

Sequent Calculus, Derivability, Provability. Gödel's Completeness Theorem¹

Marco B. Caminati²
Mathematics Department “G.Castelnuovo”
Sapienza University of Rome
Piazzale Aldo Moro 5, 00185 Roma, Italy

Summary. Fifth of a series of articles laying down the bases for classical first order model theory. This paper presents multiple themes: first it introduces sequents, rules and sets of rules for a first order language L as L -dependent types. Then defines derivability and provability according to a set of rules, and gives several technical lemmas binding all those concepts. Following that, it introduces a fixed set D of derivation rules, and proceeds to convert them to Mizar functorial cluster registrations to give the user a slick interface to apply them.

The remaining goals summon all the definitions and results introduced in this series of articles. First: D is shown to be correct and having the requisites to deliver a sensible definition of Henkin model (see [18]). Second: as a particular application of all the machinery built thus far, the satisfiability and Gödel completeness theorems are shown when restricting to countable languages. The techniques used to attain this are inspired from [18], then heavily modified with the twofold goal of embedding them into the more flexible framework of a variable ruleset here introduced, and of proving completeness of a set of rules more sparing than the one there used; in particular the simpler ruleset allowed to avoid the definition and tractation of free occurrence of a literal, a fact which, along with shortening proofs, is remarkable in its own right. A preparatory account of some of the ideas used in the proofs given here can be found in [15].

MML identifier: FOMODEL4, version: 7.11.07 4.160.1126

The notation and terminology used here have been introduced in the following papers: [1], [3], [23], [22], [4], [6], [17], [11], [12], [13], [14], [7], [8], [5], [19], [16], [24], [2], [21], [9], [26], [28], [27], [20], [25], and [10].

¹The author wrote this paper as part of his PhD thesis research and would like to thank Marco Pedicini for his encouragement and support.

²The author is the winner of the Mizar Prize for Young Researchers in 2011 for this article.

1. FORMALIZATION OF THE NOTION OF DERIVABILITY AND PROVABILITY.
HENKIN'S THEOREM FOR ARBITRARY LANGUAGES

For simplicity, we adopt the following convention: k, m, n denote natural numbers, m_1 denotes an element of \mathbb{N} , U denotes a non empty set, A, B, X, Y, Z, x, y, z denote sets, S denotes a language, s denotes an element of S , f, g denote functions, p_1, p_2, p_3, p_4 denote w.f.f. strings of S , P_1, P_2, P_3 denote subsets of AllFormulasOf S , t, t_1, t_2 denote termal strings of S , a denotes an of-atomic-formula element of S , l, l_1, l_2 denote literal elements of S , p denotes a finite sequence, and m_2 denotes a non zero natural number.

Let S be a language. The functor S -sequents is defined as follows:

(Def. 1) S -sequents = $\{(p_5, c_1); p_5 \text{ ranges over subsets of AllFormulasOf } S, c_1 \text{ ranges over w.f.f. strings of } S; p_5 \text{ is finite}\}$.

Let S be a language. Note that S -sequents is non empty.

Let us consider S . Observe that S -sequents is relation-like.

Let S be a language and let x be a set. We say that x is S -sequent-like if and only if:

(Def. 2) $x \in S$ -sequents .

Let us consider S, X . We say that X is S -sequents-like if and only if:

(Def. 3) $X \subseteq S$ -sequents .

Let us consider S . One can check that every subset of S -sequents is S -sequents-like and every element of S -sequents is S -sequent-like.

Let S be a language. One can verify that there exists an element of S -sequents which is S -sequent-like and there exists a subset of S -sequents which is S -sequents-like.

Let us consider S . One can check that there exists a set which is S -sequent-like and there exists a set which is S -sequents-like.

Let S be a language. A rule of S is an element of $(2^{S\text{-sequents}})^{2^{S\text{-sequents}}}$.

Let S be a language. A rule set of S is a subset of $(2^{S\text{-sequents}})^{2^{S\text{-sequents}}}$.

For simplicity, we adopt the following rules: D, D_1, D_2, D_3 denote rule sets of S , R denotes a rule of S , S_1, S_2, S_3 denote subsets of S -sequents, s_1, s_2, s_3 denote elements of S -sequents, S_4, S_5 denote S -sequents-like sets, and S_6, S_7 denote S -sequent-like sets.

Let us consider A, B and let X be a subset of B^A . One can check that $\cup X$ is relation-like.

Let S be a language and let D be a rule set of S . One can check that $\cup D$ is relation-like.

Let us consider S, D . The functor OneStep D yielding a rule of S is defined as follows:

(Def. 4) For every element S_8 of $2^{S\text{-sequents}}$ holds $(\text{OneStep } D)(S_8) = \cup((\cup D)^\circ\{S_8\})$.

Let us consider S, D, m . The functor (m, D) -derivables yields a rule of S and is defined by:

(Def. 5) (m, D) -derivables = $(\text{OneStep } D)^m$.

Let S be a language, let D be a rule set of S , and let S_9, S_{10} be sets. We say that S_{10} is (S_9, D) -derivable if and only if:

(Def. 6) $S_{10} \subseteq \cup(((\text{OneStep } D)^*)^\circ \{S_9\})$.

Let us consider m, S, D and let S_1, s_1 be sets. We say that s_1 is (m, S_1, D) -derivable if and only if:

(Def. 7) $s_1 \in (m, D)$ -derivables(S_1).

Let us consider S, D . The functor D -iterators yielding a family of subsets of $2^{S\text{-sequents}} \times 2^{S\text{-sequents}}$ is defined as follows:

(Def. 8) D -iterators = $\{(\text{OneStep } D)^{m_1}\}$.

Let us consider S, R . We say that R is isotone if and only if:

(Def. 9) If $S_2 \subseteq S_3$, then $R(S_2) \subseteq R(S_3)$.

Let us consider S . Observe that there exists a rule of S which is isotone.

Let us consider S, D . We say that D is isotone if and only if:

(Def. 10) For all S_2, S_3, f such that $S_2 \subseteq S_3$ and $f \in D$ there exists g such that $g \in D$ and $f(S_2) \subseteq g(S_3)$.

Let us consider S and let M be an isotone rule of S . One can verify that $\{M\}$ is isotone.

Let us consider S . One can verify that there exists a rule set of S which is isotone.

In the sequel K, K_1 are isotone rule sets of S .

Let S be a language, let D be a rule set of S , and let S_1 be a set. We say that S_1 is D -derivable if and only if:

(Def. 11) S_1 is (\emptyset, D) -derivable.

Let us consider S, D . One can verify that every set which is D -derivable is also (\emptyset, D) -derivable and every set which is (\emptyset, D) -derivable is also D -derivable.

Let us consider S, D and let S_1 be an empty set. One can verify that every set which is (S_1, D) -derivable is also D -derivable.

Let us consider S, D, X and let p_2 be a set. We say that p_2 is (X, D) -provable if and only if:

(Def. 12) $\{\{X, p_2\}\}$ is D -derivable or there exists a set s_1 such that $(s_1)_1 \subseteq X$ and $(s_1)_2 = p_2$ and $\{s_1\}$ is D -derivable.

