

# Brouwer Fixed Point Theorem in the General Case

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**Summary.** In this article we prove the Brouwer fixed point theorem for an arbitrary convex compact subset of  $\mathcal{E}^n$  with a non empty interior. This article is based on [15].

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The notation and terminology used here have been introduced in the following papers: [17], [12], [1], [4], [7], [16], [6], [13], [10], [2], [3], [14], [9], [20], [18], [8], [19], [11], [21], and [5].

## 1. PRELIMINARIES

For simplicity, we adopt the following convention:  $n$  is a natural number,  $p, q, u, w$  are points of  $\mathcal{E}_T^n$ ,  $S$  is a subset of  $\mathcal{E}_T^n$ ,  $A, B$  are convex subsets of  $\mathcal{E}_T^n$ , and  $r$  is a real number.

Next we state several propositions:

- (1)  $(1 - r) \cdot p + r \cdot q = p + r \cdot (q - p)$ .
- (2) If  $u, w \in \text{halfline}(p, q)$  and  $|u - p| = |w - p|$ , then  $u = w$ .
- (3) Let given  $S$ . Suppose  $p \in S$  and  $p \neq q$  and  $S \cap \text{halfline}(p, q)$  is Bounded. Then there exists  $w$  such that
  - (i)  $w \in \text{Fr } S \cap \text{halfline}(p, q)$ ,
  - (ii) for every  $u$  such that  $u \in S \cap \text{halfline}(p, q)$  holds  $|p - u| \leq |p - w|$ , and
  - (iii) for every  $r$  such that  $r > 0$  there exists  $u$  such that  $u \in S \cap \text{halfline}(p, q)$  and  $|w - u| < r$ .

- (4) For every  $A$  such that  $A$  is closed and  $p \in \text{Int } A$  and  $p \neq q$  and  $A \cap \text{halfline}(p, q)$  is Bounded there exists  $u$  such that  $\text{Fr } A \cap \text{halfline}(p, q) = \{u\}$ .
- (5) If  $r > 0$ , then  $\text{Fr } \overline{\text{Ball}}(p, r) = \text{Sphere}(p, r)$ .

Let  $n$  be an element of  $\mathbb{N}$ , let  $A$  be a Bounded subset of  $\mathcal{E}_{\mathbb{T}}^n$ , and let  $p$  be a point of  $\mathcal{E}_{\mathbb{T}}^n$ . One can verify that  $p + A$  is Bounded.

## 2. MAIN THEOREMS

Next we state four propositions:

- (6) Let  $n$  be an element of  $\mathbb{N}$  and  $A$  be a convex subset of  $\mathcal{E}_{\mathbb{T}}^n$ . Suppose  $A$  is compact and non boundary. Then there exists a function  $h$  from  $\mathcal{E}_{\mathbb{T}}^n \upharpoonright A$  into  $\text{Tdisk}(0_{\mathcal{E}_{\mathbb{T}}^n}, 1)$  such that  $h$  is homeomorphism and  $h^\circ \text{Fr } A = \text{Sphere}((0_{\mathcal{E}_{\mathbb{T}}^n}), 1)$ .
- (7) Let given  $A, B$ . Suppose  $A$  is compact and non boundary and  $B$  is compact and non boundary. Then there exists a function  $h$  from  $\mathcal{E}_{\mathbb{T}}^n \upharpoonright A$  into  $\mathcal{E}_{\mathbb{T}}^n \upharpoonright B$  such that  $h$  is homeomorphism and  $h^\circ \text{Fr } A = \text{Fr } B$ .
- (8)<sup>1</sup> For every  $A$  such that  $A$  is compact and non boundary holds every continuous function from  $\mathcal{E}_{\mathbb{T}}^n \upharpoonright A$  into  $\mathcal{E}_{\mathbb{T}}^n \upharpoonright A$  has a fixpoint.
- (9) Let  $A$  be a non empty convex subset of  $\mathcal{E}_{\mathbb{T}}^n$ . Suppose  $A$  is compact and non boundary. Let  $F_1$  be a non empty subspace of  $\mathcal{E}_{\mathbb{T}}^n \upharpoonright A$ . If  $\Omega_{(F_1)} = \text{Fr } A$ , then  $F_1$  is not a retract of  $\mathcal{E}_{\mathbb{T}}^n \upharpoonright A$ .

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