First Order Languages: Further Syntax and Semantics[1]

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Summary. Third of a series of articles laying down the bases for classical first order model theory. Interpretation of a language in a universe set. Evaluation of a term in a universe. Truth evaluation of an atomic formula. Reassigning the value of a symbol in a given interpretation. Syntax and semantics of a non atomic formula are then defined concurrently (this point is explained in [16], 4.2.1). As a consequence, the evaluation of any w.f.f. string and the relation of logical implication are introduced. Depth of a formula. Definition of satisfaction and entailment (aka entailment or logical implication) relations, see [18] III.3.2 and III.4.1 respectively.

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The terminology and notation used in this paper have been introduced in the following papers: [7], [1], [23], [6], [8], [17], [14], [15], [22], [9], [10], [11], [2], [21], [26], [24], [5], [3], [4], [12], [27], [28], [19], [20], [25], and [13].

For simplicity, we follow the rules: m, n denote natural numbers, m_1 denotes an element of $\mathbb{N}$, A, B, X, Y, Z, x, y denote sets, S, S_1, S_2 denote languages, s denotes an element of S, w, w_1, w_2 denote strings of S, U denotes a non empty set, f, g denote functions, and p, p_2 denote finite sequences.

Let us consider S. Then TheNorSymbOf S is an element of S.

Let U be a non empty set. The functor U-deltaInterpreter yielding a function from U^2 into *Boolean* is defined by:

(Def. 1) U-deltaInterpreter $= \chi_{(\text{the concatenation of } U)^\circ(\mathrm{id}_{U^1}), U^2}$.

Let X be a set. Then id_X is an equivalence relation of X.

Let S be a language, let U be a non empty set, and let s be an of-atomic-formula element of S. Interpreter of s and U is defined as follows:

(Def. 2)(i) It is a function from $U^{|\mathrm{ar}\, s|}$ into *Boolean* if s is relational,

(ii) it is a function from $U^{|\mathrm{ar}\, s|}$ into U, otherwise.

Let us consider S, U and let s be an of-atomic-formula element of S. We see that the interpreter of s and U is a function from $U^{|\mathrm{ar}\, s|}$ into $U \cup Boolean$.

Let us consider S, U and let s be a termal element of S. One can verify that every interpreter of s and U is U-valued.

Let S be a language. Note that every element of S which is literal is also own.

Let us consider S, U. A function is called an interpreter of S and U if:

(Def. 3) For every own element s of S holds $\mathrm{it}(s)$ is an interpreter of s and U.

Let us consider S, U, f. We say that f is (S, U)-interpreter-like if and only if:

(Def. 4) f is an interpreter of S and U and function yielding.

Let us consider S and let U be a non empty set. One can verify that every function which is (S, U)-interpreter-like is also function yielding.

Let us consider S, U and let s be an own element of S. Observe that every interpreter of s and U is non empty.

Let S be a language and let U be a non empty set. Note that there exists a function which is (S, U)-interpreter-like.

Let us consider S, U, let I be an (S, U)-interpreter-like function, and let s be an own element of S. Then $I(s)$ is an interpreter of s and U.

Let S be a language, let U be a non empty set, let I be an (S, U)-interpreter-like function, let x be an own element of S, and let f be an interpreter of x and U. One can check that $I + \cdot (x \longmapsto f)$ is (S, U)-interpreter-like.

Let us consider f, x, y. The functor $(x, y)\, \mathrm{ReassignIn}\, f$ yields a function and is defined by:

(Def. 5) $(x, y)\, \mathrm{ReassignIn}\, f = f + \cdot (x \longmapsto (\emptyset \longmapsto y))$.

Let S be a language, let U be a non empty set, let I be an (S, U)-interpreter-like function, let x be a literal element of S, and let u be an element of U. One can verify that $(x, u)\, \mathrm{ReassignIn}\, I$ is (S, U)-interpreter-like.

Let S be a language. One can check that $\mathrm{AllSymbolsOf}\, S$ is non empty.

Let Y be a set and let X, Z be non empty sets. Observe that every function from X into Z^Y is function yielding.

Let X, Y, Z be non empty sets. One can verify that there exists a function from X into Z^Y which is function yielding.

Let f be a function yielding function and let g be a function. The functor $[g, f]$ yields a function and is defined by:

(Def. 6) $\operatorname{dom}[g, f] = \operatorname{dom} f$ and for every x such that $x \in \operatorname{dom} f$ holds $[g, f](x) = g \cdot f(x)$.

Let f be an empty function and let g be a function. One can verify that $[g, f]$ is empty.

Let f be a function yielding function and let g be a function. The functor $[f, g]$ yielding a function is defined as follows:

(Def. 7) $\operatorname{dom}[f, g] = \operatorname{dom} f \cap \operatorname{dom} g$ and for every set x such that $x \in \operatorname{dom}[f, g]$ holds $[f, g](x) = f(x)(g(x))$.

Let f be a function yielding function and let g be an empty function. One can check that $[f, g]$ is empty.

Let X be a finite sequence-membered set. Observe that every function which is X-valued is also function yielding.

