

Formalization of Integral Linear Space¹

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Summary. In this article, we formalize integral linear spaces, that is a linear space with integer coefficients. Integral linear spaces are necessary for lattice problems, LLL (Lenstra-Lenstra-Lovász) base reduction algorithm that outputs short lattice base and cryptographic systems with lattice [8].

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The notation and terminology used here have been introduced in the following papers: [1], [10], [3], [9], [11], [2], [4], [6], [16], [14], [13], [12], [5], [7], [15], and [17].

1. PRELIMINARIES

The following propositions are true:

- (1) Let X be a real linear space and R_1, R_2 be finite sequences of elements of X . If $\text{len } R_1 = \text{len } R_2$, then $\sum(R_1 + R_2) = \sum R_1 + \sum R_2$.
- (2) Let X be a real linear space and R_1, R_2, R_3 be finite sequences of elements of X . If $\text{len } R_1 = \text{len } R_2$ and $R_3 = R_1 - R_2$, then $\sum R_3 = \sum R_1 - \sum R_2$.
- (3) Let X be a real linear space, R_1, R_2 be finite sequences of elements of X , and a be an element of \mathbb{R} . If $R_2 = a R_1$, then $\sum R_2 = a \cdot \sum R_1$.

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2. INTEGRAL LINEAR SPACE

For simplicity, we use the following convention: x denotes a set, a denotes a real number, i denotes an integer, V denotes a real linear space, $v, v_1, v_2, v_3, u, w, w_1, w_2, w_3$ denote vectors of V , A, B denote subsets of V , L denotes a linear combination of V , and l, l_1, l_2 denote linear combinations of A .

Let us consider V, i, L . The functor $i \cdot L$ yielding a linear combination of V is defined as follows:

(Def. 1) For every v holds $(i \cdot L)(v) = i \cdot L(v)$.

Let us consider V, A . The functor $\text{Lin}_{\mathbb{Z}} A$ yielding a subset of V is defined by:

(Def. 2) $\text{Lin}_{\mathbb{Z}} A = \{\sum l : \text{rng } l \subseteq \mathbb{Z}\}$.

One can prove the following propositions:

- (4) $(i) \cdot l = i \cdot l$.
- (5) If $\text{rng } l_1 \subseteq \mathbb{Z}$ and $\text{rng } l_2 \subseteq \mathbb{Z}$, then $\text{rng}(l_1 + l_2) \subseteq \mathbb{Z}$.
- (6) If $\text{rng } l \subseteq \mathbb{Z}$, then $\text{rng}(i \cdot l) \subseteq \mathbb{Z}$.
- (7) $\text{rng}(\mathbf{0}_{\text{LC}_V}) \subseteq \mathbb{Z}$.
- (8) $\text{Lin}_{\mathbb{Z}} A \subseteq \text{the carrier of } \text{Lin}(A)$.
- (9) If $v, u \in \text{Lin}_{\mathbb{Z}} A$, then $v + u \in \text{Lin}_{\mathbb{Z}} A$.
- (10) If $v \in \text{Lin}_{\mathbb{Z}} A$, then $i \cdot v \in \text{Lin}_{\mathbb{Z}} A$.
- (11) $0_V \in \text{Lin}_{\mathbb{Z}} A$.
- (12) If $x \in A$, then $x \in \text{Lin}_{\mathbb{Z}} A$.
- (13) If $A \subseteq B$, then $\text{Lin}_{\mathbb{Z}} A \subseteq \text{Lin}_{\mathbb{Z}} B$.
- (14) $\text{Lin}_{\mathbb{Z}}(A \cup B) = (\text{Lin}_{\mathbb{Z}} A) + \text{Lin}_{\mathbb{Z}} B$.
- (15) $\text{Lin}_{\mathbb{Z}}(A \cap B) \subseteq (\text{Lin}_{\mathbb{Z}} A) \cap \text{Lin}_{\mathbb{Z}} B$.
- (16) $x \in \text{Lin}_{\mathbb{Z}}\{v\}$ iff there exists an integer a such that $x = a \cdot v$.
- (17) $v \in \text{Lin}_{\mathbb{Z}}\{v\}$.
- (18) $x \in v + \text{Lin}_{\mathbb{Z}}\{w\}$ iff there exists an integer a such that $x = v + a \cdot w$.
- (19) $x \in \text{Lin}_{\mathbb{Z}}\{w_1, w_2\}$ iff there exist integers a, b such that $x = a \cdot w_1 + b \cdot w_2$.
- (20) $w_1 \in \text{Lin}_{\mathbb{Z}}\{w_1, w_2\}$.
- (21) $x \in v + \text{Lin}_{\mathbb{Z}}\{w_1, w_2\}$ iff there exist integers a, b such that $x = v + a \cdot w_1 + b \cdot w_2$.
- (22) $x \in \text{Lin}_{\mathbb{Z}}\{v_1, v_2, v_3\}$ iff there exist integers a, b, c such that $x = a \cdot v_1 + b \cdot v_2 + c \cdot v_3$.
- (23) $w_1, w_2, w_3 \in \text{Lin}_{\mathbb{Z}}\{w_1, w_2, w_3\}$.
- (24) $x \in v + \text{Lin}_{\mathbb{Z}}\{w_1, w_2, w_3\}$ iff there exist integers a, b, c such that $x = v + a \cdot w_1 + b \cdot w_2 + c \cdot w_3$.

