

# Valuation Theory. Part I

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**Summary.** In the article we introduce a valuation function over a field [1]. Ring of non negative elements and its ideal of positive elements have been also defined.

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The notation and terminology used here have been introduced in the following papers: [11], [19], [4], [15], [20], [8], [21], [10], [9], [16], [3], [5], [7], [18], [17], [13], [14], [6], [2], and [12].

## 1. EXTENDED REALS

We use the following convention:  $x, y, z, s$  are extended real numbers,  $i$  is an integer, and  $n, m$  are natural numbers.

The following propositions are true:

- (1) If  $x = -x$ , then  $x = 0$ .
- (2) If  $x + x = 0$ , then  $x = 0$ .

- (3) If  $0 \leq x \leq y$  and  $0 \leq s \leq z$ , then  $x \cdot s \leq y \cdot z$ .
- (4) If  $y \neq +\infty$  and  $0 < x$  and  $0 < y$ , then  $0 < \frac{x}{y}$ .
- (5) If  $y \neq +\infty$  and  $x < 0 < y$ , then  $\frac{x}{y} < 0$ .
- (6) If  $y \neq -\infty$  and  $0 < x$  and  $y < 0$ , then  $\frac{x}{y} < 0$ .
- (7) If  $x, y \in \mathbb{R}$  or  $z \in \mathbb{R}$ , then  $\frac{x+y}{z} = \frac{x}{z} + \frac{y}{z}$ .
- (8) If  $y \neq +\infty$  and  $y \neq -\infty$  and  $y \neq 0$ , then  $\frac{x}{y} \cdot y = x$ .
- (9) If  $y \neq -\infty$  and  $y \neq +\infty$  and  $x \neq 0$  and  $y \neq 0$ , then  $\frac{x}{y} \neq 0$ .

Let  $x$  be a number. We say that  $x$  is extended integer if and only if:

(Def. 1)  $x$  is integer or  $x = +\infty$ .

Let us mention that every number which is extended integer is also extended real.

One can verify the following observations:

- \*  $+\infty$  is extended integer,
- \*  $-\infty$  is non extended integer,
- \*  $\bar{1}$  is extended integer, positive, and real,
- \* every number which is integer is also extended integer, and
- \* every number which is real and extended integer is also integer.

Let us observe that there exists an element of  $\bar{\mathbb{R}}$  which is real, extended integer, and positive and there exists an extended integer number which is positive.

An extended integer is an extended integer number.

In the sequel  $x, y, v$  denote extended integers.

One can prove the following propositions:

- (10) If  $x < y$ , then  $x + 1 \leq y$ .
- (11)  $-\infty < x$ .

Let  $X$  be an extended real-membered set. Let us assume that there exists a positive extended integer  $i_0$  such that  $i_0 \in X$ . The functor least-positive  $X$  yielding a positive extended integer is defined by:

(Def. 2) least-positive  $X \in X$  and for every positive extended integer  $i$  such that  $i \in X$  holds least-positive  $X \leq i$ .

Let  $f$  be a binary relation. We say that  $f$  is extended integer valued if and only if:

(Def. 3) For every set  $x$  such that  $x \in \text{rng } f$  holds  $x$  is extended integer.

Let us note that there exists a function which is extended integer valued.

Let  $A$  be a set. Note that there exists a function from  $A$  into  $\bar{\mathbb{R}}$  which is extended integer valued.

Let  $f$  be an extended integer valued function and let  $x$  be a set. Note that  $f(x)$  is extended integer.

## 2. STRUCTURES

One can prove the following proposition

- (12) Let  $K$  be a distributive left unital add-associative right zeroed right complementable non empty double loop structure. Then  $-1_K \cdot -1_K = 1_K$ .

Let  $K$  be a non empty double loop structure, let  $S$  be a subset of  $K$ , and let  $n$  be a natural number. The functor  $S^n$  yielding a subset of  $K$  is defined by:

- (Def. 4)(i)  $S^n =$  the carrier of  $K$  if  $n = 0$ ,  
(ii) there exists a finite sequence  $f$  of elements of  $2^{\text{the carrier of } K}$  such that  $S^n = f(\text{len } f)$  and  $\text{len } f = n$  and  $f(1) = S$  and for every natural number  $i$  such that  $i, i + 1 \in \text{dom } f$  holds  $f(i + 1) = S * f_i$ , otherwise.

