

Valuation Theory. Part I

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Summary. In the article we introduce a valuation function over a field [1]. Ring of non negative elements and its ideal of positive elements have been also defined.

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The notation and terminology used here have been introduced in the following papers: [11], [19], [4], [15], [20], [8], [21], [10], [9], [16], [3], [5], [7], [18], [17], [13], [14], [6], [2], and [12].

1. EXTENDED REALS

We use the following convention: x, y, z, s are extended real numbers, i is an integer, and n, m are natural numbers.

The following propositions are true:

- (1) If $x = -x$, then $x = 0$.
- (2) If $x + x = 0$, then $x = 0$.

- (3) If $0 \leq x \leq y$ and $0 \leq s \leq z$, then $x \cdot s \leq y \cdot z$.
- (4) If $y \neq +\infty$ and $0 < x$ and $0 < y$, then $0 < \frac{x}{y}$.
- (5) If $y \neq +\infty$ and $x < 0 < y$, then $\frac{x}{y} < 0$.
- (6) If $y \neq -\infty$ and $0 < x$ and $y < 0$, then $\frac{x}{y} < 0$.
- (7) If $x, y \in \mathbb{R}$ or $z \in \mathbb{R}$, then $\frac{x+y}{z} = \frac{x}{z} + \frac{y}{z}$.
- (8) If $y \neq +\infty$ and $y \neq -\infty$ and $y \neq 0$, then $\frac{x}{y} \cdot y = x$.
- (9) If $y \neq -\infty$ and $y \neq +\infty$ and $x \neq 0$ and $y \neq 0$, then $\frac{x}{y} \neq 0$.

Let x be a number. We say that x is extended integer if and only if:

(Def. 1) x is integer or $x = +\infty$.

Let us mention that every number which is extended integer is also extended real.

One can verify the following observations:

- * $+\infty$ is extended integer,
- * $-\infty$ is non extended integer,
- * $\bar{1}$ is extended integer, positive, and real,
- * every number which is integer is also extended integer, and
- * every number which is real and extended integer is also integer.

Let us observe that there exists an element of $\bar{\mathbb{R}}$ which is real, extended integer, and positive and there exists an extended integer number which is positive.

An extended integer is an extended integer number.

In the sequel x, y, v denote extended integers.

One can prove the following propositions:

- (10) If $x < y$, then $x + 1 \leq y$.
- (11) $-\infty < x$.

Let X be an extended real-membered set. Let us assume that there exists a positive extended integer i_0 such that $i_0 \in X$. The functor least-positive X yielding a positive extended integer is defined by:

(Def. 2) least-positive $X \in X$ and for every positive extended integer i such that $i \in X$ holds least-positive $X \leq i$.

Let f be a binary relation. We say that f is extended integer valued if and only if:

(Def. 3) For every set x such that $x \in \text{rng } f$ holds x is extended integer.

Let us note that there exists a function which is extended integer valued.

Let A be a set. Note that there exists a function from A into $\bar{\mathbb{R}}$ which is extended integer valued.

Let f be an extended integer valued function and let x be a set. Note that $f(x)$ is extended integer.

2. STRUCTURES

One can prove the following proposition

- (12) Let K be a distributive left unital add-associative right zeroed right complementable non empty double loop structure. Then $-1_K \cdot -1_K = 1_K$.

Let K be a non empty double loop structure, let S be a subset of K , and let n be a natural number. The functor S^n yielding a subset of K is defined by:

- (Def. 4)(i) $S^n =$ the carrier of K if $n = 0$,
(ii) there exists a finite sequence f of elements of $2^{\text{the carrier of } K}$ such that $S^n = f(\text{len } f)$ and $\text{len } f = n$ and $f(1) = S$ and for every natural number i such that $i, i + 1 \in \text{dom } f$ holds $f(i + 1) = S * f_i$, otherwise.

In the sequel A denotes a subset of D . The following propositions are true:

- (13) $A^1 = A$.
(14) $A^2 = A * A$.

Let R be a ring, let S be an ideal of R , and let n be a natural number. Observe that S^n is non empty, add closed, left ideal, and right ideal.

Let G be a non empty double loop structure, let g be an element of G , and let i be an integer. The functor g^i yielding an element of G is defined as follows:

- (Def. 5) $g^i = \begin{cases} \text{power}_G(g, |i|), & \text{if } 0 \leq i, \\ \text{power}_G(g, |i|)^{-1}, & \text{otherwise.} \end{cases}$

Let G be a non empty double loop structure, let g be an element of G , and let n be a natural number. Then g^n can be characterized by the condition:

- (Def. 6) $g^n = \text{power}_G(g, n)$.

