

# Mazur-Ulam Theorem

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**Summary.** The Mazur-Ulam theorem [15] has been formulated as two registrations: `cluster bijective isometric -> midpoints-preserving Function of E,F`; and `cluster isometric midpoints-preserving -> Affine Function of E,F`; A proof given by Jussi Väisälä [23] has been formalized.

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The notation and terminology used in this paper have been introduced in the following papers: [19], [18], [4], [5], [20], [11], [10], [14], [17], [1], [6], [16], [24], [25], [21], [13], [12], [22], [2], [9], [8], [3], and [7].

For simplicity, we use the following convention:  $E, F, G$  are real normed spaces,  $f$  is a function from  $E$  into  $F$ ,  $g$  is a function from  $F$  into  $G$ ,  $a, b$  are points of  $E$ , and  $t$  is a real number.

Let us note that  $\mathbb{I}$  is closed.

Next we state four propositions:

- (1) `DYADIC` is a dense subset of  $\mathbb{I}$ .
- (2)  $\overline{\text{DYADIC}} = [0, 1]$ .
- (3)  $a + a = 2 \cdot a$ .
- (4)  $(a + b) - b = a$ .

Let  $A$  be an upper bounded real-membered set and let  $r$  be a non negative real number. Observe that  $r \circ A$  is upper bounded.

Let  $A$  be an upper bounded real-membered set and let  $r$  be a non positive real number. Note that  $r \circ A$  is lower bounded.

Let  $A$  be a lower bounded real-membered set and let  $r$  be a non negative real number. Observe that  $r \circ A$  is lower bounded.

Let  $A$  be a lower bounded non empty real-membered set and let  $r$  be a non positive real number. One can check that  $r \circ A$  is upper bounded.

Next we state three propositions:

- (5) For every sequence  $f$  of real numbers holds  $f + (\mathbb{N} \mapsto t) = t + f$ .
- (6) For every real number  $r$  holds  $\lim(\mathbb{N} \mapsto r) = r$ .
- (7) For every convergent sequence  $f$  of real numbers holds  $\lim(t + f) = t + \lim f$ .

Let  $f$  be a convergent sequence of real numbers and let us consider  $t$ . One can check that  $t + f$  is convergent.

Next we state three propositions:

- (8) For every sequence  $f$  of real numbers holds  $f \cdot (\mathbb{N} \mapsto a) = f \cdot a$ .
- (9)  $\lim(\mathbb{N} \mapsto a) = a$ .
- (10) For every convergent sequence  $f$  of real numbers holds  $\lim(f \cdot a) = \lim f \cdot a$ .

Let  $f$  be a convergent sequence of real numbers and let us consider  $E, a$ . Note that  $f \cdot a$  is convergent.

Let  $E, F$  be non empty normed structures and let  $f$  be a function from  $E$  into  $F$ . We say that  $f$  is isometric if and only if:

- (Def. 1) For all points  $a, b$  of  $E$  holds  $\|f(a) - f(b)\| = \|a - b\|$ .

Let  $E, F$  be non empty RLS structures and let  $f$  be a function from  $E$  into  $F$ . We say that  $f$  is affine if and only if:

- (Def. 2) For all points  $a, b$  of  $E$  and for every real number  $t$  such that  $0 \leq t \leq 1$  holds  $f((1 - t) \cdot a + t \cdot b) = (1 - t) \cdot f(a) + t \cdot f(b)$ .

We say that  $f$  preserves midpoints if and only if:

- (Def. 3) For all points  $a, b$  of  $E$  holds  $f(\frac{1}{2} \cdot (a + b)) = \frac{1}{2} \cdot (f(a) + f(b))$ .

Let  $E$  be a non empty normed structure. Observe that  $\text{id}_E$  is isometric.

Let  $E$  be a non empty RLS structure. Note that  $\text{id}_E$  is affine and preserves midpoints.

Let  $E$  be a non empty normed structure. Observe that there exists a unary operation on  $E$  which is bijective, isometric, and affine and preserves midpoints.

Next we state the proposition

- (11) If  $f$  is isometric and  $g$  is isometric, then  $g \cdot f$  is isometric.

Let us consider  $E$  and let  $f, g$  be isometric unary operations on  $E$ . One can verify that  $g \cdot f$  is isometric.

The following proposition is true

- (12) If  $f$  is bijective and isometric, then  $f^{-1}$  is isometric.

Let us consider  $E$  and let  $f$  be a bijective isometric unary operation on  $E$ . One can check that  $f^{-1}$  is isometric.

We now state the proposition

- (13) If  $f$  preserves midpoints and  $g$  preserves midpoints, then  $g \cdot f$  preserves midpoints.

Let us consider  $E$  and let  $f, g$  be unary operations on  $E$  preserving midpoints. Note that  $g \cdot f$  preserves midpoints.

The following proposition is true

- (14) If  $f$  is bijective and preserves midpoints, then  $f^{-1}$  preserves midpoints.

Let us consider  $E$  and let  $f$  be a bijective unary operation on  $E$  preserving midpoints. Observe that  $f^{-1}$  preserves midpoints.

Next we state the proposition

- (15) If  $f$  is affine and  $g$  is affine, then  $g \cdot f$  is affine.

Let us consider  $E$  and let  $f, g$  be affine unary operations on  $E$ . Observe that  $g \cdot f$  is affine.

One can prove the following proposition

- (16) If  $f$  is bijective and affine, then  $f^{-1}$  is affine.

Let us consider  $E$  and let  $f$  be a bijective affine unary operation on  $E$ . Observe that  $f^{-1}$  is affine.

Let  $E$  be a non empty RLS structure and let  $a$  be a point of  $E$ . The functor  $a$ -reflection yields a unary operation on  $E$  and is defined as follows:

- (Def. 4) For every point  $b$  of  $E$  holds  $a$ -reflection( $b$ ) =  $2 \cdot a - b$ .

The following proposition is true

- (17)  $a$ -reflection  $\cdot$   $a$ -reflection =  $\text{id}_E$ .

Let us consider  $E, a$ . Note that  $a$ -reflection is bijective.

We now state several propositions:

- (18)  $a$ -reflection( $a$ ) =  $a$  and for every  $b$  such that  $a$ -reflection( $b$ ) =  $b$  holds  $a = b$ .

- (19)  $a$ -reflection( $b$ ) -  $a = a - b$ .

- (20)  $\|a$ -reflection( $b$ ) -  $a\| = \|b - a\|$ .

- (21)  $a$ -reflection( $b$ ) -  $b = 2 \cdot (a - b)$ .

- (22)  $\|a$ -reflection( $b$ ) -  $b\| = 2 \cdot \|b - a\|$ .

- (23)  $a$ -reflection $^{-1}$  =  $a$ -reflection.

Let us consider  $E, a$ . Observe that  $a$ -reflection is isometric.

Next we state the proposition

- (24) If  $f$  is isometric, then  $f$  is continuous on  $\text{dom } f$ .

Let us consider  $E, F$ . Observe that every function from  $E$  into  $F$  which is bijective and isometric also preserves midpoints.

Let us consider  $E, F$ . One can check that every function from  $E$  into  $F$  which is isometric and preserves midpoints is also affine.

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