Let us consider S, D, X, x . Let us observe that x is (X, D) -provable if and only if:

(Def. 13) There exists a set s_1 such that $(s_1)_1 \subseteq X$ and $(s_1)_2 = x$ and $\{s_1\}$ is D -derivable.

Let us consider S, D, R . We say that R is D -macro if and only if:

(Def. 14) For every subset S_8 of S -sequents holds $R(S_8)$ is (S_8, D) -derivable.

Let us consider S, D and let P_1 be a set. The functor (P_1, D) -termEq is defined as follows:

(Def. 15) (P_1, D) -termEq = $\{\langle t_1, t_2 \rangle; t_1 \text{ ranges over termal strings of } S, t_2 \text{ ranges over termal strings of } S: \langle \text{TheEqSymbOf } S \rangle \wedge t_1 \wedge t_2 \text{ is } (P_1, D)\text{-provable}\}$.

Let us consider S, D and let P_1 be a set. We say that P_1 is D -expanded if and only if:

(Def. 16) If x is (P_1, D) -provable, then $\{x\} \subseteq P_1$.

Let us consider S, x . We say that x is S -null if and only if:

(Def. 17) Not contradiction.

Let us consider S, D and let P_1 be a set. Then (P_1, D) -termEq is a binary relation on AllTermsOf S .

Let us consider S, p_2 and let P_2, P_3 be finite subsets of AllFormulasOf S . One can check that $\langle P_2 \cup P_3, p_2 \rangle$ is S -sequent-like.

Let us consider S , let x be an empty set, and let p_2 be a w.f.f. string of S . Then $\langle x, p_2 \rangle$ is an element of S -sequents.

Let us consider S . Note that $\emptyset \cap S$ is S -sequents-like.

Let us consider S . One can verify that there exists a set which is S -null.

Let us consider S . One can check that every set which is S -sequent-like is also S -null.

Let us consider S . One can check that every element of S -sequents is S -null.

Let us consider m, S, D, X . One can verify that (m, D) -derivables(X) is S -sequents-like.

Let us consider S, Y and let X be an S -sequents-like set. One can verify that $X \cap Y$ is S -sequents-like.

Let us consider S, D, m, X . Note that every set which is (m, X, D) -derivable is also S -sequent-like.

Let us consider S, D and let P_2, P_3 be sets. Observe that every set which is $(P_2 \setminus P_3, D)$ -provable is also (P_2, D) -provable.

Let us consider S, D and let P_2, P_3 be sets. Observe that every set which is $(P_2 \setminus P_3, D)$ -provable is also $(P_2 \cup P_3, D)$ -provable.

Let us consider S, D and let P_2, P_3 be sets. Observe that every set which is $(P_2 \cap P_3, D)$ -provable is also (P_2, D) -provable.

Let us consider S, D , let X be a set, and let x be a subset of X . Note that every set which is (x, D) -provable is also (X, D) -provable.

Let us consider S , let p_5 be a finite subset of AllFormulasOf S , and let p_2 be a w.f.f. string of S . One can check that $\langle p_5, p_2 \rangle$ is S -sequent-like.

Let us consider S and let p_3, p_4 be w.f.f. strings of S . Observe that $\{\langle p_3 \rangle, p_4 \rangle$ is S -sequent-like. Let p_6 be a w.f.f. string of S . Note that $\{\langle p_3, p_4 \rangle, p_6 \rangle$ is S -sequent-like.

Let us consider S , p_3 , p_4 and let P_1 be a finite subset of $\text{AllFormulasOf } S$. One can verify that $\langle P_1 \cup \{p_3\}, p_4 \rangle$ is S -sequent-like.

Let us consider S , D . Note that there exists a subset of $\text{AllFormulasOf } S$ which is D -expanded.

Let us consider S , D . Observe that there exists a set which is D -expanded.

Let S_1 be a set, let S be a language, and let s_1 be an S -null set. We say that s_1 rule 0 S_1 if and only if:

(Def. 18) $(s_1)_2 \in (s_1)_1$.

We say that s_1 rule 1 S_1 if and only if:

(Def. 19) There exists a set y such that $y \in S_1$ and $y_1 \subseteq (s_1)_1$ and $(s_1)_2 = y_2$.

We say that s_1 rule 2 S_1 if and only if:

(Def. 20) $(s_1)_1$ is empty and there exists a termal string t of S such that $(s_1)_2 = \langle \text{TheEqSymbOf } S \rangle \wedge t \wedge t$.

We say that s_1 rule 3a S_1 if and only if the condition (Def. 21) is satisfied.

(Def. 21) There exist termal strings t , t_1 , t_2 of S and there exists a set x such that $x \in S_1$ and $(s_1)_1 = x_1 \cup \{ \langle \text{TheEqSymbOf } S \rangle \wedge t_1 \wedge t_2 \}$ and $x_2 = \langle \text{TheEqSymbOf } S \rangle \wedge t \wedge t_1$ and $(s_1)_2 = \langle \text{TheEqSymbOf } S \rangle \wedge t \wedge t_2$.

We say that s_1 rule 3b S_1 if and only if:

(Def. 22) There exist termal strings t_1 , t_2 of S such that $(s_1)_1 = \{ \langle \text{TheEqSymbOf } S \rangle \wedge t_1 \wedge t_2 \}$ and $(s_1)_2 = \langle \text{TheEqSymbOf } S \rangle \wedge t_2 \wedge t_1$.

We say that s_1 rule 3d S_1 if and only if the condition (Def. 23) is satisfied.

(Def. 23) There exists a low-compounding element s of S and there exist $|ar s|$ -element elements T , U of $(\text{AllTermsOf } S)^*$ such that

(i) s is operational,

(ii) $(s_1)_1 = \{ \langle \text{TheEqSymbOf } S \rangle \wedge T_1(j) \wedge U_1(j); j \text{ ranges over elements of } \text{Seg}|ar s|, T_1 \text{ ranges over functions from } \text{Seg}|ar s| \text{ into } (\text{AllSymbolsOf } S)^* \setminus \{ \emptyset \}, U_1 \text{ ranges over functions from } \text{Seg}|ar s| \text{ into } (\text{AllSymbolsOf } S)^* \setminus \{ \emptyset \} : T_1 = T \wedge U_1 = U \}$, and

(iii) $(s_1)_2 = \langle \text{TheEqSymbOf } S \rangle \wedge (s\text{-compound } T) \wedge (s\text{-compound } U)$.

We say that s_1 rule 3e S_1 if and only if the condition (Def. 24) is satisfied.

(Def. 24) There exists a relational element s of S and there exist $|ar s|$ -element elements T , U of $(\text{AllTermsOf } S)^*$ such that

(i) $(s_1)_1 = \{ s\text{-compound } T \} \cup \{ \langle \text{TheEqSymbOf } S \rangle \wedge T_1(j) \wedge U_1(j); j \text{ ranges over elements of } \text{Seg}|ar s|, T_1 \text{ ranges over functions from } \text{Seg}|ar s| \text{ into } (\text{AllSymbolsOf } S)^* \setminus \{ \emptyset \}, U_1 \text{ ranges over functions from } \text{Seg}|ar s| \text{ into } (\text{AllSymbolsOf } S)^* \setminus \{ \emptyset \} : T_1 = T \wedge U_1 = U \}$, and

(ii) $(s_1)_2 = s\text{-compound } U$.

