then $-x = \langle -x_1, -x_2 \rangle$.

Let G, F be vector distributive non empty RLS structures. Note that $G \times F$ is vector distributive.

Let G, F be scalar distributive non empty RLS structures. Note that $G \times F$ is scalar distributive.

Let G, F be scalar associative non empty RLS structures. Observe that $G \times F$ is scalar associative.

Let G, F be scalar unital non empty RLS structures. One can verify that $G \times F$ is scalar unital.

Let G be an Abelian add-associative right zeroed right complementable scalar distributive vector distributive scalar associative scalar unital non empty RLS structure. Note that $\langle G \rangle$ is real-linear-space-yielding.

Let G, F be Abelian add-associative right zeroed right complementable scalar distributive vector distributive scalar associative scalar unital non empty RLS structures. Note that $\langle G, F \rangle$ is real-linear-space-yielding.

2. CARTESIAN PRODUCTS OF REAL LINEAR SPACES

One can prove the following proposition

- (11) Let X be a real linear space. Then there exists a function I from X into $\prod \langle X \rangle$ such that
- (i) I is one-to-one and onto,
 - (ii) for every point x of X holds $I(x) = \langle x \rangle$,
 - (iii) for all points v, w of X holds $I(v + w) = I(v) + I(w)$,
 - (iv) for every point v of X and for every element r of \mathbb{R} holds $I(r \cdot v) = r \cdot I(v)$,
and
 - (v) $I(0_X) = 0_{\prod \langle X \rangle}$.

Let G, F be non empty real-linear-space-yielding finite sequences. Observe that $G \cap F$ is real-linear-space-yielding.

We now state three propositions:

- (12) Let X, Y be real linear spaces. Then there exists a function I from $X \times Y$ into $\prod \langle X, Y \rangle$ such that
- (i) I is one-to-one and onto,
 - (ii) for every point x of X and for every point y of Y holds $I(x, y) = \langle x, y \rangle$,
 - (iii) for all points v, w of $X \times Y$ holds $I(v + w) = I(v) + I(w)$,

- (iv) for every point v of $X \times Y$ and for every element r of \mathbb{R} holds $I(r \cdot v) = r \cdot I(v)$, and
- (v) $I(0_{X \times Y}) = 0_{\prod \langle X, Y \rangle}$.
- (13) Let X, Y be non empty real linear space-sequences. Then there exists a function I from $\prod X \times \prod Y$ into $\prod (X \wedge Y)$ such that
 - (i) I is one-to-one and onto,
 - (ii) for every point x of $\prod X$ and for every point y of $\prod Y$ there exist finite sequences x_1, y_1 such that $x = x_1$ and $y = y_1$ and $I(x, y) = x_1 \wedge y_1$,
 - (iii) for all points v, w of $\prod X \times \prod Y$ holds $I(v + w) = I(v) + I(w)$,
 - (iv) for every point v of $\prod X \times \prod Y$ and for every element r of \mathbb{R} holds $I(r \cdot v) = r \cdot I(v)$, and
 - (v) $I(0_{\prod X \times \prod Y}) = 0_{\prod (X \wedge Y)}$.
- (14) Let G, F be real linear spaces. Then
 - (i) for every set x holds x is a point of $\prod \langle G, F \rangle$ iff there exists a point x_1 of G and there exists a point x_2 of F such that $x = \langle x_1, x_2 \rangle$,
 - (ii) for all points x, y of $\prod \langle G, F \rangle$ and for all points x_1, y_1 of G and for all points x_2, y_2 of F such that $x = \langle x_1, x_2 \rangle$ and $y = \langle y_1, y_2 \rangle$ holds $x + y = \langle x_1 + y_1, x_2 + y_2 \rangle$,
 - (iii) $0_{\prod \langle G, F \rangle} = \langle 0_G, 0_F \rangle$,
 - (iv) for every point x of $\prod \langle G, F \rangle$ and for every point x_1 of G and for every point x_2 of F such that $x = \langle x_1, x_2 \rangle$ holds $-x = \langle -x_1, -x_2 \rangle$, and
 - (v) for every point x of $\prod \langle G, F \rangle$ and for every point x_1 of G and for every point x_2 of F and for every real number a such that $x = \langle x_1, x_2 \rangle$ holds $a \cdot x = \langle a \cdot x_1, a \cdot x_2 \rangle$.