In the sequel K is a field-like non degenerated associative add-associative right zeroed right complementable distributive Abelian non empty double loop structure and a, b, c are elements of K . We now state two propositions:

- (15) $a^{n+m} = a^n \cdot a^m$.
(16) If $a \neq 0_K$, then $a^i \neq 0_K$.

3. VALUATION

Let K be a double loop structure. We say that K has a valuation if and only if the condition (Def. 7) is satisfied.

- (Def. 7) There exists an extended integer valued function f from K into $\overline{\mathbb{R}}$ such that

- (i) $f(0_K) = +\infty$,
(ii) for every element a of K such that $a \neq 0_K$ holds $f(a) \in \mathbb{Z}$,
(iii) for all elements a, b of K holds $f(a \cdot b) = f(a) + f(b)$,
(iv) for every element a of K such that $0 \leq f(a)$ holds $0 \leq f(1_K + a)$, and
(v) there exists an element a of K such that $f(a) \neq 0$ and $f(a) \neq +\infty$.

Let K be a double loop structure. Let us assume that K has a valuation. An extended integer valued function from K into $\overline{\mathbb{R}}$ is said to be a valuation of K if it satisfies the conditions (Def. 8).

- (Def. 8)(i) $\text{It}(0_K) = +\infty$,
- (ii) for every element a of K such that $a \neq 0_K$ holds $\text{it}(a) \in \mathbb{Z}$,
 - (iii) for all elements a, b of K holds $\text{it}(a \cdot b) = \text{it}(a) + \text{it}(b)$,
 - (iv) for every element a of K such that $0 \leq \text{it}(a)$ holds $0 \leq \text{it}(1_K + a)$, and
 - (v) there exists an element a of K such that $\text{it}(a) \neq 0$ and $\text{it}(a) \neq +\infty$.

In the sequel v denotes a valuation of K .

One can prove the following propositions:

- (17) If K has a valuation, then $v(1_K) = 0$.
- (18) If K has a valuation and $a \neq 0_K$, then $v(a) \neq +\infty$.
- (19) If K has a valuation, then $v(-1_K) = 0$.
- (20) If K has a valuation, then $v(-a) = v(a)$.
- (21) If K has a valuation and $a \neq 0_K$, then $v(a^{-1}) = -v(a)$.
- (22) If K has a valuation and $b \neq 0_K$, then $v(\frac{a}{b}) = v(a) - v(b)$.
- (23) If K has a valuation and $a \neq 0_K$ and $b \neq 0_K$, then $v(\frac{a}{b}) = -v(\frac{b}{a})$.
- (24) If K has a valuation and $b \neq 0_K$ and $0 \leq v(\frac{a}{b})$, then $v(b) \leq v(a)$.
- (25) If K has a valuation and $a \neq 0_K$ and $b \neq 0_K$ and $v(\frac{a}{b}) \leq 0$, then $0 \leq v(\frac{b}{a})$.
- (26) If K has a valuation and $b \neq 0_K$ and $v(\frac{a}{b}) \leq 0$, then $v(a) \leq v(b)$.
- (27) If K has a valuation, then $\min(v(a), v(b)) \leq v(a + b)$.
- (28) If K has a valuation and $v(a) < v(b)$, then $v(a) = v(a + b)$.
- (29) If K has a valuation and $a \neq 0_K$, then $v(a^i) = i \cdot v(a)$.
- (30) If K has a valuation and $0 \leq v(1_K + a)$, then $0 \leq v(a)$.
- (31) If K has a valuation and $0 \leq v(1_K - a)$, then $0 \leq v(a)$.
- (32) If K has a valuation and $a \neq 0_K$ and $v(a) \leq v(b)$, then $0 \leq v(\frac{b}{a})$.
- (33) If K has a valuation, then $+\infty \in \text{rng } v$.
- (34) If $v(a) = 1$, then least-positive $\text{rng } v = 1$.
- (35) If K has a valuation, then least-positive $\text{rng } v$ is integer.
- (36) If K has a valuation, then for every element x of K such that $x \neq 0_K$ there exists an integer i such that $v(x) = i \cdot \text{least-positive } \text{rng } v$.

Let us consider K, v . Let us assume that K has a valuation. The functor Pgenerator v yielding an element of K is defined as follows:

- (Def. 9) Pgenerator $v = \text{the element of } v^{-1}(\{\text{least-positive } \text{rng } v\})$.

Let us consider K, v . Let us assume that K has a valuation. The functor normal-valuation v yields a valuation of K and is defined by:

- (Def. 10) $v(a) = (\text{normal-valuation } v)(a) \cdot \text{least-positive } \text{rng } v$.

We now state a number of propositions:

- (37) If K has a valuation, then $v(a) = 0$ iff (normal-valuation v)(a) = 0.
- (38) If K has a valuation, then $v(a) = +\infty$ iff (normal-valuation v)(a) = $+\infty$.
- (39) If K has a valuation, then $v(a) = v(b)$ iff (normal-valuation v)(a) = (normal-valuation v)(b).
- (40) If K has a valuation, then $v(a)$ is positive iff (normal-valuation v)(a) is positive.
- (41) If K has a valuation, then $0 \leq v(a)$ iff $0 \leq$ (normal-valuation v)(a).
- (42) If K has a valuation, then $v(a)$ is non negative iff (normal-valuation v)(a) is non negative.
- (43) If K has a valuation, then (normal-valuation v)(Pgenerator v) = 1.
- (44) If K has a valuation and $0 \leq v(a)$, then (normal-valuation v)(a) $\leq v(a)$.
- (45) If K has a valuation and $v(a) = 1$, then normal-valuation $v = v$.
- (46) If K has a valuation, then normal-valuation(normal-valuation v) = normal-valuation v .

4. VALUATION RING

Let K be a non empty double loop structure and let v be a valuation of K . The functor $\text{NonNegElements } v$ is defined as follows:

(Def. 11) $\text{NonNegElements } v = \{x \in K: 0 \leq v(x)\}$.

The following four propositions are true:

- (47) Let K be a non empty double loop structure, v be a valuation of K , and a be an element of K . Then $a \in \text{NonNegElements } v$ if and only if $0 \leq v(a)$.
- (48) For every non empty double loop structure K and for every valuation v of K holds $\text{NonNegElements } v \subseteq$ the carrier of K .
- (49) For every non empty double loop structure K and for every valuation v of K such that K has a valuation holds $0_K \in \text{NonNegElements } v$.
- (50) If K has a valuation, then $1_K \in \text{NonNegElements } v$.

Let us consider K, v . Let us assume that K has a valuation. The functor $\text{ValuatRing } v$ yields a strict commutative non degenerated ring and is defined by the conditions (Def. 12).

- (Def. 12)(i) The carrier of $\text{ValuatRing } v = \text{NonNegElements } v$,
- (ii) the addition of $\text{ValuatRing } v =$ (the addition of K)|(NonNegElements $v \times$ NonNegElements v),
 - (iii) the multiplication of $\text{ValuatRing } v =$ (the multiplication of K)|(NonNegElements $v \times$ NonNegElements v),
 - (iv) the zero of $\text{ValuatRing } v = 0_K$, and
 - (v) the one of $\text{ValuatRing } v = 1_K$.

The following propositions are true:

- (51) If K has a valuation, then every element of $\text{ValuatRing } v$ is an element of K .
- (52) If K has a valuation, then $0 \leq v(a)$ iff a is an element of $\text{ValuatRing } v$.
- (53) If K has a valuation, then for every subset S of $\text{ValuatRing } v$ holds 0 is a lower bound of $v^\circ S$.
- (54) Suppose K has a valuation. Let x, y be elements of K and x_1, y_1 be elements of $\text{ValuatRing } v$. If $x = x_1$ and $y = y_1$, then $x + y = x_1 + y_1$.
- (55) Suppose K has a valuation. Let x, y be elements of K and x_1, y_1 be elements of $\text{ValuatRing } v$. If $x = x_1$ and $y = y_1$, then $x \cdot y = x_1 \cdot y_1$.
- (56) If K has a valuation, then $0_{\text{ValuatRing } v} = 0_K$.
- (57) If K has a valuation, then $1_{\text{ValuatRing } v} = 1_K$.
- (58) If K has a valuation, then for every element x of K and for every element y of $\text{ValuatRing } v$ such that $x = y$ holds $-x = -y$.
- (59) If K has a valuation, then $\text{ValuatRing } v$ is integral domain-like.
- (60) If K has a valuation, then for every element y of $\text{ValuatRing } v$ holds $\text{power}_K(y, n) = \text{power}_{\text{ValuatRing } v}(y, n)$.

Let us consider K, v . Let us assume that K has a valuation. The functor $\text{PosElements } v$ yields an ideal of $\text{ValuatRing } v$ and is defined as follows:

(Def. 13) $\text{PosElements } v = \{x \in K: 0 < v(x)\}$.

Let us consider K, v . We introduce $\text{vp } v$ as a synonym of $\text{PosElements } v$.

Next we state three propositions:

- (61) If K has a valuation, then $a \in \text{vp } v$ iff $0 < v(a)$.
- (62) If K has a valuation, then $0_K \in \text{vp } v$.
- (63) If K has a valuation, then $1_K \notin \text{vp } v$.

Let us consider K, v and let S be a non empty subset of K . Let us assume that K has a valuation and S is a subset of $\text{ValuatRing } v$. The functor $\text{min}(S, v)$ yielding a subset of $\text{ValuatRing } v$ is defined as follows:

(Def. 14) $\text{min}(S, v) = v^{-1}(\{\inf(v^\circ S)\}) \cap S$.

The following four propositions are true:

- (64) For every non empty subset S of K such that K has a valuation and S is a subset of $\text{ValuatRing } v$ holds $\text{min}(S, v) \subseteq S$.
- (65) Let S be a non empty subset of K . Suppose K has a valuation and S is a subset of $\text{ValuatRing } v$. Let x be an element of K . Then $x \in \text{min}(S, v)$ if and only if the following conditions are satisfied:
 - (i) $x \in S$, and
 - (ii) for every element y of K such that $y \in S$ holds $v(x) \leq v(y)$.

(66) Suppose K has a valuation. Let I be a non empty subset of K and x be an element of $\text{ValuatRing } v$. If I is an ideal of $\text{ValuatRing } v$ and $x \in \min(I, v)$, then $I = \{x\}$ -ideal.

(67) For every non empty double loop structure R holds every add closed non empty subset of R is a set closed w.r.t. the addition of R .

Let R be a ring and let P be a right ideal of R . A submodule of $\text{RightMod}(R)$ is called a submodule of P if:

(Def. 15) The carrier of it = P .

Let R be a ring and let P be a right ideal of R . Note that there exists a submodule of P which is strict. Next we state the proposition

(68) Let R be a ring, P be an ideal of R , M be a submodule of P , a be a binary operation on P , z be an element of P , and m be a function from $P \times$ the carrier of R into P . Suppose $a = (\text{the addition of } R) \upharpoonright (P \times P)$ and $m = (\text{the multiplication of } R) \upharpoonright (P \times \text{the carrier of } R)$ and $z = \text{the zero of } R$. Then the right module structure of $M = \langle P, a, z, m \rangle$.

Let R be a ring, let M_1, M_2 be right modules over R , and let h be a function from M_1 into M_2 . We say that h is scalar linear if and only if:

(Def. 16) For every element x of M_1 and for every element r of R holds $h(x \cdot r) = h(x) \cdot r$.

Let R be a ring, let M_1 be a right module over R , and let M_2 be a submodule of M_1 . Observe that $\text{incl}(M_2, M_1)$ is additive and scalar linear.

Next we state a number of propositions:

(69) If K has a valuation and b is an element of $\text{ValuatRing } v$, then $v(a) \leq v(a) + v(b)$.

(70) If K has a valuation and a is an element of $\text{ValuatRing } v$, then $\text{power}_K(a, n)$ is an element of $\text{ValuatRing } v$.

(71) If K has a valuation, then for every element x of $\text{ValuatRing } v$ such that $x \neq 0_K$ holds $\text{power}_K(x, n) \neq 0_K$.

(72) If K has a valuation and $v(a) = 0$, then a is an element of $\text{ValuatRing } v$ and a^{-1} is an element of $\text{ValuatRing } v$.

(73) If K has a valuation and $a \neq 0_K$ and a is an element of $\text{ValuatRing } v$ and a^{-1} is an element of $\text{ValuatRing } v$, then $v(a) = 0$.

(74) If K has a valuation and $v(a) = 0$, then for every ideal I of $\text{ValuatRing } v$ holds $a \in I$ iff $I = \text{the carrier of } \text{ValuatRing } v$.

(75) If K has a valuation, then $\text{Pgenerator } v$ is an element of $\text{ValuatRing } v$.

(76) If K has a valuation, then $\text{vp } v$ is proper.

(77) If K has a valuation, then for every element x of $\text{ValuatRing } v$ such that $x \notin \text{vp } v$ holds $v(x) = 0$.

(78) If K has a valuation, then $\text{vp } v$ is prime.

- (79) If K has a valuation, then for every proper ideal I of $\text{ValuatRing } v$ holds $I \subseteq \text{vp } v$.
- (80) If K has a valuation, then $\text{vp } v$ is maximal.
- (81) If K has a valuation, then for every maximal ideal I of $\text{ValuatRing } v$ holds $I = \text{vp } v$.
- (82) If K has a valuation, then $\text{NonNegElements normal-valuation } v = \text{NonNegElements } v$.
- (83) If K has a valuation, then $\text{ValuatRing normal-valuation } v = \text{ValuatRing } v$.

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