We say that s_1 rule 4 S_1 if and only if the condition (Def. 25) is satisfied.

(Def. 25) There exists a literal element l of S and there exists a w.f.f. string p_2 of S and there exists a termal string t of S such that $(s_1)_1 = \{(l, t) \text{ SubstIn } p_2\}$ and $(s_1)_2 = \langle l \rangle \wedge p_2$.

We say that s_1 rule 5 S_1 if and only if:

(Def. 26) There exist literal elements v_1, v_2 of S and there exist x, p such that $(s_1)_1 = x \cup \{\langle v_1 \rangle \wedge p\}$ and v_2 is $x \cup \{p\} \cup \{s_{12}\}$ -absent and $\langle x \cup \{(v_1 \text{ SubstWith } v_2)(p)\} \rangle, (s_1)_2 \in S_1$.

We say that s_1 rule 6 S_1 if and only if the condition (Def. 27) is satisfied.

(Def. 27) There exist sets y_1, y_2 and there exist w.f.f. strings p_3, p_4 of S such that $y_1, y_2 \in S_1$ and $(y_1)_1 = (y_2)_1 = (s_1)_1$ and $(y_1)_2 = \langle \text{TheNorSymbOf } S \rangle \wedge p_3 \wedge p_3$ and $(y_2)_2 = \langle \text{TheNorSymbOf } S \rangle \wedge p_4 \wedge p_4$ and $(s_1)_2 = \langle \text{TheNorSymbOf } S \rangle \wedge p_3 \wedge p_4$.

We say that s_1 rule 7 S_1 if and only if:

(Def. 28) There exists a set y and there exist w.f.f. strings p_3, p_4 of S such that $y \in S_1$ and $y_1 = (s_1)_1$ and $y_2 = \langle \text{TheNorSymbOf } S \rangle \wedge p_3 \wedge p_4$ and $(s_1)_2 = \langle \text{TheNorSymbOf } S \rangle \wedge p_4 \wedge p_3$.

We say that s_1 rule 8 S_1 if and only if the condition (Def. 29) is satisfied.

(Def. 29) There exist sets y_1, y_2 and there exist w.f.f. strings p_2, p_3, p_4 of S such that $y_1, y_2 \in S_1$ and $(y_1)_1 = (y_2)_1$ and $(y_1)_2 = p_3$ and $(y_2)_2 = \langle \text{TheNorSymbOf } S \rangle \wedge p_3 \wedge p_4$ and $\{p_2\} \cup (s_1)_1 = (y_1)_1$ and $(s_1)_2 = \langle \text{TheNorSymbOf } S \rangle \wedge p_2 \wedge p_2$.

We say that s_1 rule 9 S_1 if and only if:

(Def. 30) There exists a set y and there exists a w.f.f. string p_2 of S such that $y \in S_1$ and $(s_1)_2 = p_2$ and $y_1 = (s_1)_1$ and $y_2 = \text{xnot xnot } p_2$.

Let S be a language. The functor $P0 S$ yielding a relation between $2^{S\text{-sequents}}$ and S -sequents is defined by:

(Def. 31) For every element S_1 of $2^{S\text{-sequents}}$ and for every element s_1 of S -sequents holds $\langle S_1, s_1 \rangle \in P0 S$ iff s_1 rule 0 S_1 .

The functor $P1 S$ yields a relation between $2^{S\text{-sequents}}$ and S -sequents and is defined as follows:

(Def. 32) For every element S_1 of $2^{S\text{-sequents}}$ and for every element s_1 of S -sequents holds $\langle S_1, s_1 \rangle \in P1 S$ iff s_1 rule 1 S_1 .

The functor $P2 S$ yields a relation between $2^{S\text{-sequents}}$ and S -sequents and is defined as follows:

(Def. 33) For every element S_1 of $2^{S\text{-sequents}}$ and for every element s_1 of S -sequents holds $\langle S_1, s_1 \rangle \in P2 S$ iff s_1 rule 2 S_1 .

The functor $P3a S$ yielding a relation between $2^{S\text{-sequents}}$ and S -sequents is defined as follows:

(Def. 34) For every element S_1 of $2^{S\text{-sequents}}$ and for every element s_1 of $S\text{-sequents}$ holds $\langle S_1, s_1 \rangle \in P3a S$ iff s_1 rule 3a S_1 .

The functor $P3b S$ yields a relation between $2^{S\text{-sequents}}$ and $S\text{-sequents}$ and is defined as follows:

(Def. 35) For every element S_1 of $2^{S\text{-sequents}}$ and for every element s_1 of $S\text{-sequents}$ holds $\langle S_1, s_1 \rangle \in P3b S$ iff s_1 rule 3b S_1 .

The functor $P3d S$ yields a relation between $2^{S\text{-sequents}}$ and $S\text{-sequents}$ and is defined as follows:

(Def. 36) For every element S_1 of $2^{S\text{-sequents}}$ and for every element s_1 of $S\text{-sequents}$ holds $\langle S_1, s_1 \rangle \in P3d S$ iff s_1 rule 3d S_1 .

The functor $P3e S$ yielding a relation between $2^{S\text{-sequents}}$ and $S\text{-sequents}$ is defined by:

(Def. 37) For every element S_1 of $2^{S\text{-sequents}}$ and for every element s_1 of $S\text{-sequents}$ holds $\langle S_1, s_1 \rangle \in P3e S$ iff s_1 rule 3e S_1 .

The functor $P4 S$ yielding a relation between $2^{S\text{-sequents}}$ and $S\text{-sequents}$ is defined by:

(Def. 38) For every element S_1 of $2^{S\text{-sequents}}$ and for every element s_1 of $S\text{-sequents}$ holds $\langle S_1, s_1 \rangle \in P4 S$ iff s_1 rule 4 S_1 .

The functor $P5 S$ yields a relation between $2^{S\text{-sequents}}$ and $S\text{-sequents}$ and is defined by:

(Def. 39) For every element S_1 of $2^{S\text{-sequents}}$ and for every element s_1 of $S\text{-sequents}$ holds $\langle S_1, s_1 \rangle \in P5 S$ iff s_1 rule 5 S_1 .

The functor $P6 S$ yielding a relation between $2^{S\text{-sequents}}$ and $S\text{-sequents}$ is defined by:

(Def. 40) For every element S_1 of $2^{S\text{-sequents}}$ and for every element s_1 of $S\text{-sequents}$ holds $\langle S_1, s_1 \rangle \in P6 S$ iff s_1 rule 6 S_1 .

The functor $P7 S$ yielding a relation between $2^{S\text{-sequents}}$ and $S\text{-sequents}$ is defined as follows:

(Def. 41) For every element S_1 of $2^{S\text{-sequents}}$ and for every element s_1 of $S\text{-sequents}$ holds $\langle S_1, s_1 \rangle \in P7 S$ iff s_1 rule 7 S_1 .

The functor $P8 S$ yields a relation between $2^{S\text{-sequents}}$ and $S\text{-sequents}$ and is defined as follows:

(Def. 42) For every element S_1 of $2^{S\text{-sequents}}$ and for every element s_1 of $S\text{-sequents}$ holds $\langle S_1, s_1 \rangle \in P8 S$ iff s_1 rule 8 S_1 .