- (25) Let x be a set. Then $x \in \text{Lin}_{\mathbb{Z}} A$ if and only if there exist finite sequences g_1, h_1 of elements of V and there exists an integer-valued finite sequence a_1 such that $x = \sum h_1$ and $\text{rng } g_1 \subseteq A$ and $\text{len } g_1 = \text{len } h_1$ and $\text{len } g_1 = \text{len } a_1$ and for every natural number i such that $i \in \text{Seg len } g_1$ holds $(h_1)_i = a_1(i) \cdot (g_1)_i$.

Let R_4 be a real linear space and let f be a finite sequence of elements of R_4 . The functor $\text{Lin}_{\mathbb{Z}} f$ yielding a subset of R_4 is defined by the condition (Def. 3).

- (Def. 3) $\text{Lin}_{\mathbb{Z}} f = \{ \sum g; g \text{ ranges over len } f\text{-element finite sequences of elements of } R_4: \bigvee a: \text{ len } f\text{-element integer-valued finite sequence } \bigwedge i: \text{ natural number } (i \in \text{Seg len } f \Rightarrow g_i = a(i) \cdot f_i) \}$.

One can prove the following propositions:

- (26) Let R_4 be a real linear space, f be a finite sequence of elements of R_4 , and x be a set. Then $x \in \text{Lin}_{\mathbb{Z}} f$ if and only if there exists a len f -element finite sequence g of elements of R_4 and there exists a len f -element integer-valued finite sequence a such that $x = \sum g$ and for every natural number i such that $i \in \text{Seg len } f$ holds $g_i = a(i) \cdot f_i$.
- (27) Let R_4 be a real linear space, f be a finite sequence of elements of R_4 , x, y be elements of R_4 , and a, b be elements of \mathbb{Z} . If $x, y \in \text{Lin}_{\mathbb{Z}} f$, then $a \cdot x + b \cdot y \in \text{Lin}_{\mathbb{Z}} f$.
- (28) For every real linear space R_4 and for every finite sequence f of elements of R_4 such that $f = \text{Seg len } f \mapsto 0_{(R_4)}$ holds $\sum f = 0_{(R_4)}$.
- (29) Let R_4 be a real linear space, f be a finite sequence of elements of R_4 , v be an element of R_4 , and i be a natural number. If $i \in \text{Seg len } f$ and $f = (\text{Seg len } f \mapsto 0_{(R_4)}) + (\{i\} \mapsto v)$, then $\sum f = v$.
- (30) Let R_4 be a real linear space, f be a finite sequence of elements of R_4 , and i be a natural number. If $i \in \text{Seg len } f$, then $f_i \in \text{Lin}_{\mathbb{Z}} f$.
- (31) For every real linear space R_4 and for every finite sequence f of elements of R_4 holds $\text{rng } f \subseteq \text{Lin}_{\mathbb{Z}} f$.
- (32) Let R_4 be a real linear space, f be a non empty finite sequence of elements of R_4 , g, h be finite sequences of elements of R_4 , and s be an integer-valued finite sequence. Suppose $\text{rng } g \subseteq \text{Lin}_{\mathbb{Z}} f$ and $\text{len } g = \text{len } s$ and $\text{len } g = \text{len } h$ and for every natural number i such that $i \in \text{Seg len } g$ holds $h_i = s(i) \cdot g_i$. Then $\sum h \in \text{Lin}_{\mathbb{Z}} f$.
- (33) For every real linear space R_4 and for every non empty finite sequence f of elements of R_4 holds $\text{Lin}_{\mathbb{Z}} \text{rng } f = \text{Lin}_{\mathbb{Z}} f$.
- (34) $\text{Lin}(\text{Lin}_{\mathbb{Z}} A) = \text{Lin}(A)$.
- (35) Let x be a set, g_1, h_1 be finite sequences of elements of V , and a_1 be an integer-valued finite sequence. Suppose $x = \sum h_1$ and $\text{rng } g_1 \subseteq \text{Lin}_{\mathbb{Z}} A$ and $\text{len } g_1 = \text{len } h_1$ and $\text{len } g_1 = \text{len } a_1$ and for every natural number i such that $i \in \text{Seg len } g_1$ holds $(h_1)_i = a_1(i) \cdot (g_1)_i$. Then $x \in \text{Lin}_{\mathbb{Z}} A$.

- (36) $\text{Lin}_{\mathbb{Z}} \text{Lin}_{\mathbb{Z}} A = \text{Lin}_{\mathbb{Z}} A$.
 (37) If $\text{Lin}_{\mathbb{Z}} A = \text{Lin}_{\mathbb{Z}} B$, then $\text{Lin}(A) = \text{Lin}(B)$.

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