

Continuity of Barycentric Coordinates in Euclidean Topological Spaces

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Summary. In this paper we present selected properties of barycentric coordinates in the Euclidean topological space. We prove the topological correspondence between a subset of an affine closed space of \mathcal{E}^n and the set of vectors created from barycentric coordinates of points of this subset.

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The terminology and notation used here have been introduced in the following articles: [1], [3], [15], [25], [13], [18], [5], [4], [6], [12], [7], [8], [33], [21], [24], [2], [22], [20], [17], [30], [31], [23], [10], [28], [26], [11], [16], [29], [14], [19], [27], [32], and [9].

1. PRELIMINARIES

For simplicity, we adopt the following rules: x denotes a set, n, m, k denote natural numbers, r denotes a real number, V denotes a real linear space, v, w denote vectors of V , A_1 denotes a finite subset of V , and A_2 denotes a finite affinely independent subset of V .

One can prove the following propositions:

- (1) For all real-valued finite sequences f_1, f_2 and for every real number r holds $\text{Intervals}(f_1, r) \cap \text{Intervals}(f_2, r) = \text{Intervals}(f_1 \cap f_2, r)$.
- (2) Let f_1, f_2 be finite sequences. Then $x \in \prod(f_1 \cap f_2)$ if and only if there exist finite sequences p_1, p_2 such that $x = p_1 \cap p_2$ and $p_1 \in \prod f_1$ and $p_2 \in \prod f_2$.

- (3) V is finite dimensional iff Ω_V is finite dimensional.

Let V be a finite dimensional real linear space. One can verify that every affinely independent subset of V is finite.

Let us consider n . One can check that \mathcal{E}_T^n is add-continuous and mult-continuous and \mathcal{E}_T^n is finite dimensional.

In the sequel p_3 denotes a point of \mathcal{E}_T^n , A_3 denotes a subset of \mathcal{E}_T^n , A_4 denotes an affinely independent subset of \mathcal{E}_T^n , and A_5 denotes a subset of \mathcal{E}_T^k .

Next we state three propositions:

- (4) $\dim(\mathcal{E}_T^n) = n$.
- (5) Let V be a finite dimensional real linear space and A be an affinely independent subset of V . Then $\overline{A} \leq 1 + \dim(V)$.
- (6) Let V be a finite dimensional real linear space and A be an affinely independent subset of V . Then $\overline{A} = \dim(V) + 1$ if and only if $\text{Affin } A = \Omega_V$.

2. OPEN AND CLOSED SUBSETS OF A SUBSPACE OF THE EUCLIDEAN TOPOLOGICAL SPACE

One can prove the following propositions:

- (7) If $k \leq n$ and $A_3 = \{v \in \mathcal{E}_T^n : v|k \in A_5\}$, then A_3 is open iff A_5 is open.
- (8) Let A be a subset of \mathcal{E}_T^{k+n} . Suppose $A = \{v \wedge (n \mapsto 0) : v \text{ ranges over elements of } \mathcal{E}_T^k\}$. Let B be a subset of $\mathcal{E}_T^{k+n} \upharpoonright A$. Suppose $B = \{v; v \text{ ranges over points of } \mathcal{E}_T^{k+n} : v|k \in A_5 \wedge v \in A\}$. Then A_5 is open if and only if B is open.
- (9) For every affinely independent subset A of V and for every subset B of V such that $B \subseteq A$ holds $\text{conv } A \cap \text{Affin } B = \text{conv } B$.
- (10) Let V be a non empty RLS structure, A be a non empty set, f be a partial function from A to the carrier of V , and X be a set. Then $(r \cdot f)^\circ X = r \cdot f^\circ X$.
- (11) If $\underbrace{\langle 0, \dots, 0 \rangle}_n \in A_3$, then $\text{Affin } A_3 = \Omega_{\text{Lin}(A_3)}$.

Let V be a non empty additive loop structure, let A be a finite subset of V , and let v be an element of V . Note that $v + A$ is finite.

Let V be a non empty RLS structure, let A be a finite subset of V , and let us consider r . Observe that $r \cdot A$ is finite.

Next we state the proposition

- (12) For every subset A of V holds $\overline{r \cdot A} = \overline{r \cdot \overline{A}}$ iff $r \neq 0$ or A is trivial.

Let V be a non empty RLS structure, let f be a finite sequence of elements of V , and let us consider r . Note that $r \cdot f$ is finite sequence-like.

3. THE VECTOR OF BARYCENTRIC COORDINATES

Let X be a finite set. A one-to-one finite sequence is said to be an enumeration of X if:

(Def. 1) $\text{rng it} = X$.

Let X be a 1-sorted structure and let A be a finite subset of X . We see that the enumeration of A is a one-to-one finite sequence of elements of X .

In the sequel E_1 denotes an enumeration of A_2 and E_2 denotes an enumeration of A_4 .

One can prove the following three propositions:

- (13) Let V be an Abelian add-associative right zeroed right complementable non empty additive loop structure, A be a finite subset of V , E be an enumeration of A , and v be an element of V . Then $E + \overline{A} \mapsto v$ is an enumeration of $v + A$.
- (14) For every enumeration E of A_1 holds $r \cdot E$ is an enumeration of $r \cdot A_1$ iff $r \neq 0$ or A_1 is trivial.
- (15) Let M be a matrix over \mathbb{R}_F of dimension $n \times m$. Suppose $\text{rk}(M) = n$. Let A be a finite subset of \mathcal{E}_T^n and E be an enumeration of A . Then $\text{Mx2Tran } M \cdot E$ is an enumeration of $(\text{Mx2Tran } M)^\circ A$.

Let us consider V , A_1 , let E be an enumeration of A_1 , and let us consider x . The functor $x \rightarrow E$ yielding a finite sequence of elements of \mathbb{R} is defined as follows:

(Def. 2) $x \rightarrow E = (x \rightarrow A_1) \cdot E$.

The following propositions are true:

- (16) For every enumeration E of A_1 holds $\text{len}(x \rightarrow E) = \overline{A_1}$.
- (17) For every enumeration E of $v + A_2$ such that $w \in \text{Affin } A_2$ and $E = E_1 + \overline{A_2} \mapsto v$ holds $w \rightarrow E_1 = v + w \rightarrow E$.
- (18) For every enumeration r_1 of $r \cdot A_2$ such that $v \in \text{Affin } A_2$ and $r_1 = r \cdot E_1$ and $r \neq 0$ holds $v \rightarrow E_1 = r \cdot v \rightarrow r_1$.
- (19) Let M be a matrix over \mathbb{R}_F of dimension $n \times m$. Suppose $\text{rk}(M) = n$. Let M_1 be an enumeration of $(\text{Mx2Tran } M)^\circ A_4$. If $M_1 = \text{Mx2Tran } M \cdot E_2$, then for every p_3 such that $p_3 \in \text{Affin } A_4$ holds $p_3 \rightarrow E_2 = (\text{Mx2Tran } M)(p_3) \rightarrow M_1$.
- (20) Let A be a subset of V . Suppose $A \subseteq A_2$ and $x \in \text{Affin } A_2$. Then $x \in \text{Affin } A$ if and only if for every set y such that $y \in \text{dom}(x \rightarrow E_1)$ and $E_1(y) \notin A$ holds $(x \rightarrow E_1)(y) = 0$.
- (21) For every E_1 such that $x \in \text{Affin } A_2$ holds $x \in \text{Affin}(E_1^\circ \text{Seg } k)$ iff $x \rightarrow E_1 = ((x \rightarrow E_1) \upharpoonright k) \wedge ((\overline{A_2} -' k) \mapsto 0)$.
- (22) For every E_1 such that $k \leq \overline{A_2}$ and $x \in \text{Affin } A_2$ holds $x \in \text{Affin}(A_2 \setminus E_1^\circ \text{Seg } k)$ iff $x \rightarrow E_1 = (k \mapsto 0) \wedge ((x \rightarrow E_1) \upharpoonright k)$.

- (23) Suppose $\langle \underbrace{0, \dots, 0}_n \rangle \in A_4$ and $E_2(\text{len } E_2) = \langle \underbrace{0, \dots, 0}_n \rangle$. Then
- (i) $\text{rng}(E_2 \upharpoonright (\overline{A_4} -' 1)) = A_4 \setminus \{\langle \underbrace{0, \dots, 0}_n \rangle\}$, and
 - (ii) for every subset A of the n -dimension vector space over \mathbb{R}_F such that $A_4 = A$ holds $E_2 \upharpoonright (\overline{A_4} -' 1)$ is an ordered basis of $\text{Lin}(A)$.
- (24) Let A be a subset of the n -dimension vector space over \mathbb{R}_F . Suppose $A_4 = A$ and $\langle \underbrace{0, \dots, 0}_n \rangle \in A_4$ and $E_2(\text{len } E_2) = \langle \underbrace{0, \dots, 0}_n \rangle$. Let B be an ordered basis of $\text{Lin}(A)$. If $B = E_2 \upharpoonright (\overline{A_4} -' 1)$, then for every element v of $\text{Lin}(A)$ holds $v \rightarrow B = (v \rightarrow E_2) \upharpoonright (\overline{A_4} -' 1)$.
- (25) For all E_2, A_3 such that $k \leq n$ and $\overline{A_4} = n + 1$ and $A_3 = \{p_3 : (p_3 \rightarrow E_2) \upharpoonright k \in A_5\}$ holds A_5 is open iff A_3 is open.
- (26) For every E_2 such that $k \leq n$ and $\overline{A_4} = n + 1$ and $A_3 = \{p_3 : (p_3 \rightarrow E_2) \upharpoonright k \in A_5\}$ holds A_5 is closed iff A_3 is closed.

Let us consider n . One can verify that every subset of \mathcal{E}_T^n which is affine is also closed.

In the sequel p_4 denotes an element of $\mathcal{E}_T^n \upharpoonright \text{Affin } A_4$.

Next we state two propositions:

- (27) For every E_2 and for every subset B of $\mathcal{E}_T^n \upharpoonright \text{Affin } A_4$ such that $k < \overline{A_4}$ and $B = \{p_4 : (p_4 \rightarrow E_2) \upharpoonright k \in A_5\}$ holds A_5 is open iff B is open.
- (28) Let given E_2 and B be a subset of $\mathcal{E}_T^n \upharpoonright \text{Affin } A_4$. Suppose $k < \overline{A_4}$ and $B = \{p_4 : (p_4 \rightarrow E_2) \upharpoonright k \in A_5\}$. Then A_5 is closed if and only if B is closed.

Let us consider n and let p, q be points of \mathcal{E}_T^n . Observe that halfline(p, q) is closed.

4. CONTINUITY OF BARYCENTRIC COORDINATES

Let us consider V , let A be a subset of V , and let us consider x . The functor $\vdash(A, x)$ yielding a function from V into \mathbb{R}^1 is defined as follows:

(Def. 3) $(\vdash(A, x))(v) = (v \rightarrow A)(x)$.

One can prove the following four propositions:

- (29) For every subset A of V such that $x \notin A$ holds $\vdash(A, x) = \Omega_V \mapsto 0$.
- (30) For every affinely independent subset A of V such that $\vdash(A, x) = \Omega_V \mapsto 0$ holds $x \notin A$.
- (31) $\vdash(A_4, x) \upharpoonright \text{Affin } A_4$ is a continuous function from $\mathcal{E}_T^n \upharpoonright \text{Affin } A_4$ into \mathbb{R}^1 .
- (32) If $\overline{A_4} = n + 1$, then $\vdash(A_4, x)$ is continuous.

Let us consider n, A_4 . Note that $\text{conv } A_4$ is closed.

We now state the proposition

(33) If $\overline{A_4} = n + 1$, then $\text{Int } A_4$ is open.

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