Let E, D be non empty sets, let p be a D-valued finite sequence, and let h be a function from D into E. Note that $h \cdot p$ is $\operatorname{len} p$-element.

Let X, Y be non empty sets, let f be a function from X into Y, and let p be an X-valued finite sequence. One can verify that $f \cdot p$ is finite sequence-like.

Let E, D be non empty sets, let n be a natural number, let p be an n-element D-valued finite sequence, and let h be a function from D into E. Observe that $h \cdot p$ is n-element.

We now state the proposition

(1)   For every 0-termal string t_0 of S holds $t_0 = \langle S\text{-firstChar}(t_0) \rangle$.

Let us consider S, let U be a non empty set, let u be an element of U, and let I be an (S, U)-interpreter-like function. The functor (I, u)-TermEval yields a function from $\mathbb{N}$ into $U^{\operatorname{AllTermsOf} S}$ and is defined as follows:

(Def. 8) (I, u)-TermEval$(0) = \operatorname{AllTermsOf} S \longmapsto u$ and for every m_1 holds (I, u)-TermEval$(m_1 + 1) = [I \cdot S\text{-firstChar}, [((I, u)\text{-TermEval}(m_1) \text{ } \mathbf{qua}$ function$), S\text{-subTerms}]]$.

Let us consider S, U, let I be an (S, U)-interpreter-like function, and let t be an element of $\operatorname{AllTermsOf} S$. The functor I-TermEval$t$ yields an element of U and is defined as follows:

(Def. 9)   For every element u_1 of U and for every m_1 such that $t \in S\text{-termsOfMaxDepth}(m_1)$ holds $I\text{-TermEval} t = (I, u_1)\text{-TermEval}(m_1 + 1)(t)$.

Let us consider S, U and let I be an (S, U)-interpreter-like function. The functor I-TermEval yielding a function from $\operatorname{AllTermsOf} S$ into U is defined by:

(Def. 10)   For every element t of $\operatorname{AllTermsOf} S$ holds $I\text{-TermEval}(t) = I\text{-TermEval} t$.

Let us consider S, U and let I be an (S, U)-interpreter-like function. The functor $I ===$ yielding a function is defined as follows:

(Def. 11) $I ==== I + \cdot (\text{TheEqSymbOf } S \dot{\longmapsto} U\text{-deltaInterpreter})$.

Let us consider S, U, let I be an (S, U)-interpreter-like function, and let x be a set. We say that x is I-extension if and only if:

(Def. 12) $x = I === $.

Let us consider S, U and let I be an (S, U)-interpreter-like function. Note that $I === $ is I-extension and every set which is I-extension is also function-like. Observe that there exists a function which is I-extension. Observe that $I === $ is (S, U)-interpreter-like.

Let f be an I-extension function, and let s be an of-atomic-formula element of S. Then $f(s)$ is an interpreter of s and U.

Let p_1 be a 0-w.f.f. string of S. The functor I-AtomicEval p_1 is defined as follows:

(Def. 13) I-AtomicEval $p_1 = (I === (S\text{-firstChar}(p_1)))(I\text{-TermEval} \cdot \text{SubTerms}\, p_1)$.

Let us consider S, U, let I be an (S, U)-interpreter-like function, and let p_1 be a 0-w.f.f. string of S. Then I-AtomicEval p_1 is an element of $Boolean$. Note that $I{\upharpoonright}\,\text{OwnSymbolsOf}\, S$ is $(U^* \dot\to (U \cup Boolean))$-valued and $I{\upharpoonright}\,\text{OwnSymbolsOf}\, S$ is (S, U)-interpreter-like.

Let us consider S, U and let I be an (S, U)-interpreter-like function. Observe that $I{\upharpoonright}\,\text{OwnSymbolsOf}\, S$ is total.

Let us consider S, U. The functor U-InterpretersOf S is defined by:

(Def. 14) U-InterpretersOf $S = \{f \in (U^* \dot\to (U \cup Boolean))^{\text{OwnSymbolsOf}\, S} : f$ is (S, U)-interpreter-like$\}$.

Let us consider S, U. Then U-InterpretersOf S is a subset of $(U^* \dot\to (U \cup Boolean))^{\text{OwnSymbolsOf}\, S}$. Observe that U-InterpretersOf S is non empty. One can verify that every element of U-InterpretersOf S is (S, U)-interpreter-like. The functor S-TruthEval U yields a function from

$(U$-InterpretersOf $S) \times \text{AtomicFormulasOf}\, S$ into $Boolean$ and is defined by:

(Def. 15)   For every element I of U-InterpretersOf S and for every element p_1 of AtomicFormulasOf S holds $(S\text{-TruthEval}\, U)(I, p_1) = I$-AtomicEval p_1.

Let us consider S, U, let I be an element of U-InterpretersOf S, let f be a partial function from $(U$-InterpretersOf $S) \times ((\text{AllSymbolsOf}\, S)^* \setminus \{\emptyset\})$ to $Boolean$, and let p_1 be an element of $(\text{AllSymbolsOf}\, S)^* \setminus \{\emptyset\}$. The functor f-ExFunctor(I, p_1) yielding an element of $Boolean$ is defined as follows:

(Def. 16) $f\text{-ExFunctor}(I, p_1) = \begin{cases} true, & \text{if there exists an element } u \text{ of } U \text{ and} \\ & \quad \text{there exists a literal element } v \text{ of } S \text{ such} \\ & \quad \text{that } p_1(1) = v \text{ and} \\ & \quad\quad f((v, u)\,\text{ReassignIn}\, I, (p_1)_{\downarrow 1}) = true, \\ false, & \text{otherwise.} \end{cases}$

Let us consider S, U and let g be an element of $(U$-InterpretersOf $S) \times ((\text{AllSymbolsOf}\, S)^* \setminus \{\emptyset\}) \dot\to Boolean$. The functor ExIterator g yields a partial function from $(U$-InterpretersOf $S) \times ((\text{AllSymbolsOf}\, S)^* \setminus \{\emptyset\})$ to $Boolean$ and

is defined by the conditions (Def. 17).

(Def. 17)(i) For every element x of U-InterpretersOf S and for every element y of $(\text{AllSymbolsOf } S)^* \setminus \{\emptyset\}$ holds $\langle x, y \rangle \in \text{dom ExIterator } g$ iff there exists a literal element v of S and there exists a string w of S such that $\langle x, w \rangle \in \text{dom } g$ and $y = \langle v \rangle ^\frown w$, and

(ii) for every element x of U-InterpretersOf S and for every element y of $(\text{AllSymbolsOf } S)^* \setminus \{\emptyset\}$ such that $\langle x, y \rangle \in \text{dom ExIterator } g$ holds $(\text{ExIterator } g)(x, y) = g\text{-ExFunctor}(x, y)$.

Let us consider S, U, let f be a partial function from $(U\text{-InterpretersOf } S) \times ((\text{AllSymbolsOf } S)^* \setminus \{\emptyset\})$ to $Boolean$, let I be an element of U-InterpretersOf S, and let p_1 be an element of $(\text{AllSymbolsOf } S)^* \setminus \{\emptyset\}$.

The functor $f\text{-NorFunctor}(I, p_1)$ yielding an element of $Boolean$ is defined by:

(Def. 18) $f\text{-NorFunctor}(I, p_1) = \begin{cases} true, & \text{if there exist elements } w_1,\ w_2 \text{ of} \\ & (\text{AllSymbolsOf } S)^* \setminus \{\emptyset\} \text{ such that} \\ & \langle I,\ w_1 \rangle \in \text{dom } f \text{ and } f(I,\ w_1) = false \\ & \text{and } f(I,\ w_2) = false \text{ and} \\ & p_1 = \langle \text{TheNorSymbOf } S \rangle ^\frown w_1 ^\frown w_2, \\ false, & \text{otherwise.} \end{cases}$

Let us consider S, U and let g be an element of $(U\text{-InterpretersOf } S) \times ((\text{AllSymbolsOf } S)^* \setminus \{\emptyset\}) \dotrightarrow Boolean$. The functor NorIterator g yielding a partial function from $(U\text{-InterpretersOf } S) \times ((\text{AllSymbolsOf } S)^* \setminus \{\emptyset\})$ to $Boolean$ is defined by the conditions (Def. 19).

(Def. 19)(i) For every element x of U-InterpretersOf S and for every element y of $(\text{AllSymbolsOf } S)^* \setminus \{\emptyset\}$ holds $\langle x, y \rangle \in \text{dom NorIterator } g$ iff there exist elements p_3, p_4 of $(\text{AllSymbolsOf } S)^* \setminus \{\emptyset\}$ such that $y = \langle \text{TheNorSymbOf } S \rangle ^\frown p_3 ^\frown p_4$ and $\langle x, p_3 \rangle$, $\langle x, p_4 \rangle \in \text{dom } g$, and

(ii) for every element x of U-InterpretersOf S and for every element y of $(\text{AllSymbolsOf } S)^* \setminus \{\emptyset\}$ such that $\langle x, y \rangle \in \text{dom NorIterator } g$ holds $(\text{NorIterator } g)(x, y) = g\text{-NorFunctor}(x, y)$.

Let us consider S, U. The functor (S, U)-TruthEval yields a function from $\mathbb{N}$ into $(U\text{-InterpretersOf } S) \times ((\text{AllSymbolsOf } S)^* \setminus \{\emptyset\}) \dotrightarrow Boolean$ and is defined as follows:

(Def. 20) $(S, U)\text{-TruthEval}(0) = S\text{-TruthEval } U$ and for every m_1 holds $(S, U)\text{-TruthEval}(m_1+1) = \text{ExIterator}(S, U)\text{-TruthEval}(m_1) +\cdot \text{NorIterator}(S, U)\text{-TruthEval}(m_1) +\cdot (S, U)\text{-TruthEval}(m_1)$.

Next we state the proposition

(2) For every (S, U)-interpreter-like function I holds $I \upharpoonright \text{OwnSymbolsOf } S \in U\text{-InterpretersOf } S$.

Let S be a language, let m be a natural number, and let U be a non empty set.