The functor $P9 S$ yields a relation between $2^{S\text{-sequents}}$ and $S\text{-sequents}$ and is defined as follows:

(Def. 43) For every element S_1 of $2^{S\text{-sequents}}$ and for every element s_1 of $S\text{-sequents}$ holds $\langle S_1, s_1 \rangle \in P9 S$ iff s_1 rule 9 S_1 .

Let us consider S and let R be a relation between $2^{S\text{-sequents}}$ and $S\text{-sequents}$.

The functor $\text{FuncRule } R$ yields a rule of S and is defined by:

(Def. 44) For every set i_1 such that $i_1 \in 2^{S\text{-sequents}}$ holds $(\text{FuncRule } R)(i_1) = \{x \in S\text{-sequents} : \langle i_1, x \rangle \in R\}$.

Let us consider S . The functor $\text{R0 } S$ yielding a rule of S is defined as follows:

(Def. 45) $\text{R0 } S = \text{FuncRule P0 } S$.

The functor $\text{R1 } S$ yielding a rule of S is defined as follows:

(Def. 46) $\text{R1 } S = \text{FuncRule P1 } S$.

The functor $\text{R2 } S$ yielding a rule of S is defined by:

(Def. 47) $\text{R2 } S = \text{FuncRule P2 } S$.

The functor $\text{R3a } S$ yielding a rule of S is defined by:

(Def. 48) $\text{R3a } S = \text{FuncRule P3a } S$.

The functor $\text{R3b } S$ yielding a rule of S is defined as follows:

(Def. 49) $\text{R3b } S = \text{FuncRule P3b } S$.

The functor $\text{R3d } S$ yielding a rule of S is defined as follows:

(Def. 50) $\text{R3d } S = \text{FuncRule P3d } S$.

The functor $\text{R3e } S$ yielding a rule of S is defined by:

(Def. 51) $\text{R3e } S = \text{FuncRule P3e } S$.

The functor $\text{R4 } S$ yields a rule of S and is defined as follows:

(Def. 52) $\text{R4 } S = \text{FuncRule P4 } S$.

The functor $\text{R5 } S$ yielding a rule of S is defined as follows:

(Def. 53) $\text{R5 } S = \text{FuncRule P5 } S$.

The functor $\text{R6 } S$ yields a rule of S and is defined by:

(Def. 54) $\text{R6 } S = \text{FuncRule P6 } S$.

The functor $\text{R7 } S$ yields a rule of S and is defined by:

(Def. 55) $\text{R7 } S = \text{FuncRule P7 } S$.

The functor $\text{R8 } S$ yielding a rule of S is defined as follows:

(Def. 56) $\text{R8 } S = \text{FuncRule P8 } S$.

The functor $\text{R9 } S$ yields a rule of S and is defined by:

(Def. 57) $\text{R9 } S = \text{FuncRule P9 } S$.

Let us consider S and let t be a termal string of S .

Note that $\{\langle \emptyset, \langle \text{TheEqSymbOf } S \rangle \wedge t \wedge t \rangle\}$ is $\{\text{R2 } S\}$ -derivable. Note that $\text{R2 } S$ is isotone. One can verify that $\text{R3b } S$ is isotone.

Let t, t_1, t_2 be termal strings of S , and let p_5 be a finite subset of $\text{AllFormulasOf } S$. Observe that $\langle p_5 \cup \{\langle \text{TheEqSymbOf } S \rangle \wedge t_1 \wedge t_2\}, \langle \text{TheEqSymbOf } S \rangle \wedge t \wedge t_2 \rangle$ is $(1, \{\langle p_5, \langle \text{TheEqSymbOf } S \rangle \wedge t \wedge t_1 \rangle\}, \{\text{R3a } S\})$ -derivable.

Let us consider S , let t, t_1, t_2 be termal strings of S , and let p_2 be a w.f.f. string of S . Note that $\langle \{p_2, \langle \text{TheEqSymbOf } S \rangle \wedge t_1 \wedge t_2\}, \langle \text{TheEqSymbOf } S \rangle \wedge t \wedge t_2 \rangle$ is $(1, \{\{p_2\}, \langle \text{TheEqSymbOf } S \rangle \wedge t \wedge t_1\}, \{\text{R3a } S\})$ -derivable.

Let us consider S , let p_2 be a w.f.f. string of S , and let P_1 be a finite subset of $\text{AllFormulasOf } S$. One can verify that $\langle P_1 \cup \{p_2\}, p_2 \rangle$ is $(1, \emptyset, \{\text{R0 } S\})$ -derivable.

Let us consider S and let p_3, p_4 be w.f.f. strings of S . One can check that $\langle \{p_3, p_4\}, p_3 \rangle$ is $(1, \emptyset, \{\text{R0 } S\})$ -derivable.

Let us consider S, p_2 . Note that $\langle \{p_2\}, p_2 \rangle$ is $(1, \emptyset, \{\text{R0 } S\})$ -derivable.

Let us consider S and let p_2 be a w.f.f. string of S . Observe that $\langle \{p_2\}, p_2 \rangle$ is $(\emptyset, \{\text{R0 } S\})$ -derivable.

Let us consider S . One can verify the following observations:

- * $\text{R0 } S$ is isotone,
- * $\text{R3a } S$ is isotone,
- * $\text{R3d } S$ is isotone, and
- * $\text{R3e } S$ is isotone.

Let us consider K_1, K_2 . One can verify that $K_1 \cup K_2$ is isotone.

Let us consider S and let t_1, t_2 be termal strings of S .

Observe that $\langle \text{TheEqSymbOf } S \rangle \wedge t_1 \wedge t_2$ is 0-w.f.f..

Let us consider S , let m be a non zero natural number, and let T, U be m -element elements of $(\text{AllTermsOf } S)^*$. The functor $\text{PairWiseEq}(T, U)$ is defined by the condition (Def. 58).

(Def. 58) $\text{PairWiseEq}(T, U) = \{ \langle \text{TheEqSymbOf } S \rangle \wedge T_1(j) \wedge U_1(j); j \text{ ranges over elements of } \text{Seg } m, T_1 \text{ ranges over functions from } \text{Seg } m \text{ into } (\text{AllSymbolsOf } S)^* \setminus \{\emptyset\}, U_1 \text{ ranges over functions from } \text{Seg } m \text{ into } (\text{AllSymbolsOf } S)^* \setminus \{\emptyset\} : T_1 = T \wedge U_1 = U \}$.

Let us consider S , let m be a non zero natural number, and let T_2, T_3 be m -element elements of $(\text{AllTermsOf } S)^*$. Then $\text{PairWiseEq}(T_2, T_3)$ is a subset of $\text{AllFormulasOf } S$.

Let us consider S , let m be a non zero natural number, and let T, U be m -element elements of $(\text{AllTermsOf } S)^*$. Observe that $\text{PairWiseEq}(T, U)$ is finite.

Let us consider S , let s be a relational element of S , and let T_2, T_3 be $|ar s|$ -element elements of $(\text{AllTermsOf } S)^*$. Observe that $\langle \{ \text{PairWiseEq}(T_2, T_3) \cup \{s\text{-compound } T_2\}, s\text{-compound } T_3 \} \rangle$ is $(\emptyset, \{\text{R3e } S\})$ -derivable.

