

Elementary Introduction to Stochastic Finance in Discrete Time

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Summary. This article gives an elementary introduction to stochastic finance (in discrete time). A formalization of random variables is given and some elements of Borel sets are considered. Furthermore, special functions (for buying a present portfolio and the value of a portfolio in the future) and some statements about the relation between these functions are introduced. For details see: [8] (p. 185), [7] (pp. 12, 20), [6] (pp. 3–6).

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The notation and terminology used in this paper have been introduced in the following papers: [15], [2], [1], [3], [4], [11], [10], [9], [5], [14], [12], and [13].

We use the following convention: O_1, O_2 are non empty sets, S_1, F are σ -fields of subsets of O_1 , and S_2, F_2 are σ -fields of subsets of O_2 .

Let a, r be real numbers. We introduce the halfline finance of a and r as a synonym of $[a, r[$. Then the halfline finance of a and r is a subset of \mathbb{R} .

We now state two propositions:

- (1) For every real number k holds $\mathbb{R} \setminus [k, +\infty[=]-\infty, k[$.
- (2) For every real number k holds $\mathbb{R} \setminus]-\infty, k[= [k, +\infty[$.

Let a, b be real numbers. The half open sets of a and b yields a sequence of subsets of \mathbb{R} and is defined by the conditions (Def. 1).

- (Def. 1)(i) (The half open sets of a and b)(0) = the halfline finance of a and $b + 1$, and
- (ii) for every element n of \mathbb{N} holds (the half open sets of a and b)($n+1$) = the halfline finance of a and $b + \frac{1}{n+1}$.

A sequence of real numbers is said to be a price function if:

(Def. 2) $it(0) = 1$ and for every element n of \mathbb{N} holds $it(n) \geq 0$.

Let p_1, j_1 be sequences of real numbers. We introduce the elements of buy portfolio of p_1 and j_1 as a synonym of $p_1 \cdot j_1$. Then the elements of buy portfolio of p_1 and j_1 is a sequence of real numbers.

Let d be a natural number. The buy portfolio extension of p_1, j_1 , and d yields an element of \mathbb{R} and is defined as follows:

(Def. 3) The buy portfolio extension of p_1, j_1 , and $d = (\sum_{\alpha=0}^{\kappa} (\text{the elements of buy portfolio of } p_1 \text{ and } j_1)(\alpha))_{\kappa \in \mathbb{N}}(d)$.

The buy portfolio of p_1, j_1 , and d yielding an element of \mathbb{R} is defined as follows:

(Def. 4) The buy portfolio of p_1, j_1 , and $d = (\sum_{\alpha=0}^{\kappa} ((\text{the elements of buy portfolio of } p_1 \text{ and } j_1) \uparrow 1)(\alpha))_{\kappa \in \mathbb{N}}(d - 1)$.

Let O_1, O_2 be sets, let S_1 be a σ -field of subsets of O_1 , let S_2 be a σ -field of subsets of O_2 , and let X be a function. We say that X is random variable on S_1 and S_2 if and only if:

(Def. 5) For every element x of S_2 holds $\{y \in O_1: X(y) \text{ is an element of } x\}$ is an element of S_1 .

Let O_1, O_2 be sets, let F be a σ -field of subsets of O_1 , and let F_2 be a σ -field of subsets of O_2 . The set of random variables on F and F_2 is defined by:

(Def. 6) The set of random variables on F and $F_2 = \{f : O_1 \rightarrow O_2: f \text{ is random variable on } F \text{ and } F_2\}$.

Let us consider O_1, O_2, F, F_2 . One can check that the set of random variables on F and F_2 is non empty.

Let O_1, O_2 be non empty sets, let F be a σ -field of subsets of O_1 , let F_2 be a σ -field of subsets of O_2 , and let X be a set. Let us assume that $X =$ the set of random variables on F and F_2 . Let k be an element of X . The change element to function F, F_2 , and k yielding a function from O_1 into O_2 is defined by:

(Def. 7) The change element to function F, F_2 , and $k = k$.

Let O_1 be a non empty set, let F be a σ -field of subsets of O_1 , let X be a non empty set, and let k be an element of X . The random variables for future elements of portfolio value of F and k yields a function from O_1 into \mathbb{R} and is defined by the condition (Def. 8).

(Def. 8) Let w be an element of O_1 . Then (the random variables for future elements of portfolio value of F and k)(w) = (the change element to function F , the Borel sets, and k)(w).

Let p be a natural number, let O_1, O_2 be non empty sets, let F be a σ -field of subsets of O_1 , let F_2 be a σ -field of subsets of O_2 , and let X be a set. Let us assume that $X =$ the set of random variables on F and F_2 . Let G be a function from \mathbb{N} into X . The element of F, F_2, G , and p yields a function from O_1 into O_2 and is defined as follows:

(Def. 9) The element of F, F_2, G , and $p = G(p)$.

Let r be a real number, let O_1 be a non empty set, let F be a σ -field of subsets of O_1 , let X be a non empty set, let w be an element of O_1 , let G be a function from \mathbb{N} into X , and let p_1 be a sequence of real numbers. The future elements of portfolio value of r, p_1, F, w , and G yields a sequence of real numbers and is defined by the condition (Def. 10).

(Def. 10) Let n be an element of \mathbb{N} . Then (the future elements of portfolio value of r, p_1, F, w , and G)(n) = (the random variables for future elements of portfolio value of F and $G(n$))(w) $\cdot p_1(n)$.

Let r be a real number, let d be a natural number, let p_1 be a sequence of real numbers, let O_1 be a non empty set, let F be a σ -field of subsets of O_1 , let X be a non empty set, let G be a function from \mathbb{N} into X , and let w be an element of O_1 . The future portfolio value extension of r, d, p_1, F, G , and w yields an element of \mathbb{R} and is defined by the condition (Def. 11).

(Def. 11) The future portfolio value extension of r, d, p_1, F, G , and $w = (\sum_{\alpha=0}^{\kappa} (\text{the future elements of portfolio value of } r, p_1, F, w, \text{ and } G)(\alpha))_{\kappa \in \mathbb{N}}(d)$.

The future portfolio value of r, d, p_1, F, G , and w yields an element of \mathbb{R} and is defined by the condition (Def. 12).

(Def. 12) The future portfolio value of r, d, p_1, F, G , and $w = (\sum_{\alpha=0}^{\kappa} ((\text{the future elements of portfolio value of } r, p_1, F, w, \text{ and } G) \uparrow 1)(\alpha))_{\kappa \in \mathbb{N}}(d - 1)$.

Let us observe that there exists an element of the Borel sets which is non empty.

One can prove the following propositions:

- (3) For every real number k holds $[k, +\infty[$ is an element of the Borel sets and $] -\infty, k[$ is an element of the Borel sets.
- (4) For all real numbers k_1, k_2 holds $[k_2, k_1[$ is an element of the Borel sets.
- (5) For all real numbers a, b holds Intersection (the half open sets of a and b) is an element of the Borel sets.
- (6) For all real numbers a, b holds Intersection (the half open sets of a and b) = $[a, b]$.
- (7) Let a, b be real numbers and n be a natural number. Then (the partial intersections of the half open sets of a and b)(n) is an element of the Borel sets.
- (8) For all real numbers k_1, k_2 holds $[k_2, k_1]$ is an element of the Borel sets.
- (9) Let X be a function from O_1 into \mathbb{R} . Suppose X is random variable on S_1 and the Borel sets. Then for every real number k holds $\{w \in O_1: X(w) \geq k\}$ is an element of S_1 and $\{w \in O_1: X(w) < k\}$ is an element of S_1 and for all real numbers k_1, k_2 such that $k_1 < k_2$ holds $\{w \in O_1: k_1 \leq X(w) \wedge X(w) < k_2\}$ is an element of S_1 and for all real numbers k_1, k_2 such that $k_1 \leq k_2$ holds $\{w \in O_1: k_1 \leq X(w) \wedge X(w) \leq k_2\}$ is an

element of S_1 and for every real number r holds $\text{LE-dom}(X, r) = \{w \in O_1: X(w) < r\}$ and for every real number r holds $\text{GTE-dom}(X, r) = \{w \in O_1: X(w) \geq r\}$ and for every real number r holds $\text{EQ-dom}(X, r) = \{w \in O_1: X(w) = r\}$ and for every real number r holds $\text{EQ-dom}(X, r)$ is an element of S_1 .

- (10) For every real number s holds $O_1 \mapsto s$ is random variable on S_1 and the Borel sets.
- (11) Let p_1 be a sequence of real numbers, j_1 be a price function, and d be a natural number. Suppose $d > 0$. Then the buy portfolio extension of p_1 , j_1 , and $d = p_1(0) +$ the buy portfolio of p_1 , j_1 , and d .
- (12) Let d be a natural number. Suppose $d > 0$. Let r be a real number, p_1 be a sequence of real numbers, and G be a function from \mathbb{N} into the set of random variables on F and the Borel sets. Suppose the element of F , the Borel sets, G , and $0 = O_1 \mapsto 1 + r$. Let w be an element of O_1 . Then the future portfolio value extension of r , d , p_1 , F , G , and $w = (1+r) \cdot p_1(0) +$ the future portfolio value of r , d , p_1 , F , G , and w .
- (13) Let d be a natural number. Suppose $d > 0$. Let r be a real number. Suppose $r > -1$. Let p_1 be a sequence of real numbers, j_1 be a price function, and G be a function from \mathbb{N} into the set of random variables on F and the Borel sets. Suppose the element of F , the Borel sets, G , and $0 = O_1 \mapsto 1 + r$. Let w be an element of O_1 . Suppose the buy portfolio extension of p_1 , j_1 , and $d \leq 0$. Then the future portfolio value extension of r , d , p_1 , F , G , and $w \leq$ (the future portfolio value of r , d , p_1 , F , G , and w) $- (1 + r) \cdot$ the buy portfolio of p_1 , j_1 , and d .
- (14) Let d be a natural number. Suppose $d > 0$. Let r be a real number. Suppose $r > -1$. Let p_1 be a sequence of real numbers, j_1 be a price function, and G be a function from \mathbb{N} into the set of random variables on F and the Borel sets. Suppose the element of F , the Borel sets, G , and $0 = O_1 \mapsto 1 + r$. Suppose the buy portfolio extension of p_1 , j_1 , and $d \leq 0$. Then
- (i) $\{w \in O_1: \text{the future portfolio value extension of } r, d, p_1, F, G, \text{ and } w \geq 0\} \subseteq \{w \in O_1: \text{the future portfolio value of } r, d, p_1, F, G, \text{ and } w \geq (1 + r) \cdot \text{the buy portfolio of } p_1, j_1, \text{ and } d\}$, and
 - (ii) $\{w \in O_1: \text{the future portfolio value extension of } r, d, p_1, F, G, \text{ and } w > 0\} \subseteq \{w \in O_1: \text{the future portfolio value of } r, d, p_1, F, G, \text{ and } w > (1 + r) \cdot \text{the buy portfolio of } p_1, j_1, \text{ and } d\}$.
- (15) Let f be a function from O_1 into \mathbb{R} . Suppose f is random variable on S_1 and the Borel sets. Then f is measurable on $\Omega_{(S_1)}$ and f is a real-valued random variable on S_1 .
- (16) The set of random variables on S_1 and the Borel sets \subseteq the real-valued random variables set on S_1 .

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