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Continuity of Barycentric Coordinates in Euclidean Topological Spaces

Karol Pąk
Institute of Informatics
University of Białystok
Poland

Summary. In this paper we present selected properties of barycentric coordinates in the Euclidean topological space. We prove the topological correspondence between a subset of an affine closed space of $\mathcal{E}^n$ and the set of vectors created from barycentric coordinates of points of this subset.

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The terminology and notation used here have been introduced in the following articles: [1], [3], [15], [25], [13], [18], [5], [4], [6], [12], [7], [8], [33], [21], [24], [2], [22], [20], [17], [30], [31], [23], [10], [28], [26], [11], [16], [29], [14], [19], [27], [32], and [9].

1. Preliminaries

For simplicity, we adopt the following rules: x denotes a set, n, m, k denote natural numbers, r denotes a real number, V denotes a real linear space, v, w denote vectors of V, A_1 denotes a finite subset of V, and A_2 denotes a finite affinely independent subset of V.

One can prove the following propositions:

(1) For all real-valued finite sequences f_1, f_2 and for every real number r holds $(\mathrm{Intervals}(f_1, r)) \frown \mathrm{Intervals}(f_2, r) = \mathrm{Intervals}(f_1 \frown f_2, r)$.

(2) Let f_1, f_2 be finite sequences. Then $x \in \prod(f_1 \frown f_2)$ if and only if there exist finite sequences p_1, p_2 such that $x = p_1 \frown p_2$ and $p_1 \in \prod f_1$ and $p_2 \in \prod f_2$.

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(3) V is finite dimensional iff Ω_V is finite dimensional.

Let V be a finite dimensional real linear space. One can verify that every affinely independent subset of V is finite.

Let us consider n. One can check that $\mathcal{E}^n_{\mathrm{T}}$ is add-continuous and mult-continuous and $\mathcal{E}^n_{\mathrm{T}}$ is finite dimensional.

In the sequel p_3 denotes a point of $\mathcal{E}^n_{\mathrm{T}}$, A_3 denotes a subset of $\mathcal{E}^n_{\mathrm{T}}$, A_4 denotes an affinely independent subset of $\mathcal{E}^n_{\mathrm{T}}$, and A_5 denotes a subset of $\mathcal{E}^k_{\mathrm{T}}$.

Next we state three propositions:

(4) $\dim(\mathcal{E}^n_{\mathrm{T}}) = n$.

(5) Let V be a finite dimensional real linear space and A be an affinely independent subset of V. Then $\overline{\overline{A}} \leq 1 + \dim(V)$.

(6) Let V be a finite dimensional real linear space and A be an affinely independent subset of V. Then $\overline{\overline{A}} = \dim(V)+1$ if and only if $\operatorname{Affin} A = \Omega_V$.

2. Open and Closed Subsets of a Subspace of the Euclidean Topological Space

One can prove the following propositions:

(7) If $k \leq n$ and $A_3 = \{v \in \mathcal{E}^n_{\mathrm{T}}\colon v{\restriction}k \in A_5\}$, then A_3 is open iff A_5 is open.

(8) Let A be a subset of $\mathcal{E}^{k+n}_{\mathrm{T}}$. Suppose $A = \{v \frown (n \mapsto 0) : v$ ranges over elements of $\mathcal{E}^k_{\mathrm{T}}\}$. Let B be a subset of $\mathcal{E}^{k+n}_{\mathrm{T}}{\restriction}A$. Suppose $B = \{v; v$ ranges over points of $\mathcal{E}^{k+n}_{\mathrm{T}}\colon v{\restriction}k \in A_5 \ \wedge \ v \in A\}$. Then A_5 is open if and only if B is open.

(9) For every affinely independent subset A of V and for every subset B of V such that $B \subseteq A$ holds $\operatorname{conv} A \cap \operatorname{Affin} B = \operatorname{conv} B$.

(10) Let V be a non empty RLS structure, A be a non empty set, f be a partial function from A to the carrier of V, and X be a set. Then $(r \cdot f)^\circ X = r \cdot f^\circ X$.

(11) If $\langle \underbrace{0, \ldots, 0}_{n} \rangle \in A_3$, then $\operatorname{Affin} A_3 = \Omega_{\mathrm{Lin}(A_3)}$.

Let V be a non empty additive loop structure, let A be a finite subset of V, and let v be an element of V. Note that $v + A$ is finite.

Let V be a non empty RLS structure, let A be a finite subset of V, and let us consider r. Observe that $r \cdot A$ is finite.

Next we state the proposition

(12) For every subset A of V holds $\overline{\overline{A}} = \overline{\overline{r \cdot A}}$ iff $r \neq 0$ or A is trivial.

Let V be a non empty RLS structure, let f be a finite sequence of elements of V, and let us consider r. Note that $r \cdot f$ is finite sequence-like.

3. The Vector of Barycentric Coordinates

Let X be a finite set. A one-to-one finite sequence is said to be an enumeration of X if:

(Def. 1) $\operatorname{rng} \mathrm{it} = X$.

Let X be a 1-sorted structure and let A be a finite subset of X. We see that the enumeration of A is a one-to-one finite sequence of elements of X.

In the sequel E_1 denotes an enumeration of A_2 and E_2 denotes an enumeration of A_4.

One can prove the following three propositions:

(13) Let V be an Abelian add-associative right zeroed right complementable non empty additive loop structure, A be a finite subset of V, E be an enumeration of A, and v be an element of V. Then $E + \overline{\overline{A}} \mapsto v$ is an enumeration of $v + A$.

(14) For every enumeration E of A_1 holds $r \cdot E$ is an enumeration of $r \cdot A_1$ iff $r \neq 0$ or A_1 is trivial.

(15) Let M be a matrix over $\mathbb{R}_{\mathrm{F}}$ of dimension $n \times m$. Suppose $\operatorname{rk}(M) = n$. Let A be a finite subset of $\mathcal{E}^n_{\mathrm{T}}$ and E be an enumeration of A. Then $\mathrm{Mx2Tran}\, M \cdot E$ is an enumeration of $(\mathrm{Mx2Tran}\, M)^{\circ} A$.

Let us consider V, A_1, let E be an enumeration of A_1, and let us consider x. The functor $x \rightarrow E$ yielding a finite sequence of elements of $\mathbb{R}$ is defined as follows:

(Def. 2) $x \rightarrow E = (x \rightarrow A_1) \cdot E$.

The following propositions are true:

(16) For every enumeration E of A_1 holds $\operatorname{len}(x \rightarrow E) = \overline{\overline{A_1}}$.

(17) For every enumeration E of $v + A_2$ such that $w \in \operatorname{Affin} A_2$ and $E = E_1 + \overline{\overline{A_2}} \mapsto v$ holds $w \rightarrow E_1 = v + w \rightarrow E$.

(18) For every enumeration r_1 of $r \cdot A_2$ such that $v \in \operatorname{Affin} A_2$ and $r_1 = r \cdot E_1$ and $r \neq 0$ holds $v \rightarrow E_1 = r \cdot v \rightarrow r_1$.

(19) Let M be a matrix over $\mathbb{R}_{\mathrm{F}}$ of dimension $n \times m$. Suppose $\operatorname{rk}(M) = n$. Let M_1 be an enumeration of $(\mathrm{Mx2Tran}\, M)^{\circ} A_4$. If $M_1 = \mathrm{Mx2Tran}\, M \cdot E_2$, then for every p_3 such that $p_3 \in \operatorname{Affin} A_4$ holds $p_3 \rightarrow E_2 = (\mathrm{Mx2Tran}\, M)(p_3) \rightarrow M_1$.

(20) Let A be a subset of V. Suppose $A \subseteq A_2$ and $x \in \operatorname{Affin} A_2$. Then $x \in \operatorname{Affin} A$ if and only if for every set y such that $y \in \operatorname{dom}(x \rightarrow E_1)$ and $E_1(y) \notin A$ holds $(x \rightarrow E_1)(y) = 0$.

(21) For every E_1 such that $x \in \operatorname{Affin} A_2$ holds $x \in \operatorname{Affin}(E_1{}^{\circ} \operatorname{Seg} k)$ iff $x \rightarrow E_1 = ((x \rightarrow E_1){\restriction}k) \frown ((\overline{\overline{A_2}} -' k) \mapsto 0)$.

(22) For every E_1 such that $k \leq \overline{\overline{A_2}}$ and $x \in \operatorname{Affin} A_2$ holds $x \in \operatorname{Affin}(A_2 \setminus E_1{}^{\circ} \operatorname{Seg} k)$ iff $x \rightarrow E_1 = (k \mapsto 0) \frown ((x \rightarrow E_1)_{\downharpoonright k})$.

(23) Suppose $\langle\underbrace{0,\ldots,0}_{n}\rangle \in A_4$ and $E_2(\operatorname{len} E_2) = \langle\underbrace{0,\ldots,0}_{n}\rangle$. Then

(i) $\operatorname{rng}(E_2 \restriction (\overline{\overline{A_4}} -' 1)) = A_4 \setminus \{\langle\underbrace{0,\ldots,0}_{n}\rangle\}$, and

(ii) for every subset A of the n-dimension vector space over $\mathbb{R}_{\mathrm{F}}$ such that $A_4 = A$ holds $E_2 \restriction (\overline{\overline{A_4}} -' 1)$ is an ordered basis of $\operatorname{Lin}(A)$.

(24) Let A be a subset of the n-dimension vector space over $\mathbb{R}_{\mathrm{F}}$. Suppose $A_4 = A$ and $\langle\underbrace{0,\ldots,0}_{n}\rangle \in A_4$ and $E_2(\operatorname{len} E_2) = \langle\underbrace{0,\ldots,0}_{n}\rangle$. Let B be an ordered basis of $\operatorname{Lin}(A)$. If $B = E_2 \restriction (\overline{\overline{A_4}} -' 1)$, then for every element v of $\operatorname{Lin}(A)$ holds $v \to B = (v \to E_2) \restriction (\overline{\overline{A_4}} -' 1)$.

(25) For all E_2, A_3 such that $k \leq n$ and $\overline{\overline{A_4}} = n + 1$ and $A_3 = \{p_3 : (p_3 \to E_2) \restriction k \in A_5\}$ holds A_5 is open iff A_3 is open.

(26) For every E_2 such that $k \leq n$ and $\overline{\overline{A_4}} = n + 1$ and $A_3 = \{p_3 : (p_3 \to E_2) \restriction k \in A_5\}$ holds A_5 is closed iff A_3 is closed.

Let us consider n. One can verify that every subset of $\mathcal{E}^n_{\mathrm{T}}$ which is affine is also closed.

In the sequel p_4 denotes an element of $\mathcal{E}^n_{\mathrm{T}} \restriction \operatorname{Affin} A_4$.

Next we state two propositions:

(27) For every E_2 and for every subset B of $\mathcal{E}^n_{\mathrm{T}} \restriction \operatorname{Affin} A_4$ such that $k < \overline{\overline{A_4}}$ and $B = \{p_4 : (p_4 \to E_2) \restriction k \in A_5\}$ holds A_5 is open iff B is open.

(28) Let given E_2 and B be a subset of $\mathcal{E}^n_{\mathrm{T}} \restriction \operatorname{Affin} A_4$. Suppose $k < \overline{\overline{A_4}}$ and $B = \{p_4 : (p_4 \to E_2) \restriction k \in A_5\}$. Then A_5 is closed if and only if B is closed.

Let us consider n and let p, q be points of $\mathcal{E}^n_{\mathrm{T}}$. Observe that $\operatorname{halfline}(p, q)$ is closed.

4. Continuity of Barycentric Coordinates

Let us consider V, let A be a subset of V, and let us consider x. The functor $\vdash (A, x)$ yielding a function from V into $\mathbb{R}^{\mathbf{1}}$ is defined as follows:

(Def. 3) $(\vdash (A, x))(v) = (v \to A)(x)$.

One can prove the following four propositions:

(29) For every subset A of V such that $x \notin A$ holds $\vdash (A, x) = \Omega_V \longmapsto 0$.

(30) For every affinely independent subset A of V such that $\vdash (A, x) = \Omega_V \longmapsto 0$ holds $x \notin A$.

(31) $\vdash (A_4, x) \restriction \operatorname{Affin} A_4$ is a continuous function from $\mathcal{E}^n_{\mathrm{T}} \restriction \operatorname{Affin} A_4$ into $\mathbb{R}^{\mathbf{1}}$.

(32) If $\overline{\overline{A_4}} = n + 1$, then $\vdash (A_4, x)$ is continuous.

Let us consider n, A_4. Note that $\operatorname{conv} A_4$ is closed.

We now state the proposition

(33) If $\overline{\overline{A_4}} = n + 1$, then $\operatorname{Int} A_4$ is open.

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