Let us consider m, S, D . We say that D is m -ranked if and only if:

- (Def. 59)(i) $\text{R0 } S, \text{R2 } S, \text{R3a } S, \text{R3b } S \in D$ if $m = 0$,
- (ii) $\text{R0 } S, \text{R2 } S, \text{R3a } S, \text{R3b } S, \text{R3d } S, \text{R3e } S \in D$ if $m = 1$,
- (iii) $\text{R0 } S, \text{R1 } S, \text{R2 } S, \text{R3a } S, \text{R3b } S, \text{R3d } S, \text{R3e } S, \text{R4 } S, \text{R5 } S, \text{R6 } S, \text{R7 } S, \text{R8 } S \in D$ if $m = 2$,
- (iv) $D = \emptyset$, otherwise.

Let us consider S . One can verify that every rule set of S which is 1-ranked is also 0-ranked and every rule set of S which is 2-ranked is also 1-ranked.

Let us consider S . The functor S -rules yields a rule set of S and is defined by:

(Def. 60) S -rules = $\{R0\ S, R1\ S, R2\ S, R3a\ S, R3b\ S, R3d\ S, R3e\ S, R4\ S\} \cup \{R5\ S, R6\ S, R7\ S, R8\ S\}$.

Let us consider S . Observe that S -rules is 2-ranked.

Let us consider S . Note that there exists a rule set of S which is 2-ranked.

Let us consider S . Observe that there exists a rule set of S which is 1-ranked.

Let us consider S . Note that there exists a rule set of S which is 0-ranked.

Let us consider S , let D be a 1-ranked rule set of S , let X be a D -expanded set, and let us consider a . Observe that X -freeInterpreter a is (X, D) -termEq-respecting.

Let us consider S , let D be a 0-ranked rule set of S , and let X be a D -expanded set. Observe that (X, D) -termEq is total, symmetric, and transitive.

Let us consider S . Observe that there exists a 0-ranked rule set of S which is 1-ranked.

The following proposition is true

- (1) If $D_1 \subseteq D_2$ and if D_2 is isotone or D_1 is isotone and if Y is (X, D_1) -derivable, then Y is (X, D_2) -derivable.

Let us consider S, S_6 . One can verify that $\{S_6\}$ is S -sequents-like.

Let us consider S, S_{11}, S_5 . One can check that $S_{11} \cup S_5$ is S -sequents-like.

Let us consider S and let x, y be S -sequent-like sets. Observe that $\{x, y\}$ is S -sequents-like.

Let us consider S, p_3, p_4 . Note that $\langle \{\text{xnot } p_3, \text{xnot } p_4\}, \langle \text{TheNorSymbOf } S \rangle \wedge p_3 \wedge p_4 \rangle$ is $(1, \{\{\{\text{xnot } p_3, \text{xnot } p_4\}, \text{xnot } p_3\}, \{\{\text{xnot } p_3, \text{xnot } p_4\}, \text{xnot } p_4\}\}, \{R6\ S\})$ -derivable.

Let us consider S, p_3, p_4 . One can check that $\langle \{p_3, p_4\}, p_4 \rangle$ is $(1, \emptyset, \{R0\ S\})$ -derivable.

We now state two propositions:

- (2) For every relation R between $2^{S\text{-sequents}}$ and S -sequents such that $\langle S_4, S_6 \rangle \in R$ holds $S_6 \in (\text{FuncRule } R)(S_4)$.
- (3) If $x \in R(X)$, then x is $(1, X, \{R\})$ -derivable.

Let us consider S, D, X . Let us observe that X is D -expanded if and only if:

(Def. 61) If x is (X, D) -provable, then $x \in X$.

The following four propositions are true:

- (4) If $p_2 \in X$, then p_2 is $(X, \{R0\ S\})$ -provable.
- (5) Suppose that
- (i) $D_1 \cup D_2$ is isotone,

- (ii) $D_1 \cup D_2 \cup D_3$ is isotone,
- (iii) x is (m, S_{11}, D_1) -derivable,
- (iv) y is (m, S_5, D_2) -derivable, and
- (v) z is $(n, \{x, y\}, D_3)$ -derivable.

Then z is $(m + n, S_{11} \cup S_5, D_1 \cup D_2 \cup D_3)$ -derivable.

- (6) Suppose D_1 is isotone and $D_1 \cup D_2$ is isotone and y is (m, X, D_1) -derivable and z is $(n, \{y\}, D_2)$ -derivable. Then z is $(m + n, X, D_1 \cup D_2)$ -derivable.
- (7) If x is (m, X, D) -derivable, then $\{x\}$ is (X, D) -derivable.

Let us consider S . Observe that R6 S is isotone.

One can prove the following propositions:

- (8) If $D_1 \subseteq D_2$ and if D_1 is isotone or D_2 is isotone and if x is (X, D_1) -provable, then x is (X, D_2) -provable.
- (9) If $X \subseteq Y$ and x is (X, D) -provable, then x is (Y, D) -provable.

Let us consider S . Note that R8 S is isotone.

Let us consider S . Observe that R1 S is isotone.

Next we state the proposition

- (10) If $\{y\}$ is (S_4, D) -derivable, then there exists m_1 such that y is (m_1, S_4, D) -derivable.

Let us consider S, D, X . Observe that every set which is (X, D) -derivable is also S -sequents-like.

Let us consider S, D, X, x . Let us observe that x is (X, D) -provable if and only if:

- (Def. 62) There exists a set H and there exists m such that $H \subseteq X$ and $\langle H, x \rangle$ is (m, \emptyset, D) -derivable.

The following proposition is true

- (11) If $D_1 \subseteq D_2$ and if D_2 is isotone or D_1 is isotone and if x is (m, X, D_1) -derivable, then x is (m, X, D_2) -derivable.

Let us consider S . Observe that R7 S is isotone.

Next we state the proposition

- (12) If x is (X, D) -provable, then x is a w.f.f. string of S .

In the sequel F denotes a rule set of S .

Let us consider S, D_1 and let X be a D_1 -expanded set. One can verify that (S, X) -freeInterpreter is (X, D_1) -termEq-respecting.

Let us consider S , let D be a 0-ranked rule set of S , and let X be a D -expanded set. The functor D Henkin X yielding a function is defined by:

- (Def. 63) D Henkin $X = (S, X)$ -freeInterpreter quotient (X, D) -termEq.

Let us consider S , let D be a 0-ranked rule set of S , and let X be a D -expanded set. One can check that D Henkin X is OwnSymbolsOf S -defined.

Let us consider S , D_1 and let X be a D_1 -expanded set. Observe that D_1 Henkin X is $(S, \text{Classes}(X, D_1)\text{-termEq})$ -interpreter-like.

Let us consider S , D_1 and let X be a D_1 -expanded set. Then D_1 Henkin X is an element of $\text{Classes}((X, D_1)\text{-termEq})\text{-InterpretersOf } S$.

Let us consider S , p_3 , p_4 . One can verify that $\langle \text{TheNorSymbOf } S \rangle \wedge p_3 \wedge p_4$ is $(\{\text{xnot } p_3, \text{xnot } p_4\}, \{\text{R0 } S\} \cup \{\text{R6 } S\})$ -provable.

