

Representation Theorem for Stacks

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Summary. In the paper the concept of stacks is formalized. As the main result the Theorem of Representation for Stacks is given. Formalization is done according to [13].

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The papers [6], [15], [14], [2], [4], [7], [16], [8], [9], [10], [5], [1], [17], [11], [19], [21], [20], [3], [18], and [12] provide the terminology and notation for this paper.

1. INTRODUCTIONS

In this paper i is a natural number and x is a set.

Let A be a set and let s_1, s_2 be finite sequences of elements of A . Then $s_1 \hat{\ } s_2$ is an element of A^* .

Let A be a set, let i be a natural number, and let s be a finite sequence of elements of A . Then $s_{\uparrow i}$ is an element of A^* .

The following two propositions are true:

- (1) $\emptyset_{\uparrow i} = \emptyset$.
- (2) Let D be a non empty set and s be a finite sequence of elements of D . Suppose $s \neq \emptyset$. Then there exists a finite sequence w of elements of D and there exists an element n of D such that $s = \langle n \rangle \hat{\ } w$.

The scheme *IndSeqD* deals with a non empty set \mathcal{A} and a unary predicate \mathcal{P} , and states that:

For every finite sequence p of elements of \mathcal{A} holds $\mathcal{P}[p]$ provided the following conditions are met:

- $\mathcal{P}[\varepsilon_{\mathcal{A}}]$, and

- For every finite sequence p of elements of \mathcal{A} and for every element x of \mathcal{A} such that $\mathcal{P}[p]$ holds $\mathcal{P}[\langle x \rangle \cap p]$.

Let C, D be non empty sets and let R be a binary relation. A function from $C \times D$ into D is said to be a binary operation of C and D being congruence w.r.t. R if:

- (Def. 1) For every element x of C and for all elements y_1, y_2 of D such that $\langle y_1, y_2 \rangle \in R$ holds $\langle \text{it}(x, y_1), \text{it}(x, y_2) \rangle \in R$.

The scheme *LambdaD2* deals with non empty sets $\mathcal{A}, \mathcal{B}, \mathcal{C}$ and a binary functor \mathcal{F} yielding an element of \mathcal{C} , and states that:

There exists a function M from $\mathcal{A} \times \mathcal{B}$ into \mathcal{C} such that for every element i of \mathcal{A} and for every element j of \mathcal{B} holds $M(i, j) = \mathcal{F}(i, j)$

for all values of the parameters.

Let C, D be non empty sets, let R be an equivalence relation of D , and let b be a function from $C \times D$ into D . Let us assume that b is a binary operation of C and D being congruence w.r.t. R . The functor b/R yielding a function from $C \times \text{Classes } R$ into $\text{Classes } R$ is defined as follows:

- (Def. 2) For all sets x, y, y_1 such that $x \in C$ and $y \in \text{Classes } R$ and $y_1 \in y$ holds $b/R(x, y) = [b(x, y_1)]_R$.

Let A, B be non empty sets, let C be a subset of A , let D be a subset of B , let f be a function from A into B , and let g be a function from C into D . Then $f+g$ is a function from A into B .

2. STACK ALGEBRA

We introduce stack systems which are extensions of 2-sorted and are systems \langle a carrier, a carrier', empty stacks, a push function, a pop function, a top function \rangle ,

where the carrier is a set, the carrier' is a set, the empty stacks constitute a subset of the carrier', the push function is a function from the carrier \times the carrier' into the carrier', the pop function is a function from the carrier' into the carrier', and the top function is a function from the carrier' into the carrier.

Let a_1 be a non empty set, let a_2 be a set, let a_3 be a subset of a_2 , let a_4 be a function from $a_1 \times a_2$ into a_2 , let a_5 be a function from a_2 into a_2 , and let a_6 be a function from a_2 into a_1 . Observe that stack system $\langle a_1, a_2, a_3, a_4, a_5, a_6 \rangle$ is non empty.