In the sequel  $A$  denotes a subset of  $D$ . The following propositions are true:

- (13)  $A^1 = A$ .  
(14)  $A^2 = A * A$ .

Let  $R$  be a ring, let  $S$  be an ideal of  $R$ , and let  $n$  be a natural number. Observe that  $S^n$  is non empty, add closed, left ideal, and right ideal.

Let  $G$  be a non empty double loop structure, let  $g$  be an element of  $G$ , and let  $i$  be an integer. The functor  $g^i$  yielding an element of  $G$  is defined as follows:

- (Def. 5)  $g^i = \begin{cases} \text{power}_G(g, |i|), & \text{if } 0 \leq i, \\ \text{power}_G(g, |i|)^{-1}, & \text{otherwise.} \end{cases}$

Let  $G$  be a non empty double loop structure, let  $g$  be an element of  $G$ , and let  $n$  be a natural number. Then  $g^n$  can be characterized by the condition:

- (Def. 6)  $g^n = \text{power}_G(g, n)$ .

In the sequel  $K$  is a field-like non degenerated associative add-associative right zeroed right complementable distributive Abelian non empty double loop structure and  $a, b, c$  are elements of  $K$ . We now state two propositions:

- (15)  $a^{n+m} = a^n \cdot a^m$ .  
(16) If  $a \neq 0_K$ , then  $a^i \neq 0_K$ .

## 3. VALUATION

Let  $K$  be a double loop structure. We say that  $K$  has a valuation if and only if the condition (Def. 7) is satisfied.

- (Def. 7) There exists an extended integer valued function  $f$  from  $K$  into  $\overline{\mathbb{R}}$  such that

- (i)  $f(0_K) = +\infty$ ,  
(ii) for every element  $a$  of  $K$  such that  $a \neq 0_K$  holds  $f(a) \in \mathbb{Z}$ ,  
(iii) for all elements  $a, b$  of  $K$  holds  $f(a \cdot b) = f(a) + f(b)$ ,  
(iv) for every element  $a$  of  $K$  such that  $0 \leq f(a)$  holds  $0 \leq f(1_K + a)$ , and  
(v) there exists an element  $a$  of  $K$  such that  $f(a) \neq 0$  and  $f(a) \neq +\infty$ .

Let  $K$  be a double loop structure. Let us assume that  $K$  has a valuation. An extended integer valued function from  $K$  into  $\overline{\mathbb{R}}$  is said to be a valuation of  $K$  if it satisfies the conditions (Def. 8).

- (Def. 8)(i)  $\text{It}(0_K) = +\infty$ ,
- (ii) for every element  $a$  of  $K$  such that  $a \neq 0_K$  holds  $\text{it}(a) \in \mathbb{Z}$ ,
  - (iii) for all elements  $a, b$  of  $K$  holds  $\text{it}(a \cdot b) = \text{it}(a) + \text{it}(b)$ ,
  - (iv) for every element  $a$  of  $K$  such that  $0 \leq \text{it}(a)$  holds  $0 \leq \text{it}(1_K + a)$ , and
  - (v) there exists an element  $a$  of  $K$  such that  $\text{it}(a) \neq 0$  and  $\text{it}(a) \neq +\infty$ .

In the sequel  $v$  denotes a valuation of  $K$ .

One can prove the following propositions:

- (17) If  $K$  has a valuation, then  $v(1_K) = 0$ .
- (18) If  $K$  has a valuation and  $a \neq 0_K$ , then  $v(a) \neq +\infty$ .
- (19) If  $K$  has a valuation, then  $v(-1_K) = 0$ .
- (20) If  $K$  has a valuation, then  $v(-a) = v(a)$ .
- (21) If  $K$  has a valuation and  $a \neq 0_K$ , then  $v(a^{-1}) = -v(a)$ .
- (22) If  $K$  has a valuation and  $b \neq 0_K$ , then  $v(\frac{a}{b}) = v(a) - v(b)$ .
- (23) If  $K$  has a valuation and  $a \neq 0_K$  and  $b \neq 0_K$ , then  $v(\frac{a}{b}) = -v(\frac{b}{a})$ .
- (24) If  $K$  has a valuation and  $b \neq 0_K$  and  $0 \leq v(\frac{a}{b})$ , then  $v(b) \leq v(a)$ .
- (25) If  $K$  has a valuation and  $a \neq 0_K$  and  $b \neq 0_K$  and  $v(\frac{a}{b}) \leq 0$ , then  $0 \leq v(\frac{b}{a})$ .
- (26) If  $K$  has a valuation and  $b \neq 0_K$  and  $v(\frac{a}{b}) \leq 0$ , then  $v(a) \leq v(b)$ .
- (27) If  $K$  has a valuation, then  $\min(v(a), v(b)) \leq v(a + b)$ .
- (28) If  $K$  has a valuation and  $v(a) < v(b)$ , then  $v(a) = v(a + b)$ .
- (29) If  $K$  has a valuation and  $a \neq 0_K$ , then  $v(a^i) = i \cdot v(a)$ .
- (30) If  $K$  has a valuation and  $0 \leq v(1_K + a)$ , then  $0 \leq v(a)$ .
- (31) If  $K$  has a valuation and  $0 \leq v(1_K - a)$ , then  $0 \leq v(a)$ .
- (32) If  $K$  has a valuation and  $a \neq 0_K$  and  $v(a) \leq v(b)$ , then  $0 \leq v(\frac{b}{a})$ .
- (33) If  $K$  has a valuation, then  $+\infty \in \text{rng } v$ .
- (34) If  $v(a) = 1$ , then least-positive  $\text{rng } v = 1$ .
- (35) If  $K$  has a valuation, then least-positive  $\text{rng } v$  is integer.
- (36) If  $K$  has a valuation, then for every element  $x$  of  $K$  such that  $x \neq 0_K$  there exists an integer  $i$  such that  $v(x) = i \cdot \text{least-positive } \text{rng } v$ .

Let us consider  $K, v$ . Let us assume that  $K$  has a valuation. The functor Pgenerator  $v$  yielding an element of  $K$  is defined as follows:

- (Def. 9) Pgenerator  $v = \text{the element of } v^{-1}(\{\text{least-positive } \text{rng } v\})$ .

Let us consider  $K, v$ . Let us assume that  $K$  has a valuation. The functor normal-valuation  $v$  yields a valuation of  $K$  and is defined by:

- (Def. 10)  $v(a) = (\text{normal-valuation } v)(a) \cdot \text{least-positive } \text{rng } v$ .

We now state a number of propositions:

- (37) If  $K$  has a valuation, then  $v(a) = 0$  iff (normal-valuation  $v$ )( $a$ ) = 0.
- (38) If  $K$  has a valuation, then  $v(a) = +\infty$  iff (normal-valuation  $v$ )( $a$ ) =  $+\infty$ .
- (39) If  $K$  has a valuation, then  $v(a) = v(b)$  iff (normal-valuation  $v$ )( $a$ ) = (normal-valuation  $v$ )( $b$ ).
- (40) If  $K$  has a valuation, then  $v(a)$  is positive iff (normal-valuation  $v$ )( $a$ ) is positive.
- (41) If  $K$  has a valuation, then  $0 \leq v(a)$  iff  $0 \leq$  (normal-valuation  $v$ )( $a$ ).
- (42) If  $K$  has a valuation, then  $v(a)$  is non negative iff (normal-valuation  $v$ )( $a$ ) is non negative.
- (43) If  $K$  has a valuation, then (normal-valuation  $v$ )(Pgenerator  $v$ ) = 1.
- (44) If  $K$  has a valuation and  $0 \leq v(a)$ , then (normal-valuation  $v$ )( $a$ )  $\leq v(a)$ .
- (45) If  $K$  has a valuation and  $v(a) = 1$ , then normal-valuation  $v = v$ .
- (46) If  $K$  has a valuation, then normal-valuation(normal-valuation  $v$ ) = normal-valuation  $v$ .

#### 4. VALUATION RING

Let  $K$  be a non empty double loop structure and let  $v$  be a valuation of  $K$ . The functor  $\text{NonNegElements } v$  is defined as follows:

(Def. 11)  $\text{NonNegElements } v = \{x \in K : 0 \leq v(x)\}$ .

The following four propositions are true:

- (47) Let  $K$  be a non empty double loop structure,  $v$  be a valuation of  $K$ , and  $a$  be an element of  $K$ . Then  $a \in \text{NonNegElements } v$  if and only if  $0 \leq v(a)$ .
- (48) For every non empty double loop structure  $K$  and for every valuation  $v$  of  $K$  holds  $\text{NonNegElements } v \subseteq$  the carrier of  $K$ .
- (49) For every non empty double loop structure  $K$  and for every valuation  $v$  of  $K$  such that  $K$  has a valuation holds  $0_K \in \text{NonNegElements } v$ .
- (50) If  $K$  has a valuation, then  $1_K \in \text{NonNegElements } v$ .