Let us consider S . Note that every 0-ranked rule set of S is non empty.

Let us consider S , x . We say that x is S -premises-like if and only if:

(Def. 64) $x \subseteq \text{AllFormulasOf } S$ and x is finite.

Let us consider S . One can verify that every set which is S -premises-like is also finite.

Let us consider S , p_2 . Note that $\{p_2\}$ is S -premises-like.

Let us consider S and let e be an empty set. One can check that $e \text{ null } S$ is S -premises-like.

Let us consider X , S . Observe that there exists a subset of X which is S -premises-like.

Let us consider S . Observe that there exists a set which is S -premises-like.

Let us consider S and let X be an S -premises-like set. Observe that every subset of X is S -premises-like.

In the sequel H_3 denotes an S -premises-like set.

Let us consider S , H_2 , H_1 . Then $H_1 \text{ null } H_2$ is a subset of $\text{AllFormulasOf } S$.

Let us consider S , H , x . Note that $H \text{ null } x$ is S -premises-like.

Let us consider S , H_1 , H_2 . Note that $H_1 \cup H_2$ is S -premises-like.

Let us consider S , H , p_2 . Observe that $\langle H, p_2 \rangle$ is S -sequent-like.

Let us consider S , H_1 , H_2 , p_2 . One can verify that $\langle H_1 \cup H_2, p_2 \rangle$ is $(1, \{\langle H_1, p_2 \rangle\}, \{\text{R1 } S\})$ -derivable.

Let us consider S , H , p_2 , p_3 , p_4 . One can check that $\langle H \text{ null } p_3 \wedge p_4, \text{xnot } p_2 \rangle$ is $(1, \{\langle H \cup \{p_2\}, p_3 \rangle, \langle H \cup \{p_2\}, \langle \text{TheNorSymbOf } S \rangle \wedge p_3 \wedge p_4 \rangle\}, \{\text{R8 } S\})$ -derivable.

Let us consider S . One can verify that $\emptyset \text{ null } S$ is S -sequents-like.

Let us consider S , H , p_2 . Observe that $\langle H \cup \{p_2\}, p_2 \rangle$ is $(1, \emptyset, \{\text{R0 } S\})$ -derivable. Let us consider p_3 , p_4 . Note that $\langle H \text{ null } p_4, \text{xnot } p_3 \rangle$ is

$(2, \{\langle H, \langle \text{TheNorSymbOf } S \rangle \wedge p_3 \wedge p_4 \rangle\}, \{\text{R0 } S\} \cup \{\text{R1 } S\} \cup \{\text{R8 } S\})$ -derivable.

Let us consider S , H , p_3 , p_4 . Note that $\langle H, \langle \text{TheNorSymbOf } S \rangle \wedge p_4 \wedge p_3 \rangle$ is $(1, \{\langle H, \langle \text{TheNorSymbOf } S \rangle \wedge p_3 \wedge p_4 \rangle\}, \{\text{R7 } S\})$ -derivable.

Let us consider S , H , p_3 , p_4 . Observe that $\langle H \text{ null } p_3, \text{xnot } p_4 \rangle$ is $(3, \{\langle H, \langle \text{TheNorSymbOf } S \rangle \wedge p_3 \wedge p_4 \rangle\}, \{\text{R0 } S\} \cup \{\text{R1 } S\} \cup \{\text{R8 } S\} \cup \{\text{R7 } S\})$ -derivable.

Let us consider S , S_6 . Observe that $(S_6)_1$ is S -premises-like.

Let us consider S , X , D . Then $D \text{ null } X$ is a rule set of S .

Let us consider S , p_3 , p_4 , l_1 , H and let l_2 be an $H \cup \{p_3\} \cup \{p_4\}$ -absent literal element of S .

Note that $\langle (H \cup \{ \langle l_1 \rangle \wedge p_3 \}) \text{ null } l_2, p_4 \rangle$ is $(1, \{ \langle H \cup \{ (l_1, l_2) \text{-SymbolSubstIn } p_3 \}, p_4 \rangle \}, \{ \text{R5 } S \})$ -derivable.

Let us consider S, D, X . We say that X is D -inconsistent if and only if:

(Def. 65) There exist p_3, p_4 such that p_3 is (X, D) -provable and $\langle \text{TheNorSymbOf } S \rangle \wedge p_3 \wedge p_4$ is (X, D) -provable.

Let us consider m_2, S, H_1, H_2, p_2 . Note that $\langle (H_1 \cup H_2) \text{ null } m_2, p_2 \rangle$ is $(m_2, \{ \langle H_1, p_2 \rangle \}, \{ \text{R1 } S \})$ -derivable.

Let us consider S . Observe that there exists an isotone rule set of S which is non empty.

We now state the proposition

(13) If X is D -inconsistent and D is isotone and $\text{R1 } S, \text{R8 } S \in D$, then $\text{xnot } p_2$ is (X, D) -provable.

Let us consider S . Observe that $\text{R5 } S$ is isotone.

Let us consider S, l, t, p_2 . Observe that $\{ \langle (l, t) \text{SubstIn } p_2 \rangle, \langle l \rangle \wedge p_2 \}$ is $(1, \emptyset, \{ \text{R4 } S \})$ -derivable.

Let us consider S . One can verify that $\text{R4 } S$ is isotone.

Let us consider S, X . We say that X is S -witnessed if and only if:

(Def. 66) For all l_1, p_3 such that $\langle l_1 \rangle \wedge p_3 \in X$ there exists l_2 such that $\langle l_1, l_2 \rangle \text{-SymbolSubstIn } p_3 \in X$ and $l_2 \notin \text{rng } p_3$.

We now state the proposition

(14)³ Let X be a D_1 -expanded set. Suppose $\text{R1 } S, \text{R4 } S, \text{R6 } S, \text{R7 } S, \text{R8 } S \in D_1$ and X is S -mincover and S -witnessed. Then $(D_1 \text{Henkin } X)\text{-TruthEval } p_1 = 1$ if and only if $p_1 \in X$.

Let us consider S, D, X . We introduce X is D -consistent as an antonym of X is D -inconsistent.

We now state the proposition

(15) For every subset X of Y such that X is D -inconsistent holds Y is D -inconsistent.

Let us consider S, D , let X be a functional set, and let p_2 be an element of $\text{ExFormulasOf } S$. The functor $(D, p_2) \text{AddAsWitnessTo } X$ is defined by:

(Def. 67) $(D, p_2) \text{AddAsWitnessTo } X = \begin{cases} X \cup \{ (S\text{-firstChar}(p_2), \text{ the element of } \text{LettersOf } S \setminus \text{SymbolsOf } (((\text{AllSymbolsOf } S)^* \setminus \{ \emptyset \}) \cap (X \cup \{ \text{head } p_2 \}))) \text{-SymbolSubstIn head } p_2 \}, \\ \text{ if } X \cup \{ p_2 \} \text{ is } D\text{-consistent and } \\ \text{ LettersOf } S \setminus \text{SymbolsOf } (((\text{AllSym} \\ \text{bolsOf } S)^* \setminus \{ \emptyset \}) \cap (X \cup \{ \text{head } p_2 \})) \neq \emptyset, \\ X, \text{ otherwise.} \end{cases}$

³Henkin's Theorem

Let us consider S, D , let X be a functional set, and let p_2 be an element of $\text{ExFormulasOf } S$. One can check that $X \setminus ((D, p_2) \text{ AddAsWitnessTo } X)$ is empty.

Let us consider S, D , let X be a functional set, and let p_2 be an element of $\text{ExFormulasOf } S$. One can check that $((D, p_2) \text{ AddAsWitnessTo } X) \setminus X$ is trivial.

Let us consider S, D , let X be a functional set, and let p_2 be an element of $\text{ExFormulasOf } S$. Then $(D, p_2) \text{ AddAsWitnessTo } X$ is a subset of $X \cup \text{AllFormulasOf } S$.

Let us consider S, D . We say that D is correct if and only if the condition (Def. 68) is satisfied.

(Def. 68) Let given p_2, X . Suppose p_2 is (X, D) -provable. Let given U and I be an element of $U\text{-InterpretersOf } S$. If X is I -satisfied, then $I\text{-TruthEval } p_2 = 1$.

Let us consider S, t_1, t_2 . One can check that $\text{SubTerms}(\langle \text{TheEqSymbOf } S \rangle \wedge t_1 \wedge t_2) \div \langle t_1, t_2 \rangle$ is empty.

Let us consider S and let R be a rule of S . We say that R is correct if and only if:

(Def. 69) If X is S -correct, then $R(X)$ is S -correct.

Let us consider S . Observe that every set which is S -sequent-like is also S -null.

Let us consider S . Note that $R0 S$ is correct.

Let us consider S . Note that there exists a rule of S which is correct.

Let us consider S . One can check that $R1 S$ is correct.

Let us consider S . Note that $R2 S$ is correct.

Let us consider S . One can check that $R3a S$ is correct.

Let us consider S . Observe that $R3b S$ is correct.

Let us consider S . Observe that $R3d S$ is correct.

Let us consider S . Note that $R3e S$ is correct.

Let us consider S . One can check that $R4 S$ is correct.

Let us consider S . One can check that $R5 S$ is correct.

Let us consider S . One can verify that $R6 S$ is correct.

Let us consider S . Observe that $R7 S$ is correct.

Let us consider S . Observe that $R8 S$ is correct.

Next we state the proposition

(16) If for every rule R of S such that $R \in D$ holds R is correct, then D is correct.

Let us consider S and let R be a correct rule of S . Note that $\{R\}$ is correct. Observe that S -rules is correct. One can check that $R9 S$ is isotone. Let us consider H, p_2 . Observe that $\langle H, p_2 \rangle \text{ null } 1$ is $(1, \{\langle H, \text{xnot xnot } p_2 \rangle\}, \{R9 S\})$ -derivable.

Let us consider X, S . Observe that there exists an 0-w.f.f. string of S which is X -implied.

Let us consider X, S . Observe that there exists a w.f.f. string of S which is X -implied.

Let us consider S, X and let p_2 be an X -implied w.f.f. string of S . Observe that $\text{xnot xnot } p_2$ is X -implied.

Let us consider X, S, p_2 . We say that p_2 is X -provable if and only if:

(Def. 70) p_2 is $(X, \{\text{R9 } S\} \cup S\text{-rules})$ -provable.

2. CONSTRUCTIONS FOR COUNTABLE LANGUAGES: WITNESS ADJOINING

Let X be a functional set, let us consider S, D , and let n_1 be a function from \mathbb{N} into $\text{ExFormulasOf } S$. The functor $(D, n_1) \text{ AddWitnessesTo } X$ yields a function from \mathbb{N} into $2^{X \cup \text{AllFormulasOf } S}$ and is defined by:

(Def. 71) $((D, n_1) \text{ AddWitnessesTo } X)(0) = X$ and
 for every m_1 holds $((D, n_1) \text{ AddWitnessesTo } X)(m_1 + 1) =$
 $(D, n_1(m_1)) \text{ AddAsWitnessTo}((D, n_1) \text{ AddWitnessesTo } X)(m_1)$.

Let X be a functional set, let us consider S, D , and let n_1 be a function from \mathbb{N} into $\text{ExFormulasOf } S$. We introduce $(D, n_1) \text{ addw } X$ as a synonym of $(D, n_1) \text{ AddWitnessesTo } X$.

We now state the proposition

(17) Let X be a functional set and n_1 be a function from \mathbb{N} into $\text{ExFormulasOf } S$. Suppose D is isotone and $\text{R1 } S, \text{R8 } S, \text{R2 } S, \text{R5 } S \in D$ and $\text{LettersOf } S \setminus \text{SymbolsOf}(X \cap ((\text{AllSymbolsOf } S)^* \setminus \{\emptyset\}))$ is infinite and X is D -consistent. Then $((D, n_1) \text{ addw } X)(k) \subseteq ((D, n_1) \text{ addw } X)(k + m)$ and $\text{LettersOf } S \setminus \text{SymbolsOf}(((D, n_1) \text{ addw } X)(m) \cap ((\text{AllSymbolsOf } S)^* \setminus \{\emptyset\}))$ is infinite and $((D, n_1) \text{ addw } X)(m)$ is D -consistent.

Let X be a functional set, let us consider S, D , and let n_1 be a function from \mathbb{N} into $\text{ExFormulasOf } S$. The functor $X \text{ WithWitnessesFrom}(D, n_1)$ yielding a subset of $X \cup \text{AllFormulasOf } S$ is defined by:

(Def. 72) $X \text{ WithWitnessesFrom}(D, n_1) = \bigcup \text{rng}((D, n_1) \text{ AddWitnessesTo } X)$.

Let X be a functional set, let us consider S, D , and let n_1 be a function from \mathbb{N} into $\text{ExFormulasOf } S$. We introduce $X \text{ addw}(D, n_1)$ as a synonym of $X \text{ WithWitnessesFrom}(D, n_1)$.

Let X be a functional set, let us consider S, D , and let n_1 be a function from \mathbb{N} into $\text{ExFormulasOf } S$. One can verify that $X \setminus (X \text{ addw}(D, n_1))$ is empty.

The following proposition is true

(18) Let X be a functional set and n_1 be a function from \mathbb{N} into $\text{ExFormulasOf } S$. Suppose that D is isotone and $\text{R1 } S, \text{R8 } S, \text{R2 } S, \text{R5 } S \in D$ and $\text{LettersOf } S \setminus \text{SymbolsOf}(X \cap ((\text{AllSymbolsOf } S)^* \setminus \{\emptyset\}))$ is infinite and $X \text{ addw}(D, n_1) \subseteq Z$ and Z is D -consistent and $\text{rng } n_1 = \text{ExFormulasOf } S$. Then Z is S -witnessed.

3. CONSTRUCTIONS FOR COUNTABLE LANGUAGES: CONSISTENTLY
MAXIMIZING A SET OF FORMULAS OF A COUNTABLE LANGUAGE
(LINDENBAUM'S LEMMA)

Let us consider X, S, D and let p_2 be an element of $\text{AllFormulasOf } S$. The functor $(D, p_2) \text{ AddFormulaTo } X$ is defined by:

$$\text{(Def. 73)} \quad (D, p_2) \text{ AddFormulaTo } X = \begin{cases} X \cup \{p_2\}, \\ \quad \text{if } \text{xnot } p_2 \text{ is not } (X, D)\text{-provable,} \\ X \cup \{\text{xnot } p_2\}, \text{ otherwise.} \end{cases}$$

Let us consider X, S, D and let p_2 be an element of $\text{AllFormulasOf } S$. Then $(D, p_2) \text{ AddFormulaTo } X$ is a subset of $X \cup \text{AllFormulasOf } S$.

