

Mazur-Ulam Theorem

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Summary. The Mazur-Ulam theorem [15] has been formulated as two registrations: cluster bijective isometric \rightarrow midpoints-preserving Function of E, F ; and cluster isometric midpoints-preserving \rightarrow Affine Function of E, F ; A proof given by Jussi Väisälä [23] has been formalized.

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The notation and terminology used in this paper have been introduced in the following papers: [19], [18], [4], [5], [20], [11], [10], [14], [17], [1], [6], [16], [24], [25], [21], [13], [12], [22], [2], [9], [8], [3], and [7].

For simplicity, we use the following convention: E, F, G are real normed spaces, f is a function from E into F , g is a function from F into G , a, b are points of E , and t is a real number.

Let us note that \mathbb{I} is closed.

Next we state four propositions:

- (1) DYADIC is a dense subset of \mathbb{I} .
- (2) $\overline{\text{DYADIC}} = [0, 1]$.
- (3) $a + a = 2 \cdot a$.
- (4) $(a + b) - b = a$.

Let A be an upper bounded real-membered set and let r be a non negative real number. Observe that $r \circ A$ is upper bounded.

Let A be an upper bounded real-membered set and let r be a non positive real number. Note that $r \circ A$ is lower bounded.

Let A be a lower bounded real-membered set and let r be a non negative real number. Observe that $r \circ A$ is lower bounded.

Let A be a lower bounded non empty real-membered set and let r be a non positive real number. One can check that $r \circ A$ is upper bounded.

Next we state three propositions:

- (5) For every sequence f of real numbers holds $f + (\mathbb{N} \mapsto t) = t + f$.
- (6) For every real number r holds $\lim(\mathbb{N} \mapsto r) = r$.
- (7) For every convergent sequence f of real numbers holds $\lim(t + f) = t + \lim f$.

Let f be a convergent sequence of real numbers and let us consider t . One can check that $t + f$ is convergent.

Next we state three propositions:

- (8) For every sequence f of real numbers holds $f \cdot (\mathbb{N} \mapsto a) = f \cdot a$.
- (9) $\lim(\mathbb{N} \mapsto a) = a$.
- (10) For every convergent sequence f of real numbers holds $\lim(f \cdot a) = \lim f \cdot a$.

Let f be a convergent sequence of real numbers and let us consider E, a . Note that $f \cdot a$ is convergent.

Let E, F be non empty normed structures and let f be a function from E into F . We say that f is isometric if and only if:

- (Def. 1) For all points a, b of E holds $\|f(a) - f(b)\| = \|a - b\|$.

Let E, F be non empty RLS structures and let f be a function from E into F . We say that f is affine if and only if:

- (Def. 2) For all points a, b of E and for every real number t such that $0 \leq t \leq 1$ holds $f((1 - t) \cdot a + t \cdot b) = (1 - t) \cdot f(a) + t \cdot f(b)$.

We say that f preserves midpoints if and only if:

- (Def. 3) For all points a, b of E holds $f(\frac{1}{2} \cdot (a + b)) = \frac{1}{2} \cdot (f(a) + f(b))$.

Let E be a non empty normed structure. Observe that id_E is isometric.

Let E be a non empty RLS structure. Note that id_E is affine and preserves midpoints.

Let E be a non empty normed structure. Observe that there exists a unary operation on E which is bijective, isometric, and affine and preserves midpoints.

Next we state the proposition

- (11) If f is isometric and g is isometric, then $g \cdot f$ is isometric.

Let us consider E and let f, g be isometric unary operations on E . One can verify that $g \cdot f$ is isometric.

The following proposition is true

- (12) If f is bijective and isometric, then f^{-1} is isometric.

Let us consider E and let f be a bijective isometric unary operation on E . One can check that f^{-1} is isometric.

We now state the proposition

- (13) If f preserves midpoints and g preserves midpoints, then $g \cdot f$ preserves midpoints.

Let us consider E and let f, g be unary operations on E preserving midpoints. Note that $g \cdot f$ preserves midpoints.

The following proposition is true

- (14) If f is bijective and preserves midpoints, then f^{-1} preserves midpoints.

Let us consider E and let f be a bijective unary operation on E preserving midpoints. Observe that f^{-1} preserves midpoints.

Next we state the proposition

- (15) If f is affine and g is affine, then $g \cdot f$ is affine.

Let us consider E and let f, g be affine unary operations on E . Observe that $g \cdot f$ is affine.

One can prove the following proposition

- (16) If f is bijective and affine, then f^{-1} is affine.

Let us consider E and let f be a bijective affine unary operation on E . Observe that f^{-1} is affine.

Let E be a non empty RLS structure and let a be a point of E . The functor a -reflection yields a unary operation on E and is defined as follows:

- (Def. 4) For every point b of E holds a -reflection(b) = $2 \cdot a - b$.

The following proposition is true

- (17) a -reflection \cdot a -reflection = id_E .

Let us consider E, a . Note that a -reflection is bijective.

We now state several propositions:

- (18) a -reflection(a) = a and for every b such that a -reflection(b) = b holds $a = b$.

- (19) a -reflection(b) - $a = a - b$.

- (20) $\|a$ -reflection(b) - $a\| = \|b - a\|$.

- (21) a -reflection(b) - $b = 2 \cdot (a - b)$.

- (22) $\|a$ -reflection(b) - $b\| = 2 \cdot \|b - a\|$.

- (23) a -reflection $^{-1}$ = a -reflection.

Let us consider E, a . Observe that a -reflection is isometric.

Next we state the proposition

- (24) If f is isometric, then f is continuous on $\text{dom } f$.

Let us consider E, F . Observe that every function from E into F which is bijective and isometric also preserves midpoints.

Let us consider E, F . One can check that every function from E into F which is isometric and preserves midpoints is also affine.

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