The functor (S, U)-TruthEval m yielding an element of $(U$-InterpretersOf $S) \times ((\text{AllSymbolsOf } S)^* \setminus \{\emptyset\}) \xrightarrow{.} Boolean$ is defined as follows:

(Def. 21)   For every m_1 such that $m = m_1$ holds (S, U)-TruthEval $m = (S, U)$-TruthEval(m_1).

Let us consider S, U, m and let I be an element of U-InterpretersOf S. The functor (I, m)-TruthEval yields an element of $((\text{AllSymbolsOf } S)^* \setminus \{\emptyset\}) \xrightarrow{.} Boolean$ and is defined by:

(Def. 22) (I, m)-TruthEval $= (\text{curry}((S, U)\text{-TruthEval } m))(I)$.

Let us consider S, m. The functor S-formulasOfMaxDepth m yielding a subset of $(\text{AllSymbolsOf } S)^* \setminus \{\emptyset\}$ is defined as follows:

(Def. 23)   For every non empty set U and for every element I of U-InterpretersOf S and for every element m_1 of $\mathbb{N}$ such that $m = m_1$ holds S-formulasOfMaxDepth $m = \text{dom}((I, m_1)\text{-TruthEval})$.

Let us consider S, m, w. We say that w is m-w.f.f. if and only if:

(Def. 24) $w \in S$-formulasOfMaxDepth m.

Let us consider S, w. We say that w is w.f.f. if and only if:

(Def. 25)   There exists m such that w is m-w.f.f..

Let us consider S. Note that every string of S which is 0-w.f.f. is also 0-w.f.f. and every string of S which is 0-w.f.f. is also 0-w.f.f.. Let us consider m. One can check that every string of S which is m-w.f.f. is also w.f.f.. Let us consider n. One can check that every string of S which is $m + 0 \cdot n$-w.f.f. is also $m + n$-w.f.f..

Let us consider S, m. Observe that there exists a string of S which is m-w.f.f.. Note that S-formulasOfMaxDepth m is non empty. One can verify that there exists a string of S which is w.f.f..

Let us consider S, U, let I be an element of U-InterpretersOf S, and let w be a w.f.f. string of S. The functor I-TruthEval w yields an element of $Boolean$ and is defined as follows:

(Def. 26)   For every natural number m such that w is m-w.f.f. holds I-TruthEval $w = (I, m)$-TruthEval(w).

Let us consider S. The functor AllFormulasOf S is defined by:

(Def. 27)   AllFormulasOf $S = \{w; w$ ranges over strings of $S: \bigvee_m \ w$ is m-w.f.f.$\}$.

Let us consider S. One can check that AllFormulasOf S is non empty.

For simplicity, we follow the rules: u, u_1, u_2 are elements of U, t is a termal string of S, I is an (S, U)-interpreter-like function, l, l_1, l_2 are literal elements of S, m_2, n_1 are non zero natural numbers, p_0 is a 0-w.f.f. string of S, and p_5, p_1, p_3, p_4 are w.f.f. strings of S.

The following propositions are true:

(3) (I, u)-TermEval$(m + 1)(t) = I(S\text{-firstChar}(t))((I, u)\text{-TermEval}(m) \cdot \text{SubTerms } t)$ and if t is 0-termal, then (I, u)-TermEval$(m + 1)(t) = I(S\text{-firstChar}(t))(\emptyset)$.

(4)   For every m-termal string t of S holds (I, u_1)-$\mathrm{TermEval}(m + 1)(t) = (I, u_2)$-$\mathrm{TermEval}(m + 1 + n)(t)$.

(5) $\mathrm{curry}((S, U)$-$\mathrm{TruthEval}\, m)$ is a function from U-$\mathrm{InterpretersOf}\, S$ into $((\mathrm{AllSymbolsOf}\, S)^* \setminus \{\emptyset\}) \dot{\to} Boolean$.

(6) $x \in X \cup Y \cup Z$ iff $x \in X$ or $x \in Y$ or $x \in Z$.

(7) S-$\mathrm{formulasOfMaxDepth}\, 0 = \mathrm{AtomicFormulasOf}\, S$.

Let us consider S, m. Then S-$\mathrm{formulasOfMaxDepth}\, m$ can be characterized by the condition:

(Def. 28)   For every non empty set U and for every element I of U-$\mathrm{InterpretersOf}\, S$ holds S-$\mathrm{formulasOfMaxDepth}\, m = \mathrm{dom}((I, m)$-$\mathrm{TruthEval})$.

Next we state the proposition

(8) (S, U)-$\mathrm{TruthEval}\, m \in Boolean^{(U\text{-}\mathrm{InterpretersOf}\, S) \times (S\text{-}\mathrm{formulasOfMaxDepth}\, m)}$ and
(S, U)-$\mathrm{TruthEval}(m) \in Boolean^{(U\text{-}\mathrm{InterpretersOf}\, S) \times (S\text{-}\mathrm{formulasOfMaxDepth}\, m)}$.

Let us consider S, m. The functor m-$\mathrm{ExFormulasOf}\, S$ is defined by:

(Def. 29) m-$\mathrm{ExFormulasOf}\, S = \{\langle v \rangle ^\frown p_1 : v$ ranges over elements of $\mathrm{LettersOf}\, S, p_1$ ranges over elements of S-$\mathrm{formulasOfMaxDepth}\, m\}$.