3. CARTESIAN PRODUCTS OF REAL NORMED LINEAR SPACES

Let G, F be non empty normed structures. The functor $\text{prodnorm}(G, F)$ yields a function from (the carrier of G) \times (the carrier of F) into \mathbb{R} and is defined by:

- (Def. 6) For every point g of G and for every point f of F there exists an element v of \mathcal{R}^2 such that $v = \langle \|g\|, \|f\| \rangle$ and $(\text{prodnorm}(G, F))(g, f) = |v|$.

Let G, F be non empty normed structures. The functor $G \times F$ yielding a strict non empty normed structure is defined as follows:

- (Def. 7) $G \times F = \langle (\text{the carrier of } G) \times (\text{the carrier of } F), \text{prodzero}(G, F), \text{prodadd}(G, F), \text{prodmlt}(G, F), \text{prodnorm}(G, F) \rangle$.

Let G, F be real normed spaces. Observe that $G \times F$ is reflexive, discernible, and real normed space-like.

Let G, F be reflexive discernible real normed space-like scalar distributive vector distributive scalar associative scalar unital Abelian add-associative right

zeroed right complementable non empty normed structures. One can verify that $G \times F$ is strict, reflexive, discernible, real normed space-like, scalar distributive, vector distributive, scalar associative, scalar unital, Abelian, add-associative, right zeroed, and right complementable.

Let G be a reflexive discernible real normed space-like scalar distributive vector distributive scalar associative scalar unital Abelian add-associative right zeroed right complementable non empty normed structure. One can verify that $\langle G \rangle$ is real-norm-space-yielding.

Let G, F be reflexive discernible real normed space-like scalar distributive vector distributive scalar associative scalar unital Abelian add-associative right zeroed right complementable non empty normed structures. Observe that $\langle G, F \rangle$ is real-norm-space-yielding.

One can prove the following propositions:

- (15) Let X, Y be real normed spaces. Then there exists a function I from $X \times Y$ into $\prod \langle X, Y \rangle$ such that
- (i) I is one-to-one and onto,
 - (ii) for every point x of X and for every point y of Y holds $I(x, y) = \langle x, y \rangle$,
 - (iii) for all points v, w of $X \times Y$ holds $I(v + w) = I(v) + I(w)$,
 - (iv) for every point v of $X \times Y$ and for every element r of \mathbb{R} holds $I(r \cdot v) = r \cdot I(v)$,
 - (v) $0_{\prod \langle X, Y \rangle} = I(0_{X \times Y})$, and
 - (vi) for every point v of $X \times Y$ holds $\|I(v)\| = \|v\|$.
- (16) Let X be a real normed space. Then there exists a function I from X into $\prod \langle X \rangle$ such that
- (i) I is one-to-one and onto,
 - (ii) for every point x of X holds $I(x) = \langle x \rangle$,
 - (iii) for all points v, w of X holds $I(v + w) = I(v) + I(w)$,
 - (iv) for every point v of X and for every element r of \mathbb{R} holds $I(r \cdot v) = r \cdot I(v)$,
 - (v) $0_{\prod \langle X \rangle} = I(0_X)$, and
 - (vi) for every point v of X holds $\|I(v)\| = \|v\|$.

Let G, F be non empty real-norm-space-yielding finite sequences. One can check that $G \hat{\ } F$ is non empty and real-norm-space-yielding.