Let us consider X, S, D and let p_2 be an element of $\text{AllFormulasOf } S$. Note that $X \setminus ((D, p_2) \text{ AddFormulaTo } X)$ is empty.

Let us consider X, S, D and let n_1 be a function from \mathbb{N} into $\text{AllFormulasOf } S$. The functor $(D, n_1) \text{ AddFormulasTo } X$ yields a function from \mathbb{N} into

$2^{X \cup \text{AllFormulasOf } S}$ and is defined by:

$$\text{(Def. 74)} \quad \begin{aligned} ((D, n_1) \text{ AddFormulasTo } X)(0) &= X \text{ and for every } m \text{ holds} \\ ((D, n_1) \text{ AddFormulasTo } X)(m+1) &= \\ (D, n_1(m)) \text{ AddFormulaTo} &((D, n_1) \text{ AddFormulasTo } X)(m). \end{aligned}$$

Let us consider X, S, D and let n_1 be a function from \mathbb{N} into $\text{AllFormulasOf } S$. The functor $(D, n_1) \text{ CompletionOf } X$ yields a subset of $X \cup \text{AllFormulasOf } S$ and is defined as follows:

$$\text{(Def. 75)} \quad (D, n_1) \text{ CompletionOf } X = \bigcup \text{rng}((D, n_1) \text{ AddFormulasTo } X).$$

Let us consider X, S, D and let n_1 be a function from \mathbb{N} into $\text{AllFormulasOf } S$. One can check that $X \setminus ((D, n_1) \text{ CompletionOf } X)$ is empty.

We now state the proposition

$$\text{(19)} \quad \text{For every relation } R \text{ between } 2^{S\text{-sequents}} \text{ and } S\text{-sequents holds } y \in (\text{FuncRule } R)(X) \text{ iff } y \in S\text{-sequents and } \langle X, y \rangle \in R.$$

In the sequel D_2 is a 2-ranked rule set of S .

Let us consider S and let r_1, r_2 be isotone rules of S . Note that $\{r_1, r_2\}$ is isotone.

Let us consider S and let r_1, r_2, r_3, r_4 be isotone rules of S . Observe that $\{r_1, r_2, r_3, r_4\}$ is isotone.

Let us consider S . Observe that S -rules is isotone.

Let us consider S . Observe that there exists an isotone rule set of S which is correct.

Let us consider S . Observe that there exists a correct isotone rule set of S which is 2-ranked.

Let S be a countable language. Observe that $\text{AllFormulasOf } S$ is countable.

We now state the proposition

- (20) Let S be a countable language and D be a rule set of S . Suppose D is 2-ranked, isotone, and correct and Z is D -consistent and $Z \subseteq \text{AllFormulasOf } S$. Then there exists a non empty set U and there exists an element I of U -InterpretersOf S such that Z is I -satisfied.

In the sequel C denotes a countable language and p_2 denotes a w.f.f. string of C .

We now state the proposition

- (21) If $X \subseteq \text{AllFormulasOf } C$ and p_2 is X -implied, then p_2 is X -provable.

REFERENCES

- [1] Grzegorz Bancerek. Cardinal numbers. *Formalized Mathematics*, 1(2):377–382, 1990.
- [2] Grzegorz Bancerek. The fundamental properties of natural numbers. *Formalized Mathematics*, 1(1):41–46, 1990.
- [3] Grzegorz Bancerek. König’s theorem. *Formalized Mathematics*, 1(3):589–593, 1990.
- [4] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Formalized Mathematics*, 1(1):107–114, 1990.
- [5] Grzegorz Bancerek and Andrzej Trybulec. Miscellaneous facts about functions. *Formalized Mathematics*, 5(4):485–492, 1996.
- [6] Czesław Byliński. Finite sequences and tuples of elements of a non-empty sets. *Formalized Mathematics*, 1(3):529–536, 1990.
- [7] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(1):55–65, 1990.
- [8] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(1):153–164, 1990.
- [9] Czesław Byliński. Partial functions. *Formalized Mathematics*, 1(2):357–367, 1990.
- [10] Czesław Byliński. Some basic properties of sets. *Formalized Mathematics*, 1(1):47–53, 1990.
- [11] Marco B. Caminati. Preliminaries to classical first order model theory. *Formalized Mathematics*, 19(3):155–167, 2011, doi: 10.2478/v10037-011-0025-2.
- [12] Marco B. Caminati. Definition of first order language with arbitrary alphabet. Syntax of terms, atomic formulas and their subterms. *Formalized Mathematics*, 19(3):169–178, 2011, doi: 10.2478/v10037-011-0026-1.
- [13] Marco B. Caminati. First order languages: Further syntax and semantics. *Formalized Mathematics*, 19(3):179–192, 2011, doi: 10.2478/v10037-011-0027-0.
- [14] Marco B. Caminati. Free interpretation, quotient interpretation and substitution of a letter with a term for first order languages. *Formalized Mathematics*, 19(3):193–203, 2011, doi: 10.2478/v10037-011-0028-z.
- [15] M.B. Caminati. Yet another proof of Goedel’s completeness theorem for first-order classical logic. *Arxiv preprint arXiv:0910.2059*, 2009.
- [16] Patricia L. Carlson and Grzegorz Bancerek. Context-free grammar – part I. *Formalized Mathematics*, 2(5):683–687, 1991.
- [17] Agata Darmochwał. Finite sets. *Formalized Mathematics*, 1(1):165–167, 1990.
- [18] H.D. Ebbinghaus, J. Flum, and W. Thomas. *Mathematical logic*. Springer, 1994.
- [19] Rafał Kwiatek and Grzegorz Zwara. The divisibility of integers and integer relative primes. *Formalized Mathematics*, 1(5):829–832, 1990.
- [20] Beata Padlewska. Families of sets. *Formalized Mathematics*, 1(1):147–152, 1990.
- [21] Marta Pruszyńska and Marek Dudzicz. On the isomorphism between finite chains. *Formalized Mathematics*, 9(2):429–430, 2001.
- [22] Konrad Raczkowski and Paweł Sadowski. Equivalence relations and classes of abstraction. *Formalized Mathematics*, 1(3):441–444, 1990.
- [23] Andrzej Trybulec. Domains and their Cartesian products. *Formalized Mathematics*, 1(1):115–122, 1990.
- [24] Andrzej Trybulec. Tuples, projections and Cartesian products. *Formalized Mathematics*, 1(1):97–105, 1990.

- [25] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(1):67–71, 1990.
- [26] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1(1):73–83, 1990.
- [27] Edmund Woronowicz. Relations defined on sets. *Formalized Mathematics*, 1(1):181–186, 1990.
- [28] Edmund Woronowicz and Anna Zalewska. Properties of binary relations. *Formalized Mathematics*, 1(1):85–89, 1990.

Received December 29, 2010