The functor m-$\mathrm{NorFormulasOf}\, S$ is defined as follows:

(Def. 30) m-$\mathrm{NorFormulasOf}\, S = \{\langle \mathrm{TheNorSymbOf}\, S \rangle ^\frown p_3 ^\frown p_4 : p_3$ ranges over elements of S-$\mathrm{formulasOfMaxDepth}\, m, p_4$ ranges over elements of S-$\mathrm{formulasOfMaxDepth}\, m\}$.

Let us consider S and let w_1, w_2 be strings of S. Then $w_1 ^\frown w_2$ is a string of S.

Let us consider S, s. Then $\langle s \rangle$ is a string of S.

One can prove the following two propositions:

(9) S-$\mathrm{formulasOfMaxDepth}(m + 1) = (m$-$\mathrm{ExFormulasOf}\, S) \cup (m$-$\mathrm{NorFormulasOf}\, S) \cup (S$-$\mathrm{formulasOfMaxDepth}\, m)$.

(10) $\mathrm{AtomicFormulasOf}\, S$ is S-prefix.

Let us consider S. Note that $\mathrm{AtomicFormulasOf}\, S$ is S-prefix. Observe that S-$\mathrm{formulasOfMaxDepth}\, 0$ is S-prefix.

Let us consider p_1. The functor $\mathrm{Depth}\, p_1$ yielding a natural number is defined by:

(Def. 31) p_1 is $\mathrm{Depth}\, p_1$-w.f.f. and for every n such that p_1 is n-w.f.f. holds $\mathrm{Depth}\, p_1 \le n$.

Let us consider S, m and let p_3, p_4 be m-w.f.f. strings of S. Note that $\langle \mathrm{TheNorSymbOf}\, S \rangle ^\frown p_3 ^\frown p_4$ is $m + 1$-w.f.f..

Let us consider S, p_3, p_4. Observe that $\langle \mathrm{TheNorSymbOf}\, S \rangle ^\frown p_3 ^\frown p_4$ is w.f.f..

Let us consider S, m, let p_1 be an m-w.f.f. string of S, and let v be a literal element of S. Note that $\langle v \rangle ^\frown p_1$ is $m + 1$-w.f.f..

Let us consider S, l, p_1. Note that $\langle l \rangle \frown p_1$ is w.f.f..

Let us consider S, w and let s be a non relational element of S. One can check that $\langle s \rangle \frown w$ is non 0-w.f.f..

Let us consider S, w_1, w_2 and let s be a non relational element of S. Observe that $\langle s \rangle \frown w_1 \frown w_2$ is non 0-w.f.f..

Let us consider S. Observe that TheNorSymbOf S is non relational.

Let us consider S, w. Observe that $\langle \text{TheNorSymbOf } S \rangle \frown w$ is non 0-w.f.f..

Let us consider S, l, w. Note that $\langle l \rangle \frown w$ is non 0-w.f.f..

Let us consider S, w. We say that w is exal if and only if:

(Def. 32) S-firstChar(w) is literal.

Let us consider S, w, l. One can verify that $\langle l \rangle \frown w$ is exal.

Let us consider S, m_2. Observe that there exists an m_2-w.f.f. string of S which is exal.

Let us consider S. Note that every string of S which is exal is also non 0-w.f.f..

Let us consider S, m_2. One can check that there exists an exal m_2-w.f.f. string of S which is non 0-w.f.f..

Let us consider S. One can verify that there exists an exal w.f.f. string of S which is non 0-w.f.f..

Let us consider S and let p_1 be a non 0-w.f.f. w.f.f. string of S. Note that Depth p_1 is non zero.

Let us consider S and let w be a non 0-w.f.f. w.f.f. string of S. Observe that S-firstChar(w) is non relational.

Let us consider S, m. Observe that S-formulasOfMaxDepth m is S-prefix. Then AllFormulasOf S is a subset of $(\text{AllSymbolsOf } S)^* \setminus \{\emptyset\}$. Observe that every element of AllFormulasOf S is w.f.f.. Note that AllFormulasOf S is S-prefix.

We now state three propositions:

(11)   dom NorIterator$((S,U)\text{-TruthEval } m) =$
    $(U\text{-InterpretersOf } S) \times (m\text{-NorFormulasOf } S)$.

(12)   dom ExIterator$((S,U)\text{-TruthEval } m) =$
    $(U\text{-InterpretersOf } S) \times (m\text{-ExFormulasOf } S)$.

(13) $U\text{-deltaInterpreter}^{-1}(\{1\}) = \{\langle u,u \rangle : u \text{ ranges over elements of } U\}$.

Let us consider S. Then TheEqSymbOf S is an element of S.

Let us consider S. One can verify that ar TheEqSymbOf $S + 2$ is zero and $|\text{ar TheEqSymbOf } S| - 2$ is zero.