Let us consider  $K, v$ . Let us assume that  $K$  has a valuation. The functor  $\text{ValuatRing } v$  yields a strict commutative non degenerated ring and is defined by the conditions (Def. 12).

- (Def. 12)(i) The carrier of  $\text{ValuatRing } v = \text{NonNegElements } v$ ,
- (ii) the addition of  $\text{ValuatRing } v =$  (the addition of  $K$ )|(NonNegElements  $v \times$  NonNegElements  $v$ ),
  - (iii) the multiplication of  $\text{ValuatRing } v =$  (the multiplication of  $K$ )|(NonNegElements  $v \times$  NonNegElements  $v$ ),
  - (iv) the zero of  $\text{ValuatRing } v = 0_K$ , and
  - (v) the one of  $\text{ValuatRing } v = 1_K$ .

The following propositions are true:

- (51) If  $K$  has a valuation, then every element of  $\text{ValuatRing } v$  is an element of  $K$ .
- (52) If  $K$  has a valuation, then  $0 \leq v(a)$  iff  $a$  is an element of  $\text{ValuatRing } v$ .
- (53) If  $K$  has a valuation, then for every subset  $S$  of  $\text{ValuatRing } v$  holds  $0$  is a lower bound of  $v^\circ S$ .
- (54) Suppose  $K$  has a valuation. Let  $x, y$  be elements of  $K$  and  $x_1, y_1$  be elements of  $\text{ValuatRing } v$ . If  $x = x_1$  and  $y = y_1$ , then  $x + y = x_1 + y_1$ .
- (55) Suppose  $K$  has a valuation. Let  $x, y$  be elements of  $K$  and  $x_1, y_1$  be elements of  $\text{ValuatRing } v$ . If  $x = x_1$  and  $y = y_1$ , then  $x \cdot y = x_1 \cdot y_1$ .
- (56) If  $K$  has a valuation, then  $0_{\text{ValuatRing } v} = 0_K$ .
- (57) If  $K$  has a valuation, then  $1_{\text{ValuatRing } v} = 1_K$ .
- (58) If  $K$  has a valuation, then for every element  $x$  of  $K$  and for every element  $y$  of  $\text{ValuatRing } v$  such that  $x = y$  holds  $-x = -y$ .
- (59) If  $K$  has a valuation, then  $\text{ValuatRing } v$  is integral domain-like.
- (60) If  $K$  has a valuation, then for every element  $y$  of  $\text{ValuatRing } v$  holds  $\text{power}_K(y, n) = \text{power}_{\text{ValuatRing } v}(y, n)$ .

Let us consider  $K, v$ . Let us assume that  $K$  has a valuation. The functor  $\text{PosElements } v$  yields an ideal of  $\text{ValuatRing } v$  and is defined as follows:

(Def. 13)  $\text{PosElements } v = \{x \in K: 0 < v(x)\}$ .

Let us consider  $K, v$ . We introduce  $\text{vp } v$  as a synonym of  $\text{PosElements } v$ .

Next we state three propositions:

- (61) If  $K$  has a valuation, then  $a \in \text{vp } v$  iff  $0 < v(a)$ .
- (62) If  $K$  has a valuation, then  $0_K \in \text{vp } v$ .
- (63) If  $K$  has a valuation, then  $1_K \notin \text{vp } v$ .

Let us consider  $K, v$  and let  $S$  be a non empty subset of  $K$ . Let us assume that  $K$  has a valuation and  $S$  is a subset of  $\text{ValuatRing } v$ . The functor  $\text{min}(S, v)$  yielding a subset of  $\text{ValuatRing } v$  is defined as follows:

(Def. 14)  $\text{min}(S, v) = v^{-1}(\{\inf(v^\circ S)\}) \cap S$ .

The following four propositions are true:

- (64) For every non empty subset  $S$  of  $K$  such that  $K$  has a valuation and  $S$  is a subset of  $\text{ValuatRing } v$  holds  $\text{min}(S, v) \subseteq S$ .
- (65) Let  $S$  be a non empty subset of  $K$ . Suppose  $K$  has a valuation and  $S$  is a subset of  $\text{ValuatRing } v$ . Let  $x$  be an element of  $K$ . Then  $x \in \text{min}(S, v)$  if and only if the following conditions are satisfied:
  - (i)  $x \in S$ , