

# Sequent Calculus, Derivability, Provability. Gödel’s Completeness Theorem<sup>1</sup>

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**Summary.** Fifth of a series of articles laying down the bases for classical first order model theory. This paper presents multiple themes: first it introduces sequents, rules and sets of rules for a first order language  $L$  as  $L$ -dependent types. Then defines derivability and provability according to a set of rules, and gives several technical lemmas binding all those concepts. Following that, it introduces a fixed set  $D$  of derivation rules, and proceeds to convert them to Mizar functorial cluster registrations to give the user a slick interface to apply them.

The remaining goals summon all the definitions and results introduced in this series of articles. First:  $D$  is shown to be correct and having the requisites to deliver a sensible definition of Henkin model (see [18]). Second: as a particular application of all the machinery built thus far, the satisfiability and Gödel completeness theorems are shown when restricting to countable languages. The techniques used to attain this are inspired from [18], then heavily modified with the twofold goal of embedding them into the more flexible framework of a variable ruleset here introduced, and of proving completeness of a set of rules more sparing than the one there used; in particular the simpler ruleset allowed to avoid the definition and tractation of free occurrence of a literal, a fact which, along with shortening proofs, is remarkable in its own right. A preparatory account of some of the ideas used in the proofs given here can be found in [15].

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The notation and terminology used here have been introduced in the following papers: [1], [3], [23], [22], [4], [6], [17], [11], [12], [13], [14], [7], [8], [5], [19], [16], [24], [2], [21], [9], [26], [28], [27], [20], [25], and [10].

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1. FORMALIZATION OF THE NOTION OF DERIVABILITY AND PROVABILITY.  
HENKIN'S THEOREM FOR ARBITRARY LANGUAGES

For simplicity, we adopt the following convention:  $k, m, n$  denote natural numbers,  $m_1$  denotes an element of  $\mathbb{N}$ ,  $U$  denotes a non empty set,  $A, B, X, Y, Z, x, y, z$  denote sets,  $S$  denotes a language,  $s$  denotes an element of  $S$ ,  $f, g$  denote functions,  $p_1, p_2, p_3, p_4$  denote w.f.f. strings of  $S$ ,  $P_1, P_2, P_3$  denote subsets of AllFormulasOf  $S$ ,  $t, t_1, t_2$  denote termal strings of  $S$ ,  $a$  denotes an of-atomic-formula element of  $S$ ,  $l, l_1, l_2$  denote literal elements of  $S$ ,  $p$  denotes a finite sequence, and  $m_2$  denotes a non zero natural number.

Let  $S$  be a language. The functor  $S$ -sequents is defined as follows:

(Def. 1)  $S$ -sequents =  $\{(p_5, c_1); p_5 \text{ ranges over subsets of AllFormulasOf } S, c_1 \text{ ranges over w.f.f. strings of } S: p_5 \text{ is finite}\}$ .

Let  $S$  be a language. Note that  $S$ -sequents is non empty.

Let us consider  $S$ . Observe that  $S$ -sequents is relation-like.

Let  $S$  be a language and let  $x$  be a set. We say that  $x$  is  $S$ -sequent-like if and only if:

(Def. 2)  $x \in S$ -sequents .

Let us consider  $S, X$ . We say that  $X$  is  $S$ -sequents-like if and only if:

(Def. 3)  $X \subseteq S$ -sequents .

Let us consider  $S$ . One can check that every subset of  $S$ -sequents is  $S$ -sequents-like and every element of  $S$ -sequents is  $S$ -sequent-like.

Let  $S$  be a language. One can verify that there exists an element of  $S$ -sequents which is  $S$ -sequent-like and there exists a subset of  $S$ -sequents which is  $S$ -sequents-like.

Let us consider  $S$ . One can check that there exists a set which is  $S$ -sequent-like and there exists a set which is  $S$ -sequents-like.

Let  $S$  be a language. A rule of  $S$  is an element of  $(2^{S\text{-sequents}})^{2^{S\text{-sequents}}}$ .

Let  $S$  be a language. A rule set of  $S$  is a subset of  $(2^{S\text{-sequents}})^{2^{S\text{-sequents}}}$ .

For simplicity, we adopt the following rules:  $D, D_1, D_2, D_3$  denote rule sets of  $S$ ,  $R$  denotes a rule of  $S$ ,  $S_1, S_2, S_3$  denote subsets of  $S$ -sequents,  $s_1, s_2, s_3$  denote elements of  $S$ -sequents,  $S_4, S_5$  denote  $S$ -sequents-like sets, and  $S_6, S_7$  denote  $S$ -sequent-like sets.

Let us consider  $A, B$  and let  $X$  be a subset of  $B^A$ . One can check that  $\cup X$  is relation-like.

Let  $S$  be a language and let  $D$  be a rule set of  $S$ . One can check that  $\cup D$  is relation-like.

Let us consider  $S, D$ . The functor OneStep  $D$  yielding a rule of  $S$  is defined as follows:

(Def. 4) For every element  $S_8$  of  $2^{S\text{-sequents}}$  holds  $(\text{OneStep } D)(S_8) = \cup((\cup D)^\circ \{S_8\})$ .

Let us consider  $S, D, m$ . The functor  $(m, D)$ -derivables yields a rule of  $S$  and is defined by:

(Def. 5)  $(m, D)$ -derivables =  $(\text{OneStep } D)^m$ .

Let  $S$  be a language, let  $D$  be a rule set of  $S$ , and let  $S_9, S_{10}$  be sets. We say that  $S_{10}$  is  $(S_9, D)$ -derivable if and only if:

(Def. 6)  $S_{10} \subseteq \cup(((\text{OneStep } D)^*)^\circ \{S_9\})$ .

Let us consider  $m, S, D$  and let  $S_1, s_1$  be sets. We say that  $s_1$  is  $(m, S_1, D)$ -derivable if and only if:

(Def. 7)  $s_1 \in (m, D)$ -derivables( $S_1$ ).

Let us consider  $S, D$ . The functor  $D$ -iterators yielding a family of subsets of  $2^{S\text{-sequents}} \times 2^{S\text{-sequents}}$  is defined as follows:

(Def. 8)  $D$ -iterators =  $\{(\text{OneStep } D)^{m_1}\}$ .

Let us consider  $S, R$ . We say that  $R$  is isotone if and only if:

(Def. 9) If  $S_2 \subseteq S_3$ , then  $R(S_2) \subseteq R(S_3)$ .

Let us consider  $S$ . Observe that there exists a rule of  $S$  which is isotone.

Let us consider  $S, D$ . We say that  $D$  is isotone if and only if:

(Def. 10) For all  $S_2, S_3, f$  such that  $S_2 \subseteq S_3$  and  $f \in D$  there exists  $g$  such that  $g \in D$  and  $f(S_2) \subseteq g(S_3)$ .

Let us consider  $S$  and let  $M$  be an isotone rule of  $S$ . One can verify that  $\{M\}$  is isotone.

Let us consider  $S$ . One can verify that there exists a rule set of  $S$  which is isotone.

In the sequel  $K, K_1$  are isotone rule sets of  $S$ .

Let  $S$  be a language, let  $D$  be a rule set of  $S$ , and let  $S_1$  be a set. We say that  $S_1$  is  $D$ -derivable if and only if:

(Def. 11)  $S_1$  is  $(\emptyset, D)$ -derivable.

Let us consider  $S, D$ . One can verify that every set which is  $D$ -derivable is also  $(\emptyset, D)$ -derivable and every set which is  $(\emptyset, D)$ -derivable is also  $D$ -derivable.

Let us consider  $S, D$  and let  $S_1$  be an empty set. One can verify that every set which is  $(S_1, D)$ -derivable is also  $D$ -derivable.

Let us consider  $S, D, X$  and let  $p_2$  be a set. We say that  $p_2$  is  $(X, D)$ -provable if and only if:

(Def. 12)  $\{\{X, p_2\}\}$  is  $D$ -derivable or there exists a set  $s_1$  such that  $(s_1)_1 \subseteq X$  and  $(s_1)_2 = p_2$  and  $\{s_1\}$  is  $D$ -derivable.

Let us consider  $S, D, X, x$ . Let us observe that  $x$  is  $(X, D)$ -provable if and only if:

(Def. 13) There exists a set  $s_1$  such that  $(s_1)_1 \subseteq X$  and  $(s_1)_2 = x$  and  $\{s_1\}$  is  $D$ -derivable.

Let us consider  $S, D, R$ . We say that  $R$  is  $D$ -macro if and only if:

(Def. 14) For every subset  $S_8$  of  $S$ -sequents holds  $R(S_8)$  is  $(S_8, D)$ -derivable.

Let us consider  $S, D$  and let  $P_1$  be a set. The functor  $(P_1, D)$ -termEq is defined as follows:

(Def. 15)  $(P_1, D)$ -termEq =  $\{\langle t_1, t_2 \rangle; t_1 \text{ ranges over termal strings of } S, t_2 \text{ ranges over termal strings of } S: \langle \text{TheEqSymbOf } S \rangle \wedge t_1 \wedge t_2 \text{ is } (P_1, D)\text{-provable}\}$ .

Let us consider  $S, D$  and let  $P_1$  be a set. We say that  $P_1$  is  $D$ -expanded if and only if:

(Def. 16) If  $x$  is  $(P_1, D)$ -provable, then  $\{x\} \subseteq P_1$ .

Let us consider  $S, x$ . We say that  $x$  is  $S$ -null if and only if:

(Def. 17) Not contradiction.

Let us consider  $S, D$  and let  $P_1$  be a set. Then  $(P_1, D)$ -termEq is a binary relation on AllTermsOf  $S$ .

Let us consider  $S, p_2$  and let  $P_2, P_3$  be finite subsets of AllFormulasOf  $S$ . One can check that  $\langle P_2 \cup P_3, p_2 \rangle$  is  $S$ -sequent-like.

Let us consider  $S$ , let  $x$  be an empty set, and let  $p_2$  be a w.f.f. string of  $S$ . Then  $\langle x, p_2 \rangle$  is an element of  $S$ -sequents.

Let us consider  $S$ . Note that  $\emptyset \cap S$  is  $S$ -sequents-like.

Let us consider  $S$ . One can verify that there exists a set which is  $S$ -null.

Let us consider  $S$ . One can check that every set which is  $S$ -sequent-like is also  $S$ -null.

Let us consider  $S$ . One can check that every element of  $S$ -sequents is  $S$ -null.

Let us consider  $m, S, D, X$ . One can verify that  $(m, D)$ -derivables( $X$ ) is  $S$ -sequents-like.

Let us consider  $S, Y$  and let  $X$  be an  $S$ -sequents-like set. One can verify that  $X \cap Y$  is  $S$ -sequents-like.

Let us consider  $S, D, m, X$ . Note that every set which is  $(m, X, D)$ -derivable is also  $S$ -sequent-like.

Let us consider  $S, D$  and let  $P_2, P_3$  be sets. Observe that every set which is  $(P_2 \setminus P_3, D)$ -provable is also  $(P_2, D)$ -provable.

Let us consider  $S, D$  and let  $P_2, P_3$  be sets. Observe that every set which is  $(P_2 \setminus P_3, D)$ -provable is also  $(P_2 \cup P_3, D)$ -provable.

Let us consider  $S, D$  and let  $P_2, P_3$  be sets. Observe that every set which is  $(P_2 \cap P_3, D)$ -provable is also  $(P_2, D)$ -provable.

Let us consider  $S, D$ , let  $X$  be a set, and let  $x$  be a subset of  $X$ . Note that every set which is  $(x, D)$ -provable is also  $(X, D)$ -provable.

Let us consider  $S$ , let  $p_5$  be a finite subset of AllFormulasOf  $S$ , and let  $p_2$  be a w.f.f. string of  $S$ . One can check that  $\langle p_5, p_2 \rangle$  is  $S$ -sequent-like.

Let us consider  $S$  and let  $p_3, p_4$  be w.f.f. strings of  $S$ . Observe that  $\{\langle p_3 \rangle, p_4 \rangle$  is  $S$ -sequent-like. Let  $p_6$  be a w.f.f. string of  $S$ . Note that  $\{\langle p_3, p_4 \rangle, p_6 \rangle$  is  $S$ -sequent-like.

Let us consider  $S, p_3, p_4$  and let  $P_1$  be a finite subset of  $\text{AllFormulasOf } S$ . One can verify that  $\langle P_1 \cup \{p_3\}, p_4 \rangle$  is  $S$ -sequent-like.

Let us consider  $S, D$ . Note that there exists a subset of  $\text{AllFormulasOf } S$  which is  $D$ -expanded.

Let us consider  $S, D$ . Observe that there exists a set which is  $D$ -expanded.

Let  $S_1$  be a set, let  $S$  be a language, and let  $s_1$  be an  $S$ -null set. We say that  $s_1$  rule 0  $S_1$  if and only if:

(Def. 18)  $(s_1)_2 \in (s_1)_1$ .

We say that  $s_1$  rule 1  $S_1$  if and only if:

(Def. 19) There exists a set  $y$  such that  $y \in S_1$  and  $y_1 \subseteq (s_1)_1$  and  $(s_1)_2 = y_2$ .

We say that  $s_1$  rule 2  $S_1$  if and only if:

(Def. 20)  $(s_1)_1$  is empty and there exists a termal string  $t$  of  $S$  such that  $(s_1)_2 = \langle \text{TheEqSymbOf } S \rangle \wedge t \wedge t$ .

We say that  $s_1$  rule 3a  $S_1$  if and only if the condition (Def. 21) is satisfied.

(Def. 21) There exist termal strings  $t, t_1, t_2$  of  $S$  and there exists a set  $x$  such that  $x \in S_1$  and  $(s_1)_1 = x_1 \cup \{\langle \text{TheEqSymbOf } S \rangle \wedge t_1 \wedge t_2\}$  and  $x_2 = \langle \text{TheEqSymbOf } S \rangle \wedge t \wedge t_1$  and  $(s_1)_2 = \langle \text{TheEqSymbOf } S \rangle \wedge t \wedge t_2$ .

We say that  $s_1$  rule 3b  $S_1$  if and only if:

(Def. 22) There exist termal strings  $t_1, t_2$  of  $S$  such that  $(s_1)_1 = \{\langle \text{TheEqSymbOf } S \rangle \wedge t_1 \wedge t_2\}$  and  $(s_1)_2 = \langle \text{TheEqSymbOf } S \rangle \wedge t_2 \wedge t_1$ .

We say that  $s_1$  rule 3d  $S_1$  if and only if the condition (Def. 23) is satisfied.

(Def. 23) There exists a low-compounding element  $s$  of  $S$  and there exist  $|ar s|$ -element elements  $T, U$  of  $(\text{AllTermsOf } S)^*$  such that

(i)  $s$  is operational,

(ii)  $(s_1)_1 = \{\langle \text{TheEqSymbOf } S \rangle \wedge T_1(j) \wedge U_1(j); j \text{ ranges over elements of } \text{Seg}|ar s|, T_1 \text{ ranges over functions from } \text{Seg}|ar s| \text{ into } (\text{AllSymbolsOf } S)^* \setminus \{\emptyset\}, U_1 \text{ ranges over functions from } \text{Seg}|ar s| \text{ into } (\text{AllSymbolsOf } S)^* \setminus \{\emptyset\} : T_1 = T \wedge U_1 = U\}$ , and

(iii)  $(s_1)_2 = \langle \text{TheEqSymbOf } S \rangle \wedge (s\text{-compound } T) \wedge (s\text{-compound } U)$ .

We say that  $s_1$  rule 3e  $S_1$  if and only if the condition (Def. 24) is satisfied.

(Def. 24) There exists a relational element  $s$  of  $S$  and there exist  $|ar s|$ -element elements  $T, U$  of  $(\text{AllTermsOf } S)^*$  such that

(i)  $(s_1)_1 = \{s\text{-compound } T\} \cup \{\langle \text{TheEqSymbOf } S \rangle \wedge T_1(j) \wedge U_1(j); j \text{ ranges over elements of } \text{Seg}|ar s|, T_1 \text{ ranges over functions from } \text{Seg}|ar s| \text{ into } (\text{AllSymbolsOf } S)^* \setminus \{\emptyset\}, U_1 \text{ ranges over functions from } \text{Seg}|ar s| \text{ into } (\text{AllSymbolsOf } S)^* \setminus \{\emptyset\} : T_1 = T \wedge U_1 = U\}$ , and

(ii)  $(s_1)_2 = s\text{-compound } U$ .

We say that  $s_1$  rule 4  $S_1$  if and only if the condition (Def. 25) is satisfied.

(Def. 25) There exists a literal element  $l$  of  $S$  and there exists a w.f.f. string  $p_2$  of  $S$  and there exists a termal string  $t$  of  $S$  such that  $(s_1)_1 = \{(l, t) \text{ SubstIn } p_2\}$  and  $(s_1)_2 = \langle l \rangle \wedge p_2$ .

We say that  $s_1$  rule 5  $S_1$  if and only if:

(Def. 26) There exist literal elements  $v_1, v_2$  of  $S$  and there exist  $x, p$  such that  $(s_1)_1 = x \cup \{\langle v_1 \rangle \wedge p\}$  and  $v_2$  is  $x \cup \{p\} \cup \{s_{12}\}$ -absent and  $\langle x \cup \{(v_1 \text{ SubstWith } v_2)(p)\}, (s_1)_2 \rangle \in S_1$ .

We say that  $s_1$  rule 6  $S_1$  if and only if the condition (Def. 27) is satisfied.

(Def. 27) There exist sets  $y_1, y_2$  and there exist w.f.f. strings  $p_3, p_4$  of  $S$  such that  $y_1, y_2 \in S_1$  and  $(y_1)_1 = (y_2)_1 = (s_1)_1$  and  $(y_1)_2 = \langle \text{TheNorSymbOf } S \rangle \wedge p_3 \wedge p_3$  and  $(y_2)_2 = \langle \text{TheNorSymbOf } S \rangle \wedge p_4 \wedge p_4$  and  $(s_1)_2 = \langle \text{TheNorSymbOf } S \rangle \wedge p_3 \wedge p_4$ .

We say that  $s_1$  rule 7  $S_1$  if and only if:

(Def. 28) There exists a set  $y$  and there exist w.f.f. strings  $p_3, p_4$  of  $S$  such that  $y \in S_1$  and  $y_1 = (s_1)_1$  and  $y_2 = \langle \text{TheNorSymbOf } S \rangle \wedge p_3 \wedge p_4$  and  $(s_1)_2 = \langle \text{TheNorSymbOf } S \rangle \wedge p_4 \wedge p_3$ .

We say that  $s_1$  rule 8  $S_1$  if and only if the condition (Def. 29) is satisfied.

(Def. 29) There exist sets  $y_1, y_2$  and there exist w.f.f. strings  $p_2, p_3, p_4$  of  $S$  such that  $y_1, y_2 \in S_1$  and  $(y_1)_1 = (y_2)_1$  and  $(y_1)_2 = p_3$  and  $(y_2)_2 = \langle \text{TheNorSymbOf } S \rangle \wedge p_3 \wedge p_4$  and  $\{p_2\} \cup (s_1)_1 = (y_1)_1$  and  $(s_1)_2 = \langle \text{TheNorSymbOf } S \rangle \wedge p_2 \wedge p_2$ .

We say that  $s_1$  rule 9  $S_1$  if and only if:

(Def. 30) There exists a set  $y$  and there exists a w.f.f. string  $p_2$  of  $S$  such that  $y \in S_1$  and  $(s_1)_2 = p_2$  and  $y_1 = (s_1)_1$  and  $y_2 = \text{xnot xnot } p_2$ .

Let  $S$  be a language. The functor  $P0 S$  yielding a relation between  $2^{S\text{-sequents}}$  and  $S$ -sequents is defined by:

(Def. 31) For every element  $S_1$  of  $2^{S\text{-sequents}}$  and for every element  $s_1$  of  $S$ -sequents holds  $\langle S_1, s_1 \rangle \in P0 S$  iff  $s_1$  rule 0  $S_1$ .

The functor  $P1 S$  yields a relation between  $2^{S\text{-sequents}}$  and  $S$ -sequents and is defined as follows:

(Def. 32) For every element  $S_1$  of  $2^{S\text{-sequents}}$  and for every element  $s_1$  of  $S$ -sequents holds  $\langle S_1, s_1 \rangle \in P1 S$  iff  $s_1$  rule 1  $S_1$ .

The functor  $P2 S$  yields a relation between  $2^{S\text{-sequents}}$  and  $S$ -sequents and is defined as follows:

(Def. 33) For every element  $S_1$  of  $2^{S\text{-sequents}}$  and for every element  $s_1$  of  $S$ -sequents holds  $\langle S_1, s_1 \rangle \in P2 S$  iff  $s_1$  rule 2  $S_1$ .

The functor  $P3a S$  yielding a relation between  $2^{S\text{-sequents}}$  and  $S$ -sequents is defined as follows:

(Def. 34) For every element  $S_1$  of  $2^{S\text{-sequents}}$  and for every element  $s_1$  of  $S\text{-sequents}$  holds  $\langle S_1, s_1 \rangle \in P3a S$  iff  $s_1$  rule 3a  $S_1$ .

The functor  $P3b S$  yields a relation between  $2^{S\text{-sequents}}$  and  $S\text{-sequents}$  and is defined as follows:

(Def. 35) For every element  $S_1$  of  $2^{S\text{-sequents}}$  and for every element  $s_1$  of  $S\text{-sequents}$  holds  $\langle S_1, s_1 \rangle \in P3b S$  iff  $s_1$  rule 3b  $S_1$ .

The functor  $P3d S$  yields a relation between  $2^{S\text{-sequents}}$  and  $S\text{-sequents}$  and is defined as follows:

(Def. 36) For every element  $S_1$  of  $2^{S\text{-sequents}}$  and for every element  $s_1$  of  $S\text{-sequents}$  holds  $\langle S_1, s_1 \rangle \in P3d S$  iff  $s_1$  rule 3d  $S_1$ .

The functor  $P3e S$  yielding a relation between  $2^{S\text{-sequents}}$  and  $S\text{-sequents}$  is defined by:

(Def. 37) For every element  $S_1$  of  $2^{S\text{-sequents}}$  and for every element  $s_1$  of  $S\text{-sequents}$  holds  $\langle S_1, s_1 \rangle \in P3e S$  iff  $s_1$  rule 3e  $S_1$ .

The functor  $P4 S$  yielding a relation between  $2^{S\text{-sequents}}$  and  $S\text{-sequents}$  is defined by:

(Def. 38) For every element  $S_1$  of  $2^{S\text{-sequents}}$  and for every element  $s_1$  of  $S\text{-sequents}$  holds  $\langle S_1, s_1 \rangle \in P4 S$  iff  $s_1$  rule 4  $S_1$ .

The functor  $P5 S$  yields a relation between  $2^{S\text{-sequents}}$  and  $S\text{-sequents}$  and is defined by:

(Def. 39) For every element  $S_1$  of  $2^{S\text{-sequents}}$  and for every element  $s_1$  of  $S\text{-sequents}$  holds  $\langle S_1, s_1 \rangle \in P5 S$  iff  $s_1$  rule 5  $S_1$ .

The functor  $P6 S$  yielding a relation between  $2^{S\text{-sequents}}$  and  $S\text{-sequents}$  is defined by:

(Def. 40) For every element  $S_1$  of  $2^{S\text{-sequents}}$  and for every element  $s_1$  of  $S\text{-sequents}$  holds  $\langle S_1, s_1 \rangle \in P6 S$  iff  $s_1$  rule 6  $S_1$ .

The functor  $P7 S$  yielding a relation between  $2^{S\text{-sequents}}$  and  $S\text{-sequents}$  is defined as follows:

(Def. 41) For every element  $S_1$  of  $2^{S\text{-sequents}}$  and for every element  $s_1$  of  $S\text{-sequents}$  holds  $\langle S_1, s_1 \rangle \in P7 S$  iff  $s_1$  rule 7  $S_1$ .

The functor  $P8 S$  yields a relation between  $2^{S\text{-sequents}}$  and  $S\text{-sequents}$  and is defined as follows:

(Def. 42) For every element  $S_1$  of  $2^{S\text{-sequents}}$  and for every element  $s_1$  of  $S\text{-sequents}$  holds  $\langle S_1, s_1 \rangle \in P8 S$  iff  $s_1$  rule 8  $S_1$ .

The functor  $P9 S$  yields a relation between  $2^{S\text{-sequents}}$  and  $S\text{-sequents}$  and is defined as follows:

(Def. 43) For every element  $S_1$  of  $2^{S\text{-sequents}}$  and for every element  $s_1$  of  $S\text{-sequents}$  holds  $\langle S_1, s_1 \rangle \in P9 S$  iff  $s_1$  rule 9  $S_1$ .

Let us consider  $S$  and let  $R$  be a relation between  $2^{S\text{-sequents}}$  and  $S$ -sequents.

The functor  $\text{FuncRule } R$  yields a rule of  $S$  and is defined by:

(Def. 44) For every set  $i_1$  such that  $i_1 \in 2^{S\text{-sequents}}$  holds  $(\text{FuncRule } R)(i_1) = \{x \in S\text{-sequents} : \langle i_1, x \rangle \in R\}$ .

Let us consider  $S$ . The functor  $\text{R0 } S$  yielding a rule of  $S$  is defined as follows:

(Def. 45)  $\text{R0 } S = \text{FuncRule P0 } S$ .

The functor  $\text{R1 } S$  yielding a rule of  $S$  is defined as follows:

(Def. 46)  $\text{R1 } S = \text{FuncRule P1 } S$ .

The functor  $\text{R2 } S$  yielding a rule of  $S$  is defined by:

(Def. 47)  $\text{R2 } S = \text{FuncRule P2 } S$ .

The functor  $\text{R3a } S$  yielding a rule of  $S$  is defined by:

(Def. 48)  $\text{R3a } S = \text{FuncRule P3a } S$ .

The functor  $\text{R3b } S$  yielding a rule of  $S$  is defined as follows:

(Def. 49)  $\text{R3b } S = \text{FuncRule P3b } S$ .

The functor  $\text{R3d } S$  yielding a rule of  $S$  is defined as follows:

(Def. 50)  $\text{R3d } S = \text{FuncRule P3d } S$ .

The functor  $\text{R3e } S$  yielding a rule of  $S$  is defined by:

(Def. 51)  $\text{R3e } S = \text{FuncRule P3e } S$ .

The functor  $\text{R4 } S$  yields a rule of  $S$  and is defined as follows:

(Def. 52)  $\text{R4 } S = \text{FuncRule P4 } S$ .

The functor  $\text{R5 } S$  yielding a rule of  $S$  is defined as follows:

(Def. 53)  $\text{R5 } S = \text{FuncRule P5 } S$ .

The functor  $\text{R6 } S$  yields a rule of  $S$  and is defined by:

(Def. 54)  $\text{R6 } S = \text{FuncRule P6 } S$ .

The functor  $\text{R7 } S$  yields a rule of  $S$  and is defined by:

(Def. 55)  $\text{R7 } S = \text{FuncRule P7 } S$ .

The functor  $\text{R8 } S$  yielding a rule of  $S$  is defined as follows:

(Def. 56)  $\text{R8 } S = \text{FuncRule P8 } S$ .

The functor  $\text{R9 } S$  yields a rule of  $S$  and is defined by:

(Def. 57)  $\text{R9 } S = \text{FuncRule P9 } S$ .

Let us consider  $S$  and let  $t$  be a termal string of  $S$ .

Note that  $\{\langle \emptyset, \langle \text{TheEqSymbOf } S \rangle \wedge t \wedge t \rangle\}$  is  $\{\text{R2 } S\}$ -derivable. Note that  $\text{R2 } S$  is isotone. One can verify that  $\text{R3b } S$  is isotone.

Let  $t, t_1, t_2$  be termal strings of  $S$ , and let  $p_5$  be a finite subset of  $\text{AllFormulasOf } S$ . Observe that  $\langle p_5 \cup \{\langle \text{TheEqSymbOf } S \rangle \wedge t_1 \wedge t_2\}, \langle \text{TheEqSymbOf } S \rangle \wedge t \wedge t_2 \rangle$  is  $(1, \{\langle p_5, \langle \text{TheEqSymbOf } S \rangle \wedge t \wedge t_1 \rangle\}, \{\text{R3a } S\})$ -derivable.



Let us consider  $S$ , let  $t, t_1, t_2$  be termal strings of  $S$ , and let  $p_2$  be a w.f.f. string of  $S$ . Note that  $\langle \{p_2, \langle \text{TheEqSymbOf } S \rangle \wedge t_1 \wedge t_2\}, \langle \text{TheEqSymbOf } S \rangle \wedge t \wedge t_2 \rangle$  is  $(1, \{\{p_2\}, \langle \text{TheEqSymbOf } S \rangle \wedge t \wedge t_1\}, \{\text{R3a } S\})$ -derivable.

Let us consider  $S$ , let  $p_2$  be a w.f.f. string of  $S$ , and let  $P_1$  be a finite subset of  $\text{AllFormulasOf } S$ . One can verify that  $\langle P_1 \cup \{p_2\}, p_2 \rangle$  is  $(1, \emptyset, \{\text{R0 } S\})$ -derivable.

Let us consider  $S$  and let  $p_3, p_4$  be w.f.f. strings of  $S$ . One can check that  $\langle \{p_3, p_4\}, p_3 \rangle$  is  $(1, \emptyset, \{\text{R0 } S\})$ -derivable.

Let us consider  $S, p_2$ . Note that  $\langle \{p_2\}, p_2 \rangle$  is  $(1, \emptyset, \{\text{R0 } S\})$ -derivable.

Let us consider  $S$  and let  $p_2$  be a w.f.f. string of  $S$ . Observe that  $\langle \{p_2\}, p_2 \rangle$  is  $(\emptyset, \{\text{R0 } S\})$ -derivable.

Let us consider  $S$ . One can verify the following observations:

- \*  $\text{R0 } S$  is isotone,
- \*  $\text{R3a } S$  is isotone,
- \*  $\text{R3d } S$  is isotone, and
- \*  $\text{R3e } S$  is isotone.

Let us consider  $K_1, K_2$ . One can verify that  $K_1 \cup K_2$  is isotone.

Let us consider  $S$  and let  $t_1, t_2$  be termal strings of  $S$ .

Observe that  $\langle \text{TheEqSymbOf } S \rangle \wedge t_1 \wedge t_2$  is 0-w.f.f..

Let us consider  $S$ , let  $m$  be a non zero natural number, and let  $T, U$  be  $m$ -element elements of  $(\text{AllTermsOf } S)^*$ . The functor  $\text{PairWiseEq}(T, U)$  is defined by the condition (Def. 58).

(Def. 58)  $\text{PairWiseEq}(T, U) = \{ \langle \text{TheEqSymbOf } S \rangle \wedge T_1(j) \wedge U_1(j); j \text{ ranges over elements of } \text{Seg } m, T_1 \text{ ranges over functions from } \text{Seg } m \text{ into } (\text{AllSymbolsOf } S)^* \setminus \{\emptyset\}, U_1 \text{ ranges over functions from } \text{Seg } m \text{ into } (\text{AllSymbolsOf } S)^* \setminus \{\emptyset\} : T_1 = T \wedge U_1 = U \}$ .

Let us consider  $S$ , let  $m$  be a non zero natural number, and let  $T_2, T_3$  be  $m$ -element elements of  $(\text{AllTermsOf } S)^*$ . Then  $\text{PairWiseEq}(T_2, T_3)$  is a subset of  $\text{AllFormulasOf } S$ .

Let us consider  $S$ , let  $m$  be a non zero natural number, and let  $T, U$  be  $m$ -element elements of  $(\text{AllTermsOf } S)^*$ . Observe that  $\text{PairWiseEq}(T, U)$  is finite.

Let us consider  $S$ , let  $s$  be a relational element of  $S$ , and let  $T_2, T_3$  be  $|ar s|$ -element elements of  $(\text{AllTermsOf } S)^*$ . Observe that  $\langle \{ \text{PairWiseEq}(T_2, T_3) \cup \{s\text{-compound } T_2\}, s\text{-compound } T_3 \} \rangle$  is  $(\emptyset, \{\text{R3e } S\})$ -derivable.

Let us consider  $m, S, D$ . We say that  $D$  is  $m$ -ranked if and only if:

- (Def. 59)(i)  $\text{R0 } S, \text{R2 } S, \text{R3a } S, \text{R3b } S \in D$  if  $m = 0$ ,
- (ii)  $\text{R0 } S, \text{R2 } S, \text{R3a } S, \text{R3b } S, \text{R3d } S, \text{R3e } S \in D$  if  $m = 1$ ,
- (iii)  $\text{R0 } S, \text{R1 } S, \text{R2 } S, \text{R3a } S, \text{R3b } S, \text{R3d } S, \text{R3e } S, \text{R4 } S, \text{R5 } S, \text{R6 } S, \text{R7 } S, \text{R8 } S \in D$  if  $m = 2$ ,
- (iv)  $D = \emptyset$ , otherwise.

Let us consider  $S$ . One can verify that every rule set of  $S$  which is 1-ranked is also 0-ranked and every rule set of  $S$  which is 2-ranked is also 1-ranked.

Let us consider  $S$ . The functor  $S$ -rules yields a rule set of  $S$  and is defined by:

(Def. 60)  $S$ -rules =  $\{R0\ S, R1\ S, R2\ S, R3a\ S, R3b\ S, R3d\ S, R3e\ S, R4\ S\} \cup \{R5\ S, R6\ S, R7\ S, R8\ S\}$ .

Let us consider  $S$ . Observe that  $S$ -rules is 2-ranked.

Let us consider  $S$ . Note that there exists a rule set of  $S$  which is 2-ranked.

Let us consider  $S$ . Observe that there exists a rule set of  $S$  which is 1-ranked.

Let us consider  $S$ . Note that there exists a rule set of  $S$  which is 0-ranked.

Let us consider  $S$ , let  $D$  be a 1-ranked rule set of  $S$ , let  $X$  be a  $D$ -expanded set, and let us consider  $a$ . Observe that  $X$ -freeInterpreter  $a$  is  $(X, D)$ -termEq-respecting.

Let us consider  $S$ , let  $D$  be a 0-ranked rule set of  $S$ , and let  $X$  be a  $D$ -expanded set. Observe that  $(X, D)$ -termEq is total, symmetric, and transitive.

Let us consider  $S$ . Observe that there exists a 0-ranked rule set of  $S$  which is 1-ranked.

The following proposition is true

(1) If  $D_1 \subseteq D_2$  and if  $D_2$  is isotone or  $D_1$  is isotone and if  $Y$  is  $(X, D_1)$ -derivable, then  $Y$  is  $(X, D_2)$ -derivable.

Let us consider  $S, S_6$ . One can verify that  $\{S_6\}$  is  $S$ -sequents-like.

Let us consider  $S, S_{11}, S_5$ . One can check that  $S_{11} \cup S_5$  is  $S$ -sequents-like.

Let us consider  $S$  and let  $x, y$  be  $S$ -sequent-like sets. Observe that  $\{x, y\}$  is  $S$ -sequents-like.

Let us consider  $S, p_3, p_4$ . Note that  $\langle \{\text{xnot } p_3, \text{xnot } p_4\}, \langle \text{TheNorSymbOf } S \rangle \wedge p_3 \wedge p_4 \rangle$  is  $(1, \{\{\{\text{xnot } p_3, \text{xnot } p_4\}, \text{xnot } p_3\}, \{\{\text{xnot } p_3, \text{xnot } p_4\}, \text{xnot } p_4\}\}, \{R6\ S\})$ -derivable.

Let us consider  $S, p_3, p_4$ . One can check that  $\langle \{p_3, p_4\}, p_4 \rangle$  is  $(1, \emptyset, \{R0\ S\})$ -derivable.

We now state two propositions:

(2) For every relation  $R$  between  $2^{S\text{-sequents}}$  and  $S$ -sequents such that  $\langle S_4, S_6 \rangle \in R$  holds  $S_6 \in (\text{FuncRule } R)(S_4)$ .

(3) If  $x \in R(X)$ , then  $x$  is  $(1, X, \{R\})$ -derivable.

Let us consider  $S, D, X$ . Let us observe that  $X$  is  $D$ -expanded if and only if:

(Def. 61) If  $x$  is  $(X, D)$ -provable, then  $x \in X$ .

The following four propositions are true:

(4) If  $p_2 \in X$ , then  $p_2$  is  $(X, \{R0\ S\})$ -provable.

(5) Suppose that

(i)  $D_1 \cup D_2$  is isotone,

- (ii)  $D_1 \cup D_2 \cup D_3$  is isotone,
- (iii)  $x$  is  $(m, S_{11}, D_1)$ -derivable,
- (iv)  $y$  is  $(m, S_5, D_2)$ -derivable, and
- (v)  $z$  is  $(n, \{x, y\}, D_3)$ -derivable.

Then  $z$  is  $(m + n, S_{11} \cup S_5, D_1 \cup D_2 \cup D_3)$ -derivable.

- (6) Suppose  $D_1$  is isotone and  $D_1 \cup D_2$  is isotone and  $y$  is  $(m, X, D_1)$ -derivable and  $z$  is  $(n, \{y\}, D_2)$ -derivable. Then  $z$  is  $(m + n, X, D_1 \cup D_2)$ -derivable.
- (7) If  $x$  is  $(m, X, D)$ -derivable, then  $\{x\}$  is  $(X, D)$ -derivable.

Let us consider  $S$ . Observe that R6  $S$  is isotone.

One can prove the following propositions:

- (8) If  $D_1 \subseteq D_2$  and if  $D_1$  is isotone or  $D_2$  is isotone and if  $x$  is  $(X, D_1)$ -provable, then  $x$  is  $(X, D_2)$ -provable.
- (9) If  $X \subseteq Y$  and  $x$  is  $(X, D)$ -provable, then  $x$  is  $(Y, D)$ -provable.

Let us consider  $S$ . Note that R8  $S$  is isotone.

Let us consider  $S$ . Observe that R1  $S$  is isotone.

Next we state the proposition

- (10) If  $\{y\}$  is  $(S_4, D)$ -derivable, then there exists  $m_1$  such that  $y$  is  $(m_1, S_4, D)$ -derivable.

Let us consider  $S, D, X$ . Observe that every set which is  $(X, D)$ -derivable is also  $S$ -sequents-like.

Let us consider  $S, D, X, x$ . Let us observe that  $x$  is  $(X, D)$ -provable if and only if:

- (Def. 62) There exists a set  $H$  and there exists  $m$  such that  $H \subseteq X$  and  $\langle H, x \rangle$  is  $(m, \emptyset, D)$ -derivable.

The following proposition is true

- (11) If  $D_1 \subseteq D_2$  and if  $D_2$  is isotone or  $D_1$  is isotone and if  $x$  is  $(m, X, D_1)$ -derivable, then  $x$  is  $(m, X, D_2)$ -derivable.

Let us consider  $S$ . Observe that R7  $S$  is isotone.

Next we state the proposition

- (12) If  $x$  is  $(X, D)$ -provable, then  $x$  is a w.f.f. string of  $S$ .

In the sequel  $F$  denotes a rule set of  $S$ .

Let us consider  $S, D_1$  and let  $X$  be a  $D_1$ -expanded set. One can verify that  $(S, X)$ -freeInterpreter is  $(X, D_1)$ -termEq-respecting.

Let us consider  $S$ , let  $D$  be a 0-ranked rule set of  $S$ , and let  $X$  be a  $D$ -expanded set. The functor  $D$ Henkin  $X$  yielding a function is defined by:

- (Def. 63)  $D$ Henkin  $X = (S, X)$ -freeInterpreter quotient  $(X, D)$ -termEq.

Let us consider  $S$ , let  $D$  be a 0-ranked rule set of  $S$ , and let  $X$  be a  $D$ -expanded set. One can check that  $D$ Henkin  $X$  is OwnSymbolsOf  $S$ -defined.

Let us consider  $S$ ,  $D_1$  and let  $X$  be a  $D_1$ -expanded set. Observe that  $D_1$  Henkin  $X$  is  $(S, \text{Classes}(X, D_1)\text{-termEq})$ -interpreter-like.

Let us consider  $S$ ,  $D_1$  and let  $X$  be a  $D_1$ -expanded set. Then  $D_1$  Henkin  $X$  is an element of  $\text{Classes}((X, D_1)\text{-termEq})\text{-InterpretersOf } S$ .

Let us consider  $S$ ,  $p_3$ ,  $p_4$ . One can verify that  $\langle \text{TheNorSymbOf } S \rangle \wedge p_3 \wedge p_4$  is  $(\{\text{xnot } p_3, \text{xnot } p_4\}, \{\text{R0 } S\} \cup \{\text{R6 } S\})$ -provable.

Let us consider  $S$ . Note that every 0-ranked rule set of  $S$  is non empty.

Let us consider  $S$ ,  $x$ . We say that  $x$  is  $S$ -premises-like if and only if:

(Def. 64)  $x \subseteq \text{AllFormulasOf } S$  and  $x$  is finite.

Let us consider  $S$ . One can verify that every set which is  $S$ -premises-like is also finite.

Let us consider  $S$ ,  $p_2$ . Note that  $\{p_2\}$  is  $S$ -premises-like.

Let us consider  $S$  and let  $e$  be an empty set. One can check that  $e \text{ null } S$  is  $S$ -premises-like.

Let us consider  $X$ ,  $S$ . Observe that there exists a subset of  $X$  which is  $S$ -premises-like.

Let us consider  $S$ . Observe that there exists a set which is  $S$ -premises-like.

Let us consider  $S$  and let  $X$  be an  $S$ -premises-like set. Observe that every subset of  $X$  is  $S$ -premises-like.

In the sequel  $H_3$  denotes an  $S$ -premises-like set.

Let us consider  $S$ ,  $H_2$ ,  $H_1$ . Then  $H_1 \text{ null } H_2$  is a subset of  $\text{AllFormulasOf } S$ .

Let us consider  $S$ ,  $H$ ,  $x$ . Note that  $H \text{ null } x$  is  $S$ -premises-like.

Let us consider  $S$ ,  $H_1$ ,  $H_2$ . Note that  $H_1 \cup H_2$  is  $S$ -premises-like.

Let us consider  $S$ ,  $H$ ,  $p_2$ . Observe that  $\langle H, p_2 \rangle$  is  $S$ -sequent-like.

Let us consider  $S$ ,  $H_1$ ,  $H_2$ ,  $p_2$ . One can verify that  $\langle H_1 \cup H_2, p_2 \rangle$  is  $(1, \{\langle H_1, p_2 \rangle\}, \{\text{R1 } S\})$ -derivable.

Let us consider  $S$ ,  $H$ ,  $p_2$ ,  $p_3$ ,  $p_4$ . One can check that  $\langle H \text{ null } p_3 \wedge p_4, \text{xnot } p_2 \rangle$  is  $(1, \{\langle H \cup \{p_2\}, p_3 \rangle, \langle H \cup \{p_2\}, \langle \text{TheNorSymbOf } S \rangle \wedge p_3 \wedge p_4 \rangle\}, \{\text{R8 } S\})$ -derivable.

Let us consider  $S$ . One can verify that  $\emptyset \text{ null } S$  is  $S$ -sequents-like.

Let us consider  $S$ ,  $H$ ,  $p_2$ . Observe that  $\langle H \cup \{p_2\}, p_2 \rangle$  is  $(1, \emptyset, \{\text{R0 } S\})$ -derivable. Let us consider  $p_3$ ,  $p_4$ . Note that  $\langle H \text{ null } p_4, \text{xnot } p_3 \rangle$  is

$(2, \{\langle H, \langle \text{TheNorSymbOf } S \rangle \wedge p_3 \wedge p_4 \rangle\}, \{\text{R0 } S\} \cup \{\text{R1 } S\} \cup \{\text{R8 } S\})$ -derivable.

Let us consider  $S$ ,  $H$ ,  $p_3$ ,  $p_4$ . Note that  $\langle H, \langle \text{TheNorSymbOf } S \rangle \wedge p_4 \wedge p_3 \rangle$  is  $(1, \{\langle H, \langle \text{TheNorSymbOf } S \rangle \wedge p_3 \wedge p_4 \rangle\}, \{\text{R7 } S\})$ -derivable.

Let us consider  $S$ ,  $H$ ,  $p_3$ ,  $p_4$ . Observe that  $\langle H \text{ null } p_3, \text{xnot } p_4 \rangle$  is  $(3, \{\langle H, \langle \text{TheNorSymbOf } S \rangle \wedge p_3 \wedge p_4 \rangle\}, \{\text{R0 } S\} \cup \{\text{R1 } S\} \cup \{\text{R8 } S\} \cup \{\text{R7 } S\})$ -derivable.

Let us consider  $S$ ,  $S_6$ . Observe that  $(S_6)_1$  is  $S$ -premises-like.

Let us consider  $S$ ,  $X$ ,  $D$ . Then  $D \text{ null } X$  is a rule set of  $S$ .

Let us consider  $S$ ,  $p_3$ ,  $p_4$ ,  $l_1$ ,  $H$  and let  $l_2$  be an  $H \cup \{p_3\} \cup \{p_4\}$ -absent literal element of  $S$ .

Note that  $\langle (H \cup \{ \langle l_1 \rangle \wedge p_3 \}) \text{ null } l_2, p_4 \rangle$  is  $(1, \{ \langle H \cup \{ (l_1, l_2) \text{-SymbolSubstIn } p_3 \}, p_4 \rangle \}, \{ \text{R5 } S \})$ -derivable.

Let us consider  $S, D, X$ . We say that  $X$  is  $D$ -inconsistent if and only if:

(Def. 65) There exist  $p_3, p_4$  such that  $p_3$  is  $(X, D)$ -provable and  $\langle \text{TheNorSymbOf } S \rangle \wedge p_3 \wedge p_4$  is  $(X, D)$ -provable.

Let us consider  $m_2, S, H_1, H_2, p_2$ . Note that  $\langle (H_1 \cup H_2) \text{ null } m_2, p_2 \rangle$  is  $(m_2, \{ \langle H_1, p_2 \rangle \}, \{ \text{R1 } S \})$ -derivable.

Let us consider  $S$ . Observe that there exists an isotone rule set of  $S$  which is non empty.

We now state the proposition

(13) If  $X$  is  $D$ -inconsistent and  $D$  is isotone and  $\text{R1 } S, \text{R8 } S \in D$ , then  $\text{xnot } p_2$  is  $(X, D)$ -provable.

Let us consider  $S$ . Observe that  $\text{R5 } S$  is isotone.

Let us consider  $S, l, t, p_2$ . Observe that  $\{ \langle (l, t) \text{SubstIn } p_2 \rangle, \langle l \rangle \wedge p_2 \}$  is  $(1, \emptyset, \{ \text{R4 } S \})$ -derivable.

Let us consider  $S$ . One can verify that  $\text{R4 } S$  is isotone.

Let us consider  $S, X$ . We say that  $X$  is  $S$ -witnessed if and only if:

(Def. 66) For all  $l_1, p_3$  such that  $\langle l_1 \rangle \wedge p_3 \in X$  there exists  $l_2$  such that  $\langle l_1, l_2 \rangle \text{-SymbolSubstIn } p_3 \in X$  and  $l_2 \notin \text{rng } p_3$ .

We now state the proposition

(14)<sup>3</sup> Let  $X$  be a  $D_1$ -expanded set. Suppose  $\text{R1 } S, \text{R4 } S, \text{R6 } S, \text{R7 } S, \text{R8 } S \in D_1$  and  $X$  is  $S$ -mincover and  $S$ -witnessed. Then  $(D_1 \text{Henkin } X)\text{-TruthEval } p_1 = 1$  if and only if  $p_1 \in X$ .

Let us consider  $S, D, X$ . We introduce  $X$  is  $D$ -consistent as an antonym of  $X$  is  $D$ -inconsistent.

We now state the proposition

(15) For every subset  $X$  of  $Y$  such that  $X$  is  $D$ -inconsistent holds  $Y$  is  $D$ -inconsistent.

Let us consider  $S, D$ , let  $X$  be a functional set, and let  $p_2$  be an element of  $\text{ExFormulasOf } S$ . The functor  $(D, p_2) \text{AddAsWitnessTo } X$  is defined by:

(Def. 67)  $(D, p_2) \text{AddAsWitnessTo } X = \begin{cases} X \cup \{ (S\text{-firstChar}(p_2), \text{ the element of LettersOf } S \setminus \text{SymbolsOf } (((\text{AllSymbolsOf } S)^* \setminus \{ \emptyset \}) \cap (X \cup \{ \text{head } p_2 \}))) \text{-SymbolSubstIn head } p_2 \}, \\ \text{ if } X \cup \{ p_2 \} \text{ is } D\text{-consistent and} \\ \text{ LettersOf } S \setminus \text{SymbolsOf } (((\text{AllSym} \\ \text{bolsOf } S)^* \setminus \{ \emptyset \}) \cap (X \cup \{ \text{head } p_2 \})) \neq \emptyset, \\ X, \text{ otherwise.} \end{cases}$

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<sup>3</sup>Henkin's Theorem

Let us consider  $S, D$ , let  $X$  be a functional set, and let  $p_2$  be an element of  $\text{ExFormulasOf } S$ . One can check that  $X \setminus ((D, p_2) \text{ AddAsWitnessTo } X)$  is empty.

Let us consider  $S, D$ , let  $X$  be a functional set, and let  $p_2$  be an element of  $\text{ExFormulasOf } S$ . One can check that  $((D, p_2) \text{ AddAsWitnessTo } X) \setminus X$  is trivial.

Let us consider  $S, D$ , let  $X$  be a functional set, and let  $p_2$  be an element of  $\text{ExFormulasOf } S$ . Then  $(D, p_2) \text{ AddAsWitnessTo } X$  is a subset of  $X \cup \text{AllFormulasOf } S$ .

Let us consider  $S, D$ . We say that  $D$  is correct if and only if the condition (Def. 68) is satisfied.

(Def. 68) Let given  $p_2, X$ . Suppose  $p_2$  is  $(X, D)$ -provable. Let given  $U$  and  $I$  be an element of  $U\text{-InterpretersOf } S$ . If  $X$  is  $I$ -satisfied, then  $I\text{-TruthEval } p_2 = 1$ .

Let us consider  $S, t_1, t_2$ . One can check that  $\text{SubTerms}(\langle \text{TheEqSymbOf } S \rangle \wedge t_1 \wedge t_2) \dot{-} \langle t_1, t_2 \rangle$  is empty.

Let us consider  $S$  and let  $R$  be a rule of  $S$ . We say that  $R$  is correct if and only if:

(Def. 69) If  $X$  is  $S$ -correct, then  $R(X)$  is  $S$ -correct.

Let us consider  $S$ . Observe that every set which is  $S$ -sequent-like is also  $S$ -null.

Let us consider  $S$ . Note that  $R0 S$  is correct.

Let us consider  $S$ . Note that there exists a rule of  $S$  which is correct.

Let us consider  $S$ . One can check that  $R1 S$  is correct.

Let us consider  $S$ . Note that  $R2 S$  is correct.

Let us consider  $S$ . One can check that  $R3a S$  is correct.

Let us consider  $S$ . Observe that  $R3b S$  is correct.

Let us consider  $S$ . Observe that  $R3d S$  is correct.

Let us consider  $S$ . Note that  $R3e S$  is correct.

Let us consider  $S$ . One can check that  $R4 S$  is correct.

Let us consider  $S$ . One can check that  $R5 S$  is correct.

Let us consider  $S$ . One can verify that  $R6 S$  is correct.

Let us consider  $S$ . Observe that  $R7 S$  is correct.

Let us consider  $S$ . Observe that  $R8 S$  is correct.

Next we state the proposition

(16) If for every rule  $R$  of  $S$  such that  $R \in D$  holds  $R$  is correct, then  $D$  is correct.

Let us consider  $S$  and let  $R$  be a correct rule of  $S$ . Note that  $\{R\}$  is correct. Observe that  $S$ -rules is correct. One can check that  $R9 S$  is isotone. Let us consider  $H, p_2$ . Observe that  $\langle H, p_2 \rangle \text{ null } 1$  is  $(1, \{\langle H, \text{xnot xnot } p_2 \rangle\}, \{R9 S\})$ -derivable.

Let us consider  $X, S$ . Observe that there exists an 0-w.f.f. string of  $S$  which is  $X$ -implied.

Let us consider  $X, S$ . Observe that there exists a w.f.f. string of  $S$  which is  $X$ -implied.

Let us consider  $S, X$  and let  $p_2$  be an  $X$ -implied w.f.f. string of  $S$ . Observe that  $\text{xnot xnot } p_2$  is  $X$ -implied.

Let us consider  $X, S, p_2$ . We say that  $p_2$  is  $X$ -provable if and only if:

(Def. 70)  $p_2$  is  $(X, \{\text{R9 } S\} \cup S\text{-rules})$ -provable.

## 2. CONSTRUCTIONS FOR COUNTABLE LANGUAGES: WITNESS ADJOINING

Let  $X$  be a functional set, let us consider  $S, D$ , and let  $n_1$  be a function from  $\mathbb{N}$  into  $\text{ExFormulasOf } S$ . The functor  $(D, n_1) \text{ AddWitnessesTo } X$  yields a function from  $\mathbb{N}$  into  $2^{X \cup \text{AllFormulasOf } S}$  and is defined by:

(Def. 71)  $((D, n_1) \text{ AddWitnessesTo } X)(0) = X$  and  
 for every  $m_1$  holds  $((D, n_1) \text{ AddWitnessesTo } X)(m_1 + 1) =$   
 $(D, n_1(m_1)) \text{ AddAsWitnessTo}((D, n_1) \text{ AddWitnessesTo } X)(m_1)$ .

Let  $X$  be a functional set, let us consider  $S, D$ , and let  $n_1$  be a function from  $\mathbb{N}$  into  $\text{ExFormulasOf } S$ . We introduce  $(D, n_1) \text{ addw } X$  as a synonym of  $(D, n_1) \text{ AddWitnessesTo } X$ .

We now state the proposition

(17) Let  $X$  be a functional set and  $n_1$  be a function from  $\mathbb{N}$  into  $\text{ExFormulasOf } S$ . Suppose  $D$  is isotone and  $\text{R1 } S, \text{R8 } S, \text{R2 } S, \text{R5 } S \in D$  and  $\text{LettersOf } S \setminus \text{SymbolsOf}(X \cap ((\text{AllSymbolsOf } S)^* \setminus \{\emptyset\}))$  is infinite and  $X$  is  $D$ -consistent. Then  $((D, n_1) \text{ addw } X)(k) \subseteq ((D, n_1) \text{ addw } X)(k + m)$  and  $\text{LettersOf } S \setminus \text{SymbolsOf}(((D, n_1) \text{ addw } X)(m) \cap ((\text{AllSymbolsOf } S)^* \setminus \{\emptyset\}))$  is infinite and  $((D, n_1) \text{ addw } X)(m)$  is  $D$ -consistent.

Let  $X$  be a functional set, let us consider  $S, D$ , and let  $n_1$  be a function from  $\mathbb{N}$  into  $\text{ExFormulasOf } S$ . The functor  $X \text{ WithWitnessesFrom}(D, n_1)$  yielding a subset of  $X \cup \text{AllFormulasOf } S$  is defined by:

(Def. 72)  $X \text{ WithWitnessesFrom}(D, n_1) = \bigcup \text{rng}((D, n_1) \text{ AddWitnessesTo } X)$ .

Let  $X$  be a functional set, let us consider  $S, D$ , and let  $n_1$  be a function from  $\mathbb{N}$  into  $\text{ExFormulasOf } S$ . We introduce  $X \text{ addw}(D, n_1)$  as a synonym of  $X \text{ WithWitnessesFrom}(D, n_1)$ .

Let  $X$  be a functional set, let us consider  $S, D$ , and let  $n_1$  be a function from  $\mathbb{N}$  into  $\text{ExFormulasOf } S$ . One can verify that  $X \setminus (X \text{ addw}(D, n_1))$  is empty.

The following proposition is true

(18) Let  $X$  be a functional set and  $n_1$  be a function from  $\mathbb{N}$  into  $\text{ExFormulasOf } S$ . Suppose that  $D$  is isotone and  $\text{R1 } S, \text{R8 } S, \text{R2 } S, \text{R5 } S \in D$  and  $\text{LettersOf } S \setminus \text{SymbolsOf}(X \cap ((\text{AllSymbolsOf } S)^* \setminus \{\emptyset\}))$  is infinite and  $X \text{ addw}(D, n_1) \subseteq Z$  and  $Z$  is  $D$ -consistent and  $\text{rng } n_1 = \text{ExFormulasOf } S$ . Then  $Z$  is  $S$ -witnessed.

3. CONSTRUCTIONS FOR COUNTABLE LANGUAGES: CONSISTENTLY  
MAXIMIZING A SET OF FORMULAS OF A COUNTABLE LANGUAGE  
(LINDENBAUM'S LEMMA)

Let us consider  $X, S, D$  and let  $p_2$  be an element of  $\text{AllFormulasOf } S$ . The functor  $(D, p_2) \text{ AddFormulaTo } X$  is defined by:

$$\text{(Def. 73)} \quad (D, p_2) \text{ AddFormulaTo } X = \begin{cases} X \cup \{p_2\}, \\ \quad \text{if } \text{xnot } p_2 \text{ is not } (X, D)\text{-provable,} \\ X \cup \{\text{xnot } p_2\}, \text{ otherwise.} \end{cases}$$

Let us consider  $X, S, D$  and let  $p_2$  be an element of  $\text{AllFormulasOf } S$ . Then  $(D, p_2) \text{ AddFormulaTo } X$  is a subset of  $X \cup \text{AllFormulasOf } S$ .

Let us consider  $X, S, D$  and let  $p_2$  be an element of  $\text{AllFormulasOf } S$ . Note that  $X \setminus ((D, p_2) \text{ AddFormulaTo } X)$  is empty.

Let us consider  $X, S, D$  and let  $n_1$  be a function from  $\mathbb{N}$  into  $\text{AllFormulasOf } S$ . The functor  $(D, n_1) \text{ AddFormulasTo } X$  yields a function from  $\mathbb{N}$  into

$2^{X \cup \text{AllFormulasOf } S}$  and is defined by:

$$\text{(Def. 74)} \quad \begin{aligned} ((D, n_1) \text{ AddFormulasTo } X)(0) &= X \text{ and for every } m \text{ holds} \\ ((D, n_1) \text{ AddFormulasTo } X)(m+1) &= \\ &= (D, n_1(m)) \text{ AddFormulaTo } ((D, n_1) \text{ AddFormulasTo } X)(m). \end{aligned}$$

Let us consider  $X, S, D$  and let  $n_1$  be a function from  $\mathbb{N}$  into  $\text{AllFormulasOf } S$ . The functor  $(D, n_1) \text{ CompletionOf } X$  yields a subset of  $X \cup \text{AllFormulasOf } S$  and is defined as follows:

$$\text{(Def. 75)} \quad (D, n_1) \text{ CompletionOf } X = \bigcup \text{rng}((D, n_1) \text{ AddFormulasTo } X).$$

Let us consider  $X, S, D$  and let  $n_1$  be a function from  $\mathbb{N}$  into  $\text{AllFormulasOf } S$ . One can check that  $X \setminus ((D, n_1) \text{ CompletionOf } X)$  is empty.

We now state the proposition

$$\text{(19)} \quad \text{For every relation } R \text{ between } 2^{S\text{-sequents}} \text{ and } S\text{-sequents holds } y \in (\text{FuncRule } R)(X) \text{ iff } y \in S\text{-sequents and } \langle X, y \rangle \in R.$$

In the sequel  $D_2$  is a 2-ranked rule set of  $S$ .

Let us consider  $S$  and let  $r_1, r_2$  be isotone rules of  $S$ . Note that  $\{r_1, r_2\}$  is isotone.

Let us consider  $S$  and let  $r_1, r_2, r_3, r_4$  be isotone rules of  $S$ . Observe that  $\{r_1, r_2, r_3, r_4\}$  is isotone.

Let us consider  $S$ . Observe that  $S$ -rules is isotone.

Let us consider  $S$ . Observe that there exists an isotone rule set of  $S$  which is correct.

Let us consider  $S$ . Observe that there exists a correct isotone rule set of  $S$  which is 2-ranked.

Let  $S$  be a countable language. Observe that  $\text{AllFormulasOf } S$  is countable.

We now state the proposition



- (20) Let  $S$  be a countable language and  $D$  be a rule set of  $S$ . Suppose  $D$  is 2-ranked, isotone, and correct and  $Z$  is  $D$ -consistent and  $Z \subseteq \text{AllFormulasOf } S$ . Then there exists a non empty set  $U$  and there exists an element  $I$  of  $U\text{-InterpretersOf } S$  such that  $Z$  is  $I$ -satisfied.

In the sequel  $C$  denotes a countable language and  $p_2$  denotes a w.f.f. string of  $C$ .

We now state the proposition

- (21) If  $X \subseteq \text{AllFormulasOf } C$  and  $p_2$  is  $X$ -implied, then  $p_2$  is  $X$ -provable.

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