Let a_1 be a set, let a_2 be a non empty set, let a_3 be a subset of a_2 , let a_4 be a function from $a_1 \times a_2$ into a_2 , let a_5 be a function from a_2 into a_2 , and let a_6 be a function from a_2 into a_1 . One can verify that stack system $\langle a_1, a_2, a_3, a_4, a_5, a_6 \rangle$ is non void.

Let us note that there exists a stack system which is non empty, non void, and strict.

Let X be a stack system. A stack of X is an element of the carrier' of X .

Let X be a non empty non void stack system and let s be a stack of X . The predicate $\text{empty}(s)$ is defined by:

(Def. 3) $s \in$ the empty stacks of X .

The functor $\text{pop } s$ yields a stack of X and is defined by:

(Def. 4) $\text{pop } s = (\text{the pop function of } X)(s)$.

The functor $\text{top } s$ yields an element of X and is defined by:

(Def. 5) $\text{top } s = (\text{the top function of } X)(s)$.

Let e be an element of X . The functor $\text{push}(e, s)$ yields a stack of X and is defined by:

(Def. 6) $\text{push}(e, s) = (\text{the push function of } X)(e, s)$.

Let A be a non empty set. Standard stack system over A yielding a non empty non void strict stack system is defined by the conditions (Def. 7).

- (Def. 7)(i) The carrier of standard stack system over $A = A$,
- (ii) the carrier' of standard stack system over $A = A^*$, and
- (iii) for every stack s of standard stack system over A holds $\text{empty}(s)$ iff s is empty and for every finite sequence g such that $g = s$ holds if not $\text{empty}(s)$, then $\text{top } s = g(1)$ and $\text{pop } s = g_{\uparrow 1}$ and if $\text{empty}(s)$, then $\text{top } s =$ the element of standard stack system over A and $\text{pop } s = \emptyset$ and for every element e of standard stack system over A holds $\text{push}(e, s) = \langle e \rangle \wedge g$.

In the sequel A denotes a non empty set, c denotes an element of standard stack system over A , and m denotes a stack of standard stack system over A .

Let us consider A . Note that every stack of standard stack system over A is relation-like and function-like.

Let us consider A . Observe that every stack of standard stack system over A is finite sequence-like.

We adopt the following convention: X denotes a non empty non void stack system, s, s_1 denote stacks of X , and e, e_1, e_2 denote elements of X .

Let us consider X . We say that X is pop-finite if and only if the condition (Def. 8) is satisfied.

(Def. 8) Let f be a function from \mathbb{N} into the carrier' of X . Then there exists a natural number i and there exists s such that $f(i) = s$ and if not $\text{empty}(s)$, then $f(i + 1) \neq \text{pop } s$.

We say that X is push-pop if and only if:

(Def. 9) If not $\text{empty}(s)$, then $s = \text{push}(\text{top } s, \text{pop } s)$.

We say that X is top-push if and only if:

(Def. 10) $e = \text{top push}(e, s)$.

We say that X is pop-push if and only if:

(Def. 11) $s = \text{pop push}(e, s)$.

We say that X is push-non-empty if and only if:

(Def. 12) not empty(push(e, s)).

Let A be a non empty set. One can verify the following observations:

- * standard stack system over A is pop-finite,
- * standard stack system over A is push-pop,
- * standard stack system over A is top-push,
- * standard stack system over A is pop-push, and
- * standard stack system over A is push-non-empty.

Let us observe that there exists a non empty non void stack system which is pop-finite, push-pop, top-push, pop-push, push-non-empty, and strict.

A stack algebra is a pop-finite push-pop top-push pop-push push-non-empty non empty non void stack system.

Next we state the proposition

- (3) For every non empty non void stack system X such that X is pop-finite there exists a stack s of X such that empty(s).

Let X be a pop-finite non empty non void stack system. Note that the empty stacks of X is non empty.

We now state two propositions:

- (4) If X is top-push and pop-push and push(e_1, s_1) = push(e_2, s_2), then $e_1 = e_2$ and $s_1 = s_2$.
- (5) If X is push-pop and not empty(s_1) and not empty(s_2) and pop $s_1 =$ pop s_2 and top $s_1 =$ top s_2 , then $s_1 = s_2$.

3. SCHEMES OF INDUCTION

Now we present three schemes. The scheme *INDsch* deals with a stack algebra \mathcal{A} , a stack \mathcal{B} of \mathcal{A} , and a unary predicate \mathcal{P} , and states that:

$$\mathcal{P}[\mathcal{B}]$$

provided the following conditions are satisfied:

- For every stack s of \mathcal{A} such that empty(s) holds $\mathcal{P}[s]$, and
- For every stack s of \mathcal{A} and for every element e of \mathcal{A} such that $\mathcal{P}[s]$ holds $\mathcal{P}[\text{push}(e, s)]$.

The scheme *EXsch* deals with a stack algebra \mathcal{A} , a stack \mathcal{B} of \mathcal{A} , a non empty set \mathcal{C} , an element \mathcal{D} of \mathcal{C} , and a binary functor \mathcal{F} yielding an element of \mathcal{C} , and states that:

There exists an element a of \mathcal{C} and there exists a function F from the carrier' of \mathcal{A} into \mathcal{C} such that

- (i) $a = F(\mathcal{B})$,
- (ii) for every stack s_1 of \mathcal{A} such that empty(s_1) holds $F(s_1) = \mathcal{D}$, and

- (iii) for every stack s_1 of \mathcal{A} and for every element e of \mathcal{A} holds $F(\text{push}(e, s_1)) = \mathcal{F}(e, F(s_1))$

for all values of the parameters.

The scheme *UNIQsch* deals with a stack algebra \mathcal{A} , a stack \mathcal{B} of \mathcal{A} , a non empty set \mathcal{C} , an element \mathcal{D} of \mathcal{C} , and a binary functor \mathcal{F} yielding an element of \mathcal{C} , and states that:

Let a_1, a_2 be elements of \mathcal{C} . Suppose that

- (i) there exists a function F from the carrier' of \mathcal{A} into \mathcal{C} such that $a_1 = F(\mathcal{B})$ and for every stack s_1 of \mathcal{A} such that $\text{empty}(s_1)$ holds $F(s_1) = \mathcal{D}$ and for every stack s_1 of \mathcal{A} and for every element e of \mathcal{A} holds $F(\text{push}(e, s_1)) = \mathcal{F}(e, F(s_1))$, and
- (ii) there exists a function F from the carrier' of \mathcal{A} into \mathcal{C} such that $a_2 = F(\mathcal{B})$ and for every stack s_1 of \mathcal{A} such that $\text{empty}(s_1)$ holds $F(s_1) = \mathcal{D}$ and for every stack s_1 of \mathcal{A} and for every element e of \mathcal{A} holds $F(\text{push}(e, s_1)) = \mathcal{F}(e, F(s_1))$.

Then $a_1 = a_2$

for all values of the parameters.

4. STACK CONGRUENCE

We adopt the following rules: X is a stack algebra, s, s_1, s_2, s_3 are stacks of X , and e, e_1, e_2, e_3 are elements of X .

Let us consider X, s . The functor $|s|$ yielding an element of (the carrier of X)^{*} is defined by the condition (Def. 13).

- (Def. 13) There exists a function F from the carrier' of X into (the carrier of X)^{*} such that $|s| = F(s)$ and for every s_1 such that $\text{empty}(s_1)$ holds $F(s_1) = \emptyset$ and for all s_1, e holds $F(\text{push}(e, s_1)) = \langle e \rangle \wedge F(s_1)$.

Next we state several propositions:

- (6) If $\text{empty}(s)$, then $|s| = \emptyset$.
- (7) If not $\text{empty}(s)$, then $|s| = \langle \text{top } s \rangle \wedge |\text{pop } s|$.
- (8) If not $\text{empty}(s)$, then $|\text{pop } s| = |s|_{\uparrow 1}$.
- (9) $|\text{push}(e, s)| = \langle e \rangle \wedge |s|$.
- (10) If not $\text{empty}(s)$, then $\text{top } s = |s|(1)$.
- (11) If $|s| = \emptyset$, then $\text{empty}(s)$.
- (12) For every stack s of standard stack system over A holds $|s| = s$.
- (13) For every element x of (the carrier of X)^{*} there exists s such that $|s| = x$.

Let us consider X, s_1, s_2 . The predicate $s_1 =_G s_2$ is defined as follows:

- (Def. 14) $|s_1| = |s_2|$.

Let us notice that the predicate $s_1 =_G s_2$ is reflexive and symmetric.

The following propositions are true:

- (14) If $s_1 =_G s_2$ and $s_2 =_G s_3$, then $s_1 =_G s_3$.
- (15) If $s_1 =_G s_2$ and $\text{empty}(s_1)$, then $\text{empty}(s_2)$.
- (16) If $\text{empty}(s_1)$ and $\text{empty}(s_2)$, then $s_1 =_G s_2$.
- (17) If $s_1 =_G s_2$, then $\text{push}(e, s_1) =_G \text{push}(e, s_2)$.
- (18) If $s_1 =_G s_2$ and $\text{not empty}(s_1)$, then $\text{pop } s_1 =_G \text{pop } s_2$.
- (19) If $s_1 =_G s_2$ and $\text{not empty}(s_1)$, then $\text{top } s_1 = \text{top } s_2$.

Let us consider X . We say that X is proper for identity if and only if:

- (Def. 15) For all s_1, s_2 such that $s_1 =_G s_2$ holds $s_1 = s_2$.

Let us consider A . Observe that standard stack system over A is proper for identity.

Let us consider X . The functor $==_X$ yields a binary relation on the carrier' of X and is defined as follows:

- (Def. 16) $\langle s_1, s_2 \rangle \in ==_X$ iff $s_1 =_G s_2$.

Let us consider X . Observe that $==_X$ is total, symmetric, and transitive.

One can prove the following proposition

- (20) If $\text{empty}(s)$, then $[s]_{==_X} =$ the empty stacks of X .

Let us consider X, s . The functor $\text{coset } s$ yielding a subset of the carrier' of X is defined by the conditions (Def. 17).

- (Def. 17)(i) $s \in \text{coset } s$,
- (ii) for all e, s_1 such that $s_1 \in \text{coset } s$ holds $\text{push}(e, s_1) \in \text{coset } s$ and if $\text{not empty}(s_1)$, then $\text{pop } s_1 \in \text{coset } s$, and
 - (iii) for every subset A of the carrier' of X such that $s \in A$ and for all e, s_1 such that $s_1 \in A$ holds $\text{push}(e, s_1) \in A$ and if $\text{not empty}(s_1)$, then $\text{pop } s_1 \in A$ holds $\text{coset } s \subseteq A$.

Next we state three propositions:

- (21) If $\text{push}(e, s) \in \text{coset } s_1$, then $s \in \text{coset } s_1$ and if $\text{not empty}(s)$ and $\text{pop } s \in \text{coset } s_1$, then $s \in \text{coset } s_1$.
- (22) $s \in \text{coset } \text{push}(e, s)$ and if $\text{not empty}(s)$, then $s \in \text{coset } \text{pop } s$.
- (23) There exists s_1 such that $\text{empty}(s_1)$ and $s_1 \in \text{coset } s$.

Let us consider A and let R be a binary relation on A . Note that there exists a reduction sequence w.r.t. R which is A -valued.

Let us consider X . The construction reduction X yielding a binary relation on the carrier' of X is defined as follows:

- (Def. 18) $\langle s_1, s_2 \rangle \in$ the construction reduction X iff $\text{not empty}(s_1)$ and $s_2 = \text{pop } s_1$ or there exists e such that $s_2 = \text{push}(e, s_1)$.

Next we state the proposition

- (24) Let R be a binary relation on A and t be a reduction sequence w.r.t. R . Then $t(1) \in A$ if and only if t is A -valued.

The scheme *PathIND* deals with a non empty set \mathcal{A} , elements \mathcal{B}, \mathcal{C} of \mathcal{A} , a binary relation \mathcal{D} on \mathcal{A} , and a unary predicate \mathcal{P} , and states that:

$$\mathcal{P}[\mathcal{C}]$$

provided the parameters meet the following conditions:

- $\mathcal{P}[\mathcal{B}]$,
- \mathcal{D} reduces \mathcal{B} to \mathcal{C} , and
- For all elements x, y of \mathcal{A} such that \mathcal{D} reduces \mathcal{B} to x and $\langle x, y \rangle \in \mathcal{D}$ and $\mathcal{P}[x]$ holds $\mathcal{P}[y]$.

One can prove the following propositions:

- (25) For every reduction sequence t w.r.t. the construction reduction X such that $s = t(1)$ holds $\text{rng } t \subseteq \text{coset } s$.
- (26) $\text{coset } s = \{s_1 : \text{the construction reduction } X \text{ reduces } s \text{ to } s_1\}$.

Let us consider X, s . The functor $\text{core } s$ yields a stack of X and is defined by the conditions (Def. 19).

- (Def. 19)(i) $\text{empty}(\text{core } s)$, and
- (ii) there exists a the carrier' of X -valued reduction sequence t w.r.t. the construction reduction X such that $t(1) = s$ and $t(\text{len } t) = \text{core } s$ and for every i such that $1 \leq i < \text{len } t$ holds $\text{not empty}(t_i)$ and $t_{i+1} = \text{pop}(t_i)$.

The following propositions are true:

- (27) If $\text{empty}(s)$, then $\text{core } s = s$.
- (28) $\text{core push}(e, s) = \text{core } s$.
- (29) If $\text{not empty}(s)$, then $\text{core pop } s = \text{core } s$.
- (30) $\text{core } s \in \text{coset } s$.
- (31) For every element x of (the carrier of X)* there exists s_1 such that $|s_1| = x$ and $s_1 \in \text{coset } s$.
- (32) If $s_1 \in \text{coset } s$, then $\text{core } s_1 = \text{core } s$.
- (33) If $s_1, s_2 \in \text{coset } s$ and $|s_1| = |s_2|$, then $s_1 = s_2$.
- (34) There exists s such that $\text{coset } s_1 \cap [s_2]_{==X} = \{s\}$.

5. QUOTIENT STACK SYSTEM

Let us consider X . The functor $X_{/==}$ yields a strict stack system and is defined by the conditions (Def. 20).

- (Def. 20)(i) The carrier of $X_{/==} = \text{the carrier of } X$,
- (ii) the carrier' of $X_{/==} = \text{Classes}_{==X}$,
- (iii) the empty stacks of $X_{/==} = \{\text{the empty stacks of } X\}$,
- (iv) the push function of $X_{/==} = (\text{the push function of } X)_{/==X}$,
- (v) the pop function of $X_{/==} =$
 $((\text{the pop function of } X) + \text{id}_{\text{the empty stacks of } X})_{/==X}$, and

- (vi) for every choice function f of $\text{Classes} =_X$ holds the top function of $X_{/} = (\text{the top function of } X) \cdot f + \cdot (\text{the empty stacks of } X, \text{ the element of the carrier of } X)$.

Let us consider X . One can verify that $X_{/} =$ is non empty and non void.

The following propositions are true:

- (35) For every stack S of $X_{/} =$ there exists s such that $S = [s]_{=} =_X$.
- (36) $[s]_{=} =_X$ is a stack of $X_{/} =$.
- (37) For every stack S of $X_{/} =$ such that $S = [s]_{=} =_X$ holds $\text{empty}(s)$ iff $\text{empty}(S)$.
- (38) For every stack S of $X_{/} =$ holds $\text{empty}(S)$ iff $S =$ the empty stacks of X .
- (39) For every stack S of $X_{/} =$ and for every element E of $X_{/} =$ such that $S = [s]_{=} =_X$ and $E = e$ holds $\text{push}(e, s) \in \text{push}(E, S)$ and $[\text{push}(e, s)]_{=} =_X = \text{push}(E, S)$.
- (40) For every stack S of $X_{/} =$ such that $S = [s]_{=} =_X$ and not $\text{empty}(s)$ holds $\text{pop } s \in \text{pop } S$ and $[\text{pop } s]_{=} =_X = \text{pop } S$.
- (41) For every stack S of $X_{/} =$ such that $S = [s]_{=} =_X$ and not $\text{empty}(s)$ holds $\text{top } S = \text{top } s$.

Let us consider X . One can verify the following observations:

- * $X_{/} =$ is pop-finite,
- * $X_{/} =$ is push-pop,
- * $X_{/} =$ is top-push,
- * $X_{/} =$ is pop-push, and
- * $X_{/} =$ is push-non-empty.

Next we state the proposition

- (42) For every stack S of $X_{/} =$ such that $S = [s]_{=} =_X$ holds $|S| = |s|$.

Let us consider X . Note that $X_{/} =$ is proper for identity.

Let us note that there exists a stack algebra which is proper for identity.

6. REPRESENTATION THEOREM FOR STACKS

Let X_1, X_2 be stack algebras and let F, G be functions. We say that F and G form isomorphism between X_1 and X_2 if and only if the conditions (Def. 21) are satisfied.

- (Def. 21) $\text{dom } F =$ the carrier of X_1 and $\text{rng } F =$ the carrier of X_2 and F is one-to-one and $\text{dom } G =$ the carrier' of X_1 and $\text{rng } G =$ the carrier' of X_2 and G is one-to-one and for every stack s_1 of X_1 and for every stack s_2 of X_2 such that $s_2 = G(s_1)$ holds $\text{empty}(s_1)$ iff $\text{empty}(s_2)$ and if not $\text{empty}(s_1)$, then $\text{pop } s_2 = G(\text{pop } s_1)$ and $\text{top } s_2 = F(\text{top } s_1)$ and for every element

e_1 of X_1 and for every element e_2 of X_2 such that $e_2 = F(e_1)$ holds $\text{push}(e_2, s_2) = G(\text{push}(e_1, s_1))$.

We use the following convention: X_1, X_2, X_3 are stack algebras and F, F_1, F_2, G, G_1, G_2 are functions.

The following propositions are true:

- (43) $\text{id}_{\text{the carrier of } X}$ and $\text{id}_{\text{the carrier}' \text{ of } X}$ form isomorphism between X and X .
- (44) If F and G form isomorphism between X_1 and X_2 , then F^{-1} and G^{-1} form isomorphism between X_2 and X_1 .
- (45) Suppose F_1 and G_1 form isomorphism between X_1 and X_2 and F_2 and G_2 form isomorphism between X_2 and X_3 . Then $F_2 \cdot F_1$ and $G_2 \cdot G_1$ form isomorphism between X_1 and X_3 .
- (46) Suppose F and G form isomorphism between X_1 and X_2 . Let s_1 be a stack of X_1 and s_2 be a stack of X_2 . If $s_2 = G(s_1)$, then $|s_2| = F \cdot |s_1|$.

Let X_1, X_2 be stack algebras. We say that X_1 and X_2 are isomorphic if and only if:

- (Def. 22) There exist functions F, G such that F and G form isomorphism between X_1 and X_2 .

Let us notice that the predicate X_1 and X_2 are isomorphic is reflexive and symmetric.

We now state four propositions:

- (47) If X_1 and X_2 are isomorphic and X_2 and X_3 are isomorphic, then X_1 and X_3 are isomorphic.
- (48) If X_1 and X_2 are isomorphic and X_1 is proper for identity, then X_2 is proper for identity.
- (49) Let X be a proper for identity stack algebra. Then there exists G such that
 - (i) for every stack s of X holds $G(s) = |s|$, and
 - (ii) $\text{id}_{\text{the carrier of } X}$ and G form isomorphism between X and standard stack system over the carrier of X .
- (50) Let X be a proper for identity stack algebra. Then X and standard stack system over the carrier of X are isomorphic.

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