One can prove the following propositions:

- (17) Let X, Y be non empty real norm space-sequences. Then there exists a function I from $\prod X \times \prod Y$ into $\prod \langle X \hat{\ } Y \rangle$ such that
- (i) I is one-to-one and onto,
 - (ii) for every point x of $\prod X$ and for every point y of $\prod Y$ there exist finite sequences x_1, y_1 such that $x = x_1$ and $y = y_1$ and $I(x, y) = x_1 \hat{\ } y_1$,
 - (iii) for all points v, w of $\prod X \times \prod Y$ holds $I(v + w) = I(v) + I(w)$,

- (iv) for every point v of $\prod X \times \prod Y$ and for every element r of \mathbb{R} holds $I(r \cdot v) = r \cdot I(v)$,
 - (v) $I(0_{\prod X \times \prod Y}) = 0_{\prod(X \cap Y)}$, and
 - (vi) for every point v of $\prod X \times \prod Y$ holds $\|I(v)\| = \|v\|$.
- (18) Let G, F be real normed spaces. Then
- (i) for every set x holds x is a point of $G \times F$ iff there exists a point x_1 of G and there exists a point x_2 of F such that $x = \langle x_1, x_2 \rangle$,
 - (ii) for all points x, y of $G \times F$ and for all points x_1, y_1 of G and for all points x_2, y_2 of F such that $x = \langle x_1, x_2 \rangle$ and $y = \langle y_1, y_2 \rangle$ holds $x + y = \langle x_1 + y_1, x_2 + y_2 \rangle$,
 - (iii) $0_{G \times F} = \langle 0_G, 0_F \rangle$,
 - (iv) for every point x of $G \times F$ and for every point x_1 of G and for every point x_2 of F such that $x = \langle x_1, x_2 \rangle$ holds $-x = \langle -x_1, -x_2 \rangle$,
 - (v) for every point x of $G \times F$ and for every point x_1 of G and for every point x_2 of F and for every real number a such that $x = \langle x_1, x_2 \rangle$ holds $a \cdot x = \langle a \cdot x_1, a \cdot x_2 \rangle$, and
 - (vi) for every point x of $G \times F$ and for every point x_1 of G and for every point x_2 of F such that $x = \langle x_1, x_2 \rangle$ there exists an element w of \mathcal{R}^2 such that $w = \langle \|x_1\|, \|x_2\| \rangle$ and $\|x\| = |w|$.
- (19) Let G, F be real normed spaces. Then
- (i) for every set x holds x is a point of $\prod \langle G, F \rangle$ iff there exists a point x_1 of G and there exists a point x_2 of F such that $x = \langle x_1, x_2 \rangle$,
 - (ii) for all points x, y of $\prod \langle G, F \rangle$ and for all points x_1, y_1 of G and for all points x_2, y_2 of F such that $x = \langle x_1, x_2 \rangle$ and $y = \langle y_1, y_2 \rangle$ holds $x + y = \langle x_1 + y_1, x_2 + y_2 \rangle$,
 - (iii) $0_{\prod \langle G, F \rangle} = \langle 0_G, 0_F \rangle$,
 - (iv) for every point x of $\prod \langle G, F \rangle$ and for every point x_1 of G and for every point x_2 of F such that $x = \langle x_1, x_2 \rangle$ holds $-x = \langle -x_1, -x_2 \rangle$,
 - (v) for every point x of $\prod \langle G, F \rangle$ and for every point x_1 of G and for every point x_2 of F and for every real number a such that $x = \langle x_1, x_2 \rangle$ holds $a \cdot x = \langle a \cdot x_1, a \cdot x_2 \rangle$, and
 - (vi) for every point x of $\prod \langle G, F \rangle$ and for every point x_1 of G and for every point x_2 of F such that $x = \langle x_1, x_2 \rangle$ there exists an element w of \mathcal{R}^2 such that $w = \langle \|x_1\|, \|x_2\| \rangle$ and $\|x\| = |w|$.

Let X, Y be complete real normed spaces. Observe that $X \times Y$ is complete.

We now state several propositions:

- (20) Let X, Y be non empty real norm space-sequences. Then there exists a function I from $\prod \langle \prod X, \prod Y \rangle$ into $\prod \langle X \cap Y \rangle$ such that
- (i) I is one-to-one and onto,
 - (ii) for every point x of $\prod X$ and for every point y of $\prod Y$ there exist finite sequences x_1, y_1 such that $x = x_1$ and $y = y_1$ and $I(\langle x, y \rangle) = x_1 \cap y_1$,

- (iii) for all points v, w of $\prod\langle\prod X, \prod Y\rangle$ holds $I(v + w) = I(v) + I(w)$,
 - (iv) for every point v of $\prod\langle\prod X, \prod Y\rangle$ and for every element r of \mathbb{R} holds $I(r \cdot v) = r \cdot I(v)$,
 - (v) $I(0_{\prod\langle\prod X, \prod Y\rangle}) = 0_{\prod(X \wedge Y)}$, and
 - (vi) for every point v of $\prod\langle\prod X, \prod Y\rangle$ holds $\|I(v)\| = \|v\|$.
- (21) Let X, Y be non empty real linear spaces. Then there exists a function I from $X \times Y$ into $X \times \prod\langle Y\rangle$ such that
- (i) I is one-to-one and onto,
 - (ii) for every point x of X and for every point y of Y holds $I(x, y) = \langle x, \langle y \rangle \rangle$,
 - (iii) for all points v, w of $X \times Y$ holds $I(v + w) = I(v) + I(w)$,
 - (iv) for every point v of $X \times Y$ and for every element r of \mathbb{R} holds $I(r \cdot v) = r \cdot I(v)$, and
 - (v) $I(0_{X \times Y}) = 0_{X \times \prod\langle Y\rangle}$.
- (22) Let X be a non empty real linear space-sequence and Y be a real linear space. Then there exists a function I from $\prod X \times Y$ into $\prod(X \wedge \langle Y \rangle)$ such that
- (i) I is one-to-one and onto,
 - (ii) for every point x of $\prod X$ and for every point y of Y there exist finite sequences x_1, y_1 such that $x = x_1$ and $\langle y \rangle = y_1$ and $I(x, y) = x_1 \wedge y_1$,
 - (iii) for all points v, w of $\prod X \times Y$ holds $I(v + w) = I(v) + I(w)$,
 - (iv) for every point v of $\prod X \times Y$ and for every element r of \mathbb{R} holds $I(r \cdot v) = r \cdot I(v)$, and
 - (v) $I(0_{\prod X \times Y}) = 0_{\prod(X \wedge \langle Y \rangle)}$.
- (23) Let X, Y be non empty real normed spaces. Then there exists a function I from $X \times Y$ into $X \times \prod\langle Y\rangle$ such that
- (i) I is one-to-one and onto,
 - (ii) for every point x of X and for every point y of Y holds $I(x, y) = \langle x, \langle y \rangle \rangle$,
 - (iii) for all points v, w of $X \times Y$ holds $I(v + w) = I(v) + I(w)$,
 - (iv) for every point v of $X \times Y$ and for every element r of \mathbb{R} holds $I(r \cdot v) = r \cdot I(v)$,
 - (v) $I(0_{X \times Y}) = 0_{X \times \prod\langle Y\rangle}$, and
 - (vi) for every point v of $X \times Y$ holds $\|I(v)\| = \|v\|$.
- (24) Let X be a non empty real norm space-sequence and Y be a real normed space. Then there exists a function I from $\prod X \times Y$ into $\prod(X \wedge \langle Y \rangle)$ such that
- (i) I is one-to-one and onto,
 - (ii) for every point x of $\prod X$ and for every point y of Y there exist finite sequences x_1, y_1 such that $x = x_1$ and $\langle y \rangle = y_1$ and $I(x, y) = x_1 \wedge y_1$,
 - (iii) for all points v, w of $\prod X \times Y$ holds $I(v + w) = I(v) + I(w)$,

- (iv) for every point v of $\prod X \times Y$ and for every element r of \mathbb{R} holds $I(r \cdot v) = r \cdot I(v)$,
- (v) $I(0_{\prod X \times Y}) = 0_{\prod (X \wedge \langle Y \rangle)}$, and
- (vi) for every point v of $\prod X \times Y$ holds $\|I(v)\| = \|v\|$.

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