Cayley's Theorem

Artur Korniłowicz
Institute of Informatics
University of Białystok
Sosnowa 64, 15-887 Białystok, Poland

Summary. The article formalizes the Cayley's theorem saying that every group G is isomorphic to a subgroup of the symmetric group on G.

MML identifier: `CAYLEY`, version: `7.11.07 4.160.1126`

The notation and terminology used in this paper have been introduced in the following papers: [3], [6], [4], [5], [10], [11], [7], [2], [1], [9], and [8].

In this paper X, Y denote sets, G denotes a group, and n denotes a natural number.

Let us consider X. Note that $\emptyset_{X,\emptyset}$ is onto.

Let us observe that every set which is permutational is also functional.

Let us consider X. The functor permutations X is defined as follows:

(Def. 1)   permutations $X = \{f : f$ ranges over permutations of $X\}$.

Next we state three propositions:

(1)   For every set f such that $f \in$ permutations X holds f is a permutation of X.

(2)   permutations $X \subseteq X^X$.

(3)   permutations $\operatorname{Seg} n =$ the permutations of n.

Let us consider X. One can verify that permutations X is non empty and functional.

Let X be a finite set. One can verify that permutations X is finite.

Next we state the proposition

(4)   permutations $\emptyset = 1$.

Let us consider X. The functor SymGroup X yields a strict constituted functions multiplicative magma and is defined by:

(Def. 2)   The carrier of $\mathrm{SymGroup}\, X = $ permutations X and for all elements x, y
of $\mathrm{SymGroup}\, X$ holds $x \cdot y = (y\ \mathbf{qua}\ \text{function}) \cdot x$.

One can prove the following proposition

(5)   Every element of $\mathrm{SymGroup}\, X$ is a permutation of X.

Let us consider X. Note that $\mathrm{SymGroup}\, X$ is non empty, associative, and
group-like.

The following propositions are true:

(6) $\mathbf{1}_{\mathrm{SymGroup}\, X} = \mathrm{id}_X$.

(7)   For every element x of $\mathrm{SymGroup}\, X$ holds $x^{-1} = (x\ \mathbf{qua}\ \text{function})^{-1}$.

Let us consider n. One can verify that A_n is constituted functions.

One can prove the following proposition

(8) $\mathrm{SymGroup}\,\mathrm{Seg}\, n = A_n$.

Let X be a finite set. Observe that $\mathrm{SymGroup}\, X$ is finite.

We now state the proposition

(9) $\mathrm{SymGroup}\, \emptyset = \mathrm{Trivial\text{-}multMagma}$.

Let us note that $\mathrm{SymGroup}\, \emptyset$ is trivial.

Let us consider X, Y and let p be a function from X into Y. Let us assume
that $X \neq \emptyset$ and $Y \neq \emptyset$ and p is bijective. The functor $\mathrm{SymGroupsIso}\, p$ yielding
a function from $\mathrm{SymGroup}\, X$ into $\mathrm{SymGroup}\, Y$ is defined by:

(Def. 3)   For every element x of $\mathrm{SymGroup}\, X$ holds $(\mathrm{SymGroupsIso}\, p)(x) = p \cdot x \cdot p^{-1}$.

We now state four propositions:

(10)   For all non empty sets X, Y and for every function p from X into Y
such that p is bijective holds $\mathrm{SymGroupsIso}\, p$ is multiplicative.

(11)   For all non empty sets X, Y and for every function p from X into Y
such that p is bijective holds $\mathrm{SymGroupsIso}\, p$ is one-to-one.

(12)   For all non empty sets X, Y and for every function p from X into Y
such that p is bijective holds $\mathrm{SymGroupsIso}\, p$ is onto.

(13)   If $X \approx Y$, then $\mathrm{SymGroup}\, X$ and $\mathrm{SymGroup}\, Y$ are isomorphic.

Let us consider G. The functor $\mathrm{CayleyIso}\, G$ yields a function from G into
$\mathrm{SymGroup}\,(\text{the carrier of } G)$ and is defined as follows:

(Def. 4)   For every element g of G holds $(\mathrm{CayleyIso}\, G)(g) = \cdot g$.

Let us consider G. One can verify that $\mathrm{CayleyIso}\, G$ is multiplicative.

Let us consider G. One can verify that $\mathrm{CayleyIso}\, G$ is one-to-one.

One can prove the following proposition

(14) G and $\mathrm{Im}\, \mathrm{CayleyIso}\, G$ are isomorphic.

References

[1] Grzegorz Bancerek. The fundamental properties of natural numbers. *Formalized Mathematics*, 1(**1**):41–46, 1990.

[2] Grzegorz Bancerek. Monoids. *Formalized Mathematics*, 3(**2**):213–225, 1992.

[3] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Formalized Mathematics*, 1(**1**):107–114, 1990.

[4] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.

[5] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.

[6] Agata Darmochwał. Finite sets. *Formalized Mathematics*, 1(**1**):165–167, 1990.

[7] Katarzyna Jankowska. Transpose matrices and groups of permutations. *Formalized Mathematics*, 2(**5**):711–717, 1991.

[8] Artur Korniłowicz. The definition and basic properties of topological groups. *Formalized Mathematics*, 7(**2**):217–225, 1998.

[9] Andrzej Trybulec. Classes of independent partitions. *Formalized Mathematics*, 9(**3**):623–625, 2001.

[10] Wojciech A. Trybulec. Groups. *Formalized Mathematics*, 1(**5**):821–827, 1990.

[11] Wojciech A. Trybulec and Michał J. Trybulec. Homomorphisms and isomorphisms of groups. Quotient group. *Formalized Mathematics*, 2(**4**):573–578, 1991.

Received December 29, 2010

————