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# More on the Continuity of Real Functions[1]

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**Summary.** In this article we demonstrate basic properties of the continuous functions from $\mathbb{R}$ to $\mathcal{R}^n$ which correspond to state space equations in control engineering.

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The terminology and notation used here have been introduced in the following articles: [3], [7], [17], [2], [4], [12], [13], [14], [16], [1], [5], [9], [15], [18], [10], [8], [20], [21], [19], [11], [22], and [6].

For simplicity, we use the following convention: $n$, $i$ denote elements of $\mathbb{N}$, $X$, $X_1$ denote sets, $r$, $p$, $s$, $x_0$, $x_1$, $x_2$ denote real numbers, $f$, $f_1$, $f_2$ denote partial functions from $\mathbb{R}$ to $\mathcal{R}^n$, and $h$ denotes a partial function from $\mathbb{R}$ to the carrier of $\langle \mathcal{E}^n, \|\cdot\| \rangle$.

Let us consider $n$, $f$, $x_0$. We say that $f$ is continuous in $x_0$ if and only if:

(Def. 1) There exists a partial function $g$ from $\mathbb{R}$ to the carrier of $\langle \mathcal{E}^n, \|\cdot\| \rangle$ such that $f = g$ and $g$ is continuous in $x_0$.

We now state four propositions:

(1) If $h = f$, then $f$ is continuous in $x_0$ iff $h$ is continuous in $x_0$.

(2) If $x_0 \in X$ and $f$ is continuous in $x_0$, then $f{\restriction}X$ is continuous in $x_0$.

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(3) $f$ is continuous in $x_0$ if and only if the following conditions are satisfied:

(i) $x_0 \in \operatorname{dom} f$, and

(ii) for every $r$ such that $0 < r$ there exists $s$ such that $0 < s$ and for every $x_1$ such that $x_1 \in \operatorname{dom} f$ and $|x_1 - x_0| < s$ holds $|f_{x_1} - f_{x_0}| < r$.

(4) Let $r$ be a real number, $z$ be an element of $\mathcal{R}^n$, and $w$ be a point of $\langle \mathcal{E}^n, \|\cdot\| \rangle$. Suppose $z = w$. Then $\{y \in \mathcal{R}^n \colon |y - z| < r\} = \{y; y$ ranges over points of $\langle \mathcal{E}^n, \|\cdot\| \rangle \colon \|y - w\| < r\}$.

Let $n$ be an element of $\mathbb{N}$, let $Z$ be a set, and let $f$ be a partial function from $Z$ to $\mathcal{R}^n$. The functor $|f|$ yielding a partial function from $Z$ to $\mathbb{R}$ is defined by:

(Def. 2) $\operatorname{dom}|f| = \operatorname{dom} f$ and for every set $x$ such that $x \in \operatorname{dom}|f|$ holds $|f|_x = |f_x|$.

Let $n$ be an element of $\mathbb{N}$, let $Z$ be a non empty set, and let $f$ be a partial function from $Z$ to $\mathcal{R}^n$. The functor $-f$ yields a partial function from $Z$ to $\mathcal{R}^n$ and is defined by:

(Def. 3) $\operatorname{dom}(-f) = \operatorname{dom} f$ and for every set $c$ such that $c \in \operatorname{dom}(-f)$ holds $(-f)_c = -f_c$.

One can prove the following propositions:

(5) Let $f_1$, $f_2$ be partial functions from $\mathbb{R}$ to the carrier of $\langle \mathcal{E}^n, \|\cdot\| \rangle$ and $g_1$, $g_2$ be partial functions from $\mathbb{R}$ to $\mathcal{R}^n$. If $f_1 = g_1$ and $f_2 = g_2$, then $f_1 + f_2 = g_1 + g_2$.

(6) Let $f_1$ be a partial function from $\mathbb{R}$ to the carrier of $\langle \mathcal{E}^n, \|\cdot\| \rangle$, $g_1$ be a partial function from $\mathbb{R}$ to $\mathcal{R}^n$, and $a$ be a real number. If $f_1 = g_1$, then $a \cdot f_1 = a \cdot g_1$.

(7) For every partial function $f_1$ from $\mathbb{R}$ to $\mathcal{R}^n$ holds $(-1) \cdot f_1 = -f_1$.

(8) Let $f_1$ be a partial function from $\mathbb{R}$ to the carrier of $\langle \mathcal{E}^n, \|\cdot\| \rangle$ and $g_1$ be a partial function from $\mathbb{R}$ to $\mathcal{R}^n$. If $f_1 = g_1$, then $-f_1 = -g_1$.

(9) Let $f_1$ be a partial function from $\mathbb{R}$ to the carrier of $\langle \mathcal{E}^n, \|\cdot\| \rangle$ and $g_1$ be a partial function from $\mathbb{R}$ to $\mathcal{R}^n$. If $f_1 = g_1$, then $\|f_1\| = |g_1|$.

(10) Let $f_1$, $f_2$ be partial functions from $\mathbb{R}$ to the carrier of $\langle \mathcal{E}^n, \|\cdot\| \rangle$ and $g_1$, $g_2$ be partial functions from $\mathbb{R}$ to $\mathcal{R}^n$. If $f_1 = g_1$ and $f_2 = g_2$, then $f_1 - f_2 = g_1 - g_2$.

(11) $f$ is continuous in $x_0$ if and only if the following conditions are satisfied:

(i) $x_0 \in \operatorname{dom} f$, and

(ii) for every subset $N_1$ of $\mathcal{R}^n$ such that there exists a real number $r$ such that $0 < r$ and $\{y \in \mathcal{R}^n \colon |y - f_{x_0}| < r\} = N_1$ there exists a neighbourhood $N$ of $x_0$ such that for every $x_1$ such that $x_1 \in \operatorname{dom} f$ and $x_1 \in N$ holds $f_{x_1} \in N_1$.

(12) $f$ is continuous in $x_0$ if and only if the following conditions are satisfied:

(i) $x_0 \in \operatorname{dom} f$, and

(ii) for every subset $N_1$ of $\mathcal{R}^n$ such that there exists a real number $r$ such that $0 < r$ and $\{y \in \mathcal{R}^n: |y - f_{x_0}| < r\} = N_1$ there exists a neighbourhood $N$ of $x_0$ such that $f^\circ N \subseteq N_1$.

(13) If there exists a neighbourhood $N$ of $x_0$ such that $\mathrm{dom}\, f \cap N = \{x_0\}$, then $f$ is continuous in $x_0$.

(14) If $x_0 \in \mathrm{dom}\, f_1 \cap \mathrm{dom}\, f_2$ and $f_1$ is continuous in $x_0$ and $f_2$ is continuous in $x_0$, then $f_1 + f_2$ is continuous in $x_0$.

(15) If $x_0 \in \mathrm{dom}\, f_1 \cap \mathrm{dom}\, f_2$ and $f_1$ is continuous in $x_0$ and $f_2$ is continuous in $x_0$, then $f_1 - f_2$ is continuous in $x_0$.

(16) If $f$ is continuous in $x_0$, then $r \cdot f$ is continuous in $x_0$.

(17) If $x_0 \in \mathrm{dom}\, f$ and $f$ is continuous in $x_0$, then $|f|$ is continuous in $x_0$.

(18) If $x_0 \in \mathrm{dom}\, f$ and $f$ is continuous in $x_0$, then $-f$ is continuous in $x_0$.

(19) Let $S$ be a real normed space, $z$ be a point of $\langle \mathcal{E}^n, \|\cdot\| \rangle$, $f_1$ be a partial function from $\mathbb{R}$ to $\mathcal{R}^n$, and $f_2$ be a partial function from the carrier of $\langle \mathcal{E}^n, \|\cdot\| \rangle$ to the carrier of $S$. Suppose $x_0 \in \mathrm{dom}(f_2 \cdot f_1)$ and $f_1$ is continuous in $x_0$ and $z = (f_1)_{x_0}$ and $f_2$ is continuous in $z$. Then $f_2 \cdot f_1$ is continuous in $x_0$.

(20) Let $S$ be a real normed space, $f_1$ be a partial function from $\mathbb{R}$ to the carrier of $S$, and $f_2$ be a partial function from the carrier of $S$ to $\mathbb{R}$. Suppose $x_0 \in \mathrm{dom}(f_2 \cdot f_1)$ and $f_1$ is continuous in $x_0$ and $f_2$ is continuous in $(f_1)_{x_0}$. Then $f_2 \cdot f_1$ is continuous in $x_0$.

Let us consider $n$, let $f$ be a partial function from $\mathcal{R}^n$ to $\mathbb{R}$, and let $x_0$ be an element of $\mathcal{R}^n$. We say that $f$ is continuous in $x_0$ if and only if the condition (Def. 4) is satisfied.

(Def. 4) There exists a point $y_0$ of $\langle \mathcal{E}^n, \|\cdot\| \rangle$ and there exists a partial function $g$ from the carrier of $\langle \mathcal{E}^n, \|\cdot\| \rangle$ to $\mathbb{R}$ such that $x_0 = y_0$ and $f = g$ and $g$ is continuous in $y_0$.

One can prove the following two propositions:

(21) Let $f$ be a partial function from $\mathcal{R}^n$ to $\mathbb{R}$, $h$ be a partial function from the carrier of $\langle \mathcal{E}^n, \|\cdot\| \rangle$ to $\mathbb{R}$, $x_0$ be an element of $\mathcal{R}^n$, and $y_0$ be a point of $\langle \mathcal{E}^n, \|\cdot\| \rangle$. Suppose $f = h$ and $x_0 = y_0$. Then $f$ is continuous in $x_0$ if and only if $h$ is continuous in $y_0$.

(22) Let $f_1$ be a partial function from $\mathbb{R}$ to $\mathcal{R}^n$ and $f_2$ be a partial function from $\mathcal{R}^n$ to $\mathbb{R}$. Suppose $x_0 \in \mathrm{dom}(f_2 \cdot f_1)$ and $f_1$ is continuous in $x_0$ and $f_2$ is continuous in $(f_1)_{x_0}$. Then $f_2 \cdot f_1$ is continuous in $x_0$.

Let us consider $n$, $f$. We say that $f$ is continuous if and only if:

(Def. 5) For every $x_0$ such that $x_0 \in \mathrm{dom}\, f$ holds $f$ is continuous in $x_0$.

One can prove the following propositions:

(23) Let $g$ be a partial function from $\mathbb{R}$ to the carrier of $\langle \mathcal{E}^n, \|\cdot\| \rangle$ and $f$ be a partial function from $\mathbb{R}$ to $\mathcal{R}^n$. If $g = f$, then $g$ is continuous iff $f$ is continuous.

(24) Suppose $X \subseteq \operatorname{dom} f$. Then $f \restriction X$ is continuous if and only if for all $x_0$, $r$ such that $x_0 \in X$ and $0 < r$ there exists $s$ such that $0 < s$ and for every $x_1$ such that $x_1 \in X$ and $|x_1 - x_0| < s$ holds $|f_{x_1} - f_{x_0}| < r$.

Let us consider $n$. Observe that every partial function from $\mathbb{R}$ to $\mathcal{R}^n$ which is constant is also continuous.

Let us consider $n$. Observe that there exists a partial function from $\mathbb{R}$ to $\mathcal{R}^n$ which is continuous.

Let us consider $n$, let $f$ be a continuous partial function from $\mathbb{R}$ to $\mathcal{R}^n$, and let $X$ be a set. One can verify that $f \restriction X$ is continuous.

One can prove the following proposition

(25) If $f \restriction X$ is continuous and $X_1 \subseteq X$, then $f \restriction X_1$ is continuous.

Let us consider $n$. Note that every partial function from $\mathbb{R}$ to $\mathcal{R}^n$ which is empty is also continuous.

Let us consider $n$, $f$ and let $X$ be a trivial set. One can verify that $f \restriction X$ is continuous.

Let us consider $n$ and let $f_1$, $f_2$ be continuous partial functions from $\mathbb{R}$ to $\mathcal{R}^n$. One can check that $f_1 + f_2$ is continuous.

The following propositions are true:

(26) If $X \subseteq \operatorname{dom} f_1 \cap \operatorname{dom} f_2$ and $f_1 \restriction X$ is continuous and $f_2 \restriction X$ is continuous, then $(f_1 + f_2) \restriction X$ is continuous and $(f_1 - f_2) \restriction X$ is continuous.

(27) If $X \subseteq \operatorname{dom} f_1$ and $X_1 \subseteq \operatorname{dom} f_2$ and $f_1 \restriction X$ is continuous and $f_2 \restriction X_1$ is continuous, then $(f_1 + f_2) \restriction (X \cap X_1)$ is continuous and $(f_1 - f_2) \restriction (X \cap X_1)$ is continuous.

Let us consider $n$, let $f$ be a continuous partial function from $\mathbb{R}$ to $\mathcal{R}^n$, and let us consider $r$. Observe that $r \cdot f$ is continuous.

The following propositions are true:

(28) If $X \subseteq \operatorname{dom} f$ and $f \restriction X$ is continuous, then $(r \cdot f) \restriction X$ is continuous.

(29) If $X \subseteq \operatorname{dom} f$ and $f \restriction X$ is continuous, then $|f| \restriction X$ is continuous and $(-f) \restriction X$ is continuous.

(30) If $f$ is total and for all $x_1$, $x_2$ holds $f_{x_1 + x_2} = f_{x_1} + f_{x_2}$ and there exists $x_0$ such that $f$ is continuous in $x_0$, then $f \restriction \mathbb{R}$ is continuous.

(31) For every subset $Y$ of $\langle \mathcal{E}^n, \|\cdot\| \rangle$ such that $\operatorname{dom} f$ is compact and $f \restriction \operatorname{dom} f$ is continuous and $Y = \operatorname{rng} f$ holds $Y$ is compact.

(32) Let $Y$ be a subset of $\mathbb{R}$ and $Z$ be a subset of $\langle \mathcal{E}^n, \|\cdot\| \rangle$. Suppose $Y \subseteq \operatorname{dom} f$ and $Z = f^\circ Y$ and $Y$ is compact and $f \restriction Y$ is continuous. Then $Z$ is compact.

Let us consider $n$, $f$. We say that $f$ is Lipschitzian if and only if:

(Def. 6) There exists a partial function $g$ from $\mathbb{R}$ to the carrier of $\langle \mathcal{E}^n, \|\cdot\| \rangle$ such that $g = f$ and $g$ is Lipschitzian.

The following propositions are true:

(33) $f$ is Lipschitzian if and only if there exists a real number $r$ such that $0 < r$ and for all $x_1$, $x_2$ such that $x_1, x_2 \in \operatorname{dom} f$ holds $|f_{x_1} - f_{x_2}| \leq r \cdot |x_1 - x_2|$.

(34) If $f = h$, then $f$ is Lipschitzian iff $h$ is Lipschitzian.

(35) $f \restriction X$ is Lipschitzian if and only if there exists a real number $r$ such that $0 < r$ and for all $x_1$, $x_2$ such that $x_1, x_2 \in \operatorname{dom}(f \restriction X)$ holds $|f_{x_1} - f_{x_2}| \leq r \cdot |x_1 - x_2|$.

Let us consider $n$. Note that every partial function from $\mathbb{R}$ to $\mathcal{R}^n$ which is empty is also Lipschitzian.

Let us consider $n$. Note that there exists a partial function from $\mathbb{R}$ to $\mathcal{R}^n$ which is empty.

Let us consider $n$, let $f$ be a Lipschitzian partial function from $\mathbb{R}$ to $\mathcal{R}^n$, and let $X$ be a set. Note that $f \restriction X$ is Lipschitzian.

We now state the proposition

(36) If $f \restriction X$ is Lipschitzian and $X_1 \subseteq X$, then $f \restriction X_1$ is Lipschitzian.

Let us consider $n$ and let $f_1$, $f_2$ be Lipschitzian partial functions from $\mathbb{R}$ to $\mathcal{R}^n$. Observe that $f_1 + f_2$ is Lipschitzian and $f_1 - f_2$ is Lipschitzian.