We now state two propositions:

(14)   Let p_1 be a 0-w.f.f. string of S and I be an (S,U)-interpreter-like function. Then
    (i)   if S-firstChar$(p_1) \neq$ TheEqSymbOf S, then I-AtomicEval $p_1 = I(S\text{-firstChar}(p_1))(I\text{-TermEval} \cdot \text{SubTerms } p_1)$, and

(ii)   if S-firstChar(p_1) = TheEqSymbOf S, then I-AtomicEval p_1 = U-deltaInterpreter(I-TermEval $\cdot$ SubTerms p_1).

(15) Let I be an (S, U)-interpreter-like function and p_1 be a 0-w.f.f. string of S. If S-firstChar(p_1) = TheEqSymbOf S, then I-AtomicEval $p_1 = 1$ iff I-TermEval((SubTerms p_1)(1)) = I-TermEval((SubTerms p_1)(2)).

Let us consider S, m. One can check that m-ExFormulasOf S is non empty. Note that m-NorFormulasOf S is non empty. Then m-NorFormulasOf S is a subset of (AllSymbolsOf S)$^* \setminus \{\emptyset\}$.

Let us consider S and let w be an exal string of S. One can verify that S-firstChar(w) is literal.

Let us consider S, m. Observe that every element of m-NorFormulasOf S is non exal. Then m-ExFormulasOf S is a subset of (AllSymbolsOf S)$^* \setminus \{\emptyset\}$.

Let us consider S, m. One can check that every element of m-ExFormulasOf S is exal.

Let us consider S. One can check that there exists an element of S which is non literal.

Let us consider S, w and let s be a non literal element of S. Note that $\langle s \rangle ^\frown w$ is non exal.

Let us consider S, w_1, w_2 and let s be a non literal element of S. Observe that $\langle s \rangle ^\frown w_1 ^\frown w_2$ is non exal.

Let us consider S. Note that TheNorSymbOf S is non literal.

Next we state the proposition

(16) $p_1 \in$ AllFormulasOf S.

Let us consider S, m, w. We introduce w is m-non-w.f.f. as an antonym of w is m-w.f.f..

Let us consider m, S. One can verify that every string of S which is non m-w.f.f. is also m-non-w.f.f..

Let us consider S, p_3, p_4. Observe that $\langle$TheNorSymbOf $S\rangle ^\frown p_3 ^\frown p_4$ is $\max($Depth p_3, Depth p_4)-non-w.f.f..

Let us consider S, p_1, l. Note that $\langle l \rangle ^\frown p_1$ is Depth p_1-non-w.f.f..

Let us consider S, p_1, l. One can check that $\langle l \rangle ^\frown p_1$ is $1 +$ Depth p_1-w.f.f..

Let us consider S, U. Observe that every element of U-InterpretersOf S is OwnSymbolsOf S-defined.

Let us consider S, U. Note that there exists an element of U-InterpretersOf S which is OwnSymbolsOf S-defined.

Let us consider S, U. Note that every OwnSymbolsOf S-defined element of U-InterpretersOf S is total.

Let us consider S, U, let I be an element of U-InterpretersOf S, let x be a literal element of S, and let u be an element of U. Then (x, u) ReassignIn I is an element of U-InterpretersOf S.

In the sequel I denotes an element of U-InterpretersOf S.

Let us consider S, w. The functor xnot w yields a string of S and is defined as follows:

(Def. 33)   xnot $w = \langle \text{TheNorSymbOf } S \rangle \,\widehat{}\, w \,\widehat{}\, w$.

Let us consider S, m and let p_1 be an m-w.f.f. string of S. Observe that xnot p_1 is $m + 1$-w.f.f..

Let us consider S, p_1. Note that xnot p_1 is w.f.f..

Let us consider S. One can verify that TheEqSymbOf S is non own.

Let us consider S, X. We say that X is S-mincover if and only if:

(Def. 34)   For every p_1 holds $p_1 \in X$ iff xnot $p_1 \notin X$.

One can prove the following propositions:

(17) $\text{Depth}(\langle \text{TheNorSymbOf } S \rangle \,\widehat{}\, p_3 \,\widehat{}\, p_4) = 1 + \max(\text{Depth } p_3, \text{Depth } p_4)$ and $\text{Depth}(\langle l \rangle \,\widehat{}\, p_3) = \text{Depth } p_3 + 1$.

(18)   If $\text{Depth } p_1 = m + 1$, then p_1 is exal iff $p_1 \in m$-ExFormulasOf S and p_1 is non exal iff $p_1 \in m$-NorFormulasOf S.

(19) I-TruthEval$\langle l \rangle \,\widehat{}\, p_1 = true$ iff there exists u such that $((l, u) \text{ ReassignIn } I)$-TruthEval $p_1 = 1$ and I-TruthEval$\langle \text{TheNorSymbOf } S \rangle \,\widehat{}\, p_3 \,\widehat{}\, p_4 = true$ iff I-TruthEval $p_3 = false$ and I-TruthEval $p_4 = false$.

In the sequel I denotes an (S, U)-interpreter-like function.

One can prove the following two propositions:

(20) (I, u)-TermEval$(m + 1) \restriction S$-termsOfMaxDepth$(m) = I$-TermEval $\restriction S$-termsOfMaxDepth(m).

(21) I-TermEval$(t) = I(S\text{-firstChar}(t))(I\text{-TermEval} \cdot \text{SubTerms } t)$.

Let us consider S, p_1. The functor SubWffsOf p_1 is defined as follows:

(Def. 35)(i)   There exist p_3, p such that p is AllSymbolsOf S-valued and SubWffsOf $p_1 = \langle p_3, p \rangle$ and $p_1 = \langle S\text{-firstChar}(p_1) \rangle \,\widehat{}\, p_3 \,\widehat{}\, p$ if p_1 is non 0-w.f.f.,

(ii)   SubWffsOf $p_1 = \langle p_1, \emptyset \rangle$, otherwise.

Let us consider S, p_1. The functor head p_1 yields a w.f.f. string of S and is defined as follows:

(Def. 36)   head $p_1 = (\text{SubWffsOf } p_1)_{\mathbf{1}}$.

The functor tail p_1 yields an element of $(\text{AllSymbolsOf } S)^*$ and is defined by:

(Def. 37)   tail $p_1 = (\text{SubWffsOf } p_1)_{\mathbf{2}}$.

Let us consider S, m. One can verify that $(S\text{-formulasOfMaxDepth } m) \setminus \text{AllFormulasOf } S$ is empty.

Let us consider S. Observe that AtomicFormulasOf $S \setminus \text{AllFormulasOf } S$ is empty.

We now state two propositions:

(22) $\text{Depth}(\langle l \rangle \,\widehat{}\, p_3) > \text{Depth } p_3$ and $\text{Depth}(\langle \text{TheNorSymbOf } S \rangle \,\widehat{}\, p_3 \,\widehat{}\, p_4) > \text{Depth } p_3$ and $\text{Depth}(\langle \text{TheNorSymbOf } S \rangle \,\widehat{}\, p_3 \,\widehat{}\, p_4) > \text{Depth } p_4$.

(23) If p_1 is not 0-w.f.f., then $p_1 = \langle x \rangle \frown p_4 \frown p_2$ iff $x = S\text{-firstChar}(p_1)$ and $p_4 = \text{head } p_1$ and $p_2 = \text{tail } p_1$.

Let us consider S, m_2. Observe that there exists a non 0-w.f.f. m_2-w.f.f. string of S which is non exal.

Let us consider S and let p_1 be an exal w.f.f. string of S. One can verify that $\text{tail } p_1$ is empty.

Let us consider S and let p_1 be a non exal non 0-w.f.f. w.f.f. string of S. Then $\text{tail } p_1$ is a w.f.f. string of S.

Let us consider S and let p_1 be a non exal non 0-w.f.f. w.f.f. string of S. One can check that $\text{tail } p_1$ is w.f.f..

Let us consider S and let p_1 be a non 0-w.f.f. non exal w.f.f. string of S. One can verify that $S\text{-firstChar}(p_1) \dot{-} \text{TheNorSymbOf } S$ is empty.

Let us consider m, S and let p_1 be an $m + 1$-w.f.f. string of S. Note that $\text{head } p_1$ is m-w.f.f..

Let us consider m, S and let p_1 be an $m + 1$-w.f.f. non exal non 0-w.f.f. string of S. Observe that $\text{tail } p_1$ is m-w.f.f..

One can prove the following proposition

(24) For every element I of $U\text{-InterpretersOf } S$ holds $(I, m)\text{-TruthEval} \in Boolean^{S\text{-formulasOfMaxDepth } m}$.

Let us consider S. One can check that there exists an of-atomic-formula element of S which is non literal.

One can prove the following proposition

(25) If $l_2 \notin \text{rng } p$, then $((l_2, u) \text{ ReassignIn } I)\text{-TermEval}(p) = I\text{-TermEval}(p)$.

Let us consider X, S, s. We say that s is X-occurring if and only if:

(Def. 38) $s \in \text{SymbolsOf}(((\text{AllSymbolsOf } S)^* \setminus \{\emptyset\}) \cap X)$.

Let us consider S, s and let us consider X. We say that X is s-containing if and only if:

(Def. 39) $s \in \text{SymbolsOf}((\text{AllSymbolsOf } S)^* \setminus \{\emptyset\} \cap X)$.

Let us consider X, S, s. We introduce s is X-absent as an antonym of s is X-occurring.

Let us consider S, s, X. We introduce X is s-free as an antonym of X is s-containing.

Let X be a finite set and let us consider S. Observe that there exists a literal element of S which is X-absent.

Let us consider S, t. Note that $\text{rng } t \cap \text{LettersOf } S$ is non empty.

Let us consider S, p_1. One can verify that $\text{rng } p_1 \cap \text{LettersOf } S$ is non empty.

Let us consider B, S and let A be a subset of B. Note that every element of S which is A-occurring is also B-occurring.

Let us consider A, B, S. Observe that every element of S which is A null B-absent is also $A \cap B$-absent.

Let F be a finite set and let us consider A, S. Note that every F-absent element of S which is A-absent is also $A \cup F$-absent.

Let us consider S, U and let I be an (S, U)-interpreter-like function. One can check that $\text{OwnSymbolsOf } S \setminus \text{dom } I$ is empty.

One can prove the following proposition

(26)   There exists u such that $u = I(l)(\emptyset)$ and $(l, u) \text{ ReassignIn } I = I$.

Let us consider S, X. We say that X is S-covering if and only if:

(Def. 40)   For every p_1 holds $p_1 \in X$ or $\text{xnot } p_1 \in X$.

Let us consider S. One can check that every set which is S-mincover is also S-covering.

Let us consider U, let p_1 be a non 0-w.f.f. non exal w.f.f. string of S, and let I be an element of U-InterpretersOf S.

One can verify that $(I\text{-TruthEval } p_1) \dot{-} ((I\text{-TruthEval head } p_1) \; '\text{nor}' \; (I\text{-TruthEval tail } p_1))$ is empty.

The functor $\text{ExFormulasOf } S$ yielding a subset of $(\text{AllSymbolsOf } S)^* \setminus \{\emptyset\}$ is defined by:

(Def. 41) $\text{ExFormulasOf } S = \{p_1; p_1 \text{ ranges over strings of } S: p_1 \text{ is w.f.f.} \land p_1 \text{ is exal}\}$.

Let us consider S. Note that $\text{ExFormulasOf } S$ is non empty.

Let us consider S. One can check that every element of $\text{ExFormulasOf } S$ is exal and w.f.f..

Let us consider S. Note that every element of $\text{ExFormulasOf } S$ is w.f.f..

Let us consider S. Observe that every element of $\text{ExFormulasOf } S$ is exal.

Let us consider S. Observe that $\text{ExFormulasOf } S \setminus \text{AllFormulasOf } S$ is empty.

Let us consider U, S_1 and let S_2 be an S_1-extending language. Note that every function which is (S_2, U)-interpreter-like is also (S_1, U)-interpreter-like.

Let us consider U, S_1, let S_2 be an S_1-extending language, and let I be an (S_2, U)-interpreter-like function. Observe that $I \upharpoonright \text{OwnSymbolsOf } S_1$ is (S_1, U)-interpreter-like.

Let us consider U, S_1, let S_2 be an S_1-extending language, let I_1 be an element of U-InterpretersOf S_1, and let I_2 be an (S_2, U)-interpreter-like function. Note that $I_2 + \cdot I_1$ is (S_2, U)-interpreter-like.

Let us consider U, S, let I be an element of U-InterpretersOf S, and let us consider X. We say that X is I-satisfied if and only if:

(Def. 42)   For every p_1 such that $p_1 \in X$ holds $I\text{-TruthEval } p_1 = 1$.

Let us consider S, U, X and let I be an element of U-InterpretersOf S. We say that I satisfies X if and only if:

(Def. 43) X is I-satisfied.

Let us consider U, S, let e be an empty set, and let I be an element of U-InterpretersOf S. Observe that $e \text{ null } I$ is I-satisfied.

Let us consider X, U, S and let I be an element of U-InterpretersOf S. Observe that there exists a subset of X which is I-satisfied.

Let us consider U, S and let I be an element of U-InterpretersOf S. One can check that there exists a set which is I-satisfied.

Let us consider U, S, let I be an element of U-InterpretersOf S, and let X be an I-satisfied set. One can check that every subset of X is I-satisfied.

Let us consider U, S, let I be an element of U-InterpretersOf S, and let X, Y be I-satisfied sets. One can verify that $X \cup Y$ is I-satisfied.

Let us consider U, S, let I be an element of U-InterpretersOf S, and let X be an I-satisfied set. Observe that I null X satisfies X .

Let us consider S, X. We say that X is S-correct if and only if the condition (Def. 44) is satisfied.

(Def. 44)   Let U be a non empty set, I be an element of U-InterpretersOf S, x be an I-satisfied set, and given p_1. If $\langle x, p_1 \rangle \in X$, then I-TruthEval $p_1 = 1$.

Let us consider S. Note that $\emptyset$ null S is S-correct.

Let us consider S, X. Observe that there exists a subset of X which is S-correct.

Next we state two propositions:

(27)   For every element I of U-InterpretersOf S holds I-TruthEval $p_1 = 1$ iff $\{p_1\}$ is I-satisfied.

(28) s is $\{w\}$-occurring iff $s \in \operatorname{rng} w$.

Let us consider U, S, let us consider p_3, p_4, and let I be an element of U-InterpretersOf S. Observe that $(I$-TruthEval$\langle$TheNorSymbOf $S\rangle \frown p_3 \frown p_4) \doteq ((I$-TruthEval $p_3)$ $'$nor$'$ $(I$-TruthEval $p_4))$ is empty.

Let us consider S, p_1, U and let I be an element of U-InterpretersOf S. Note that $(I$-TruthEval xnot $p_1) \doteq \neg(I$-TruthEval $p_1)$ is empty.

Let us consider X, S, p_1. We say that p_1 is X-implied if and only if:

(Def. 45)   For every non empty set U and for every element I of U-InterpretersOf S such that X is I-satisfied holds I-TruthEval $p_1 = 1$.

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