We now state two propositions:

(37) If $f_1 \restriction X$ is Lipschitzian and $f_2 \restriction X_1$ is Lipschitzian, then $(f_1 + f_2) \restriction (X \cap X_1)$ is Lipschitzian.

(38) If $f_1 \restriction X$ is Lipschitzian and $f_2 \restriction X_1$ is Lipschitzian, then $(f_1 - f_2) \restriction (X \cap X_1)$ is Lipschitzian.

Let us consider $n$, let $f$ be a Lipschitzian partial function from $\mathbb{R}$ to $\mathcal{R}^n$, and let us consider $p$. Observe that $p \cdot f$ is Lipschitzian.

Next we state the proposition

(39) If $f \restriction X$ is Lipschitzian and $X \subseteq \operatorname{dom} f$, then $(p \cdot f) \restriction X$ is Lipschitzian.

Let us consider $n$ and let $f$ be a Lipschitzian partial function from $\mathbb{R}$ to $\mathcal{R}^n$. Observe that $|f|$ is Lipschitzian.

Next we state the proposition

(40) If $f \restriction X$ is Lipschitzian, then $-f \restriction X$ is Lipschitzian and $|f| \restriction X$ is Lipschitzian and $(-f) \restriction X$ is Lipschitzian.

Let us consider $n$. One can check that every partial function from $\mathbb{R}$ to $\mathcal{R}^n$ which is constant is also Lipschitzian.

Let us consider $n$. One can verify that every partial function from $\mathbb{R}$ to $\mathcal{R}^n$ which is Lipschitzian is also continuous.

The following propositions are true:

(41) For all elements $r$, $p$ of $\mathcal{R}^n$ such that for every $x_0$ such that $x_0 \in X$ holds $f_{x_0} = x_0 \cdot r + p$ holds $f \restriction X$ is continuous.

(42) For every element $x_0$ of $\mathcal{R}^n$ such that $1 \leq i \leq n$ holds $\mathrm{proj}(i, n)$ is continuous in $x_0$.

(43) Let $n$ be a non empty element of $\mathbb{N}$ and $h$ be a partial function from $\mathbb{R}$ to $\mathcal{R}^n$. Then $h$ is continuous in $x_0$ if and only if the following conditions are satisfied:

(i) $x_0 \in \mathrm{dom}\, h$, and

(ii) for every element $i$ of $\mathbb{N}$ such that $i \in \mathrm{Seg}\, n$ holds $\mathrm{proj}(i, n) \cdot h$ is continuous in $x_0$.

(44) Let $n$ be a non empty element of $\mathbb{N}$ and $h$ be a partial function from $\mathbb{R}$ to $\mathcal{R}^n$. Then $h$ is continuous if and only if for every element $i$ of $\mathbb{N}$ such that $i \in \mathrm{Seg}\, n$ holds $\mathrm{proj}(i, n) \cdot h$ is continuous.

(45) For every point $x_0$ of $\langle \mathcal{E}^n, \| \cdot \| \rangle$ such that $1 \leq i \leq n$ holds $\mathrm{Proj}(i, n)$ is continuous in $x_0$.

(46) Let $n$ be a non empty element of $\mathbb{N}$ and $h$ be a partial function from $\mathbb{R}$ to the carrier of $\langle \mathcal{E}^n, \| \cdot \| \rangle$. Then $h$ is continuous in $x_0$ if and only if for every element $i$ of $\mathbb{N}$ such that $i \in \mathrm{Seg}\, n$ holds $\mathrm{Proj}(i, n) \cdot h$ is continuous in $x_0$.

(47) Let $n$ be a non empty element of $\mathbb{N}$ and $h$ be a partial function from $\mathbb{R}$ to the carrier of $\langle \mathcal{E}^n, \| \cdot \| \rangle$. Then $h$ is continuous if and only if for every element $i$ of $\mathbb{N}$ such that $i \in \mathrm{Seg}\, n$ holds $\mathrm{Proj}(i, n) \cdot h$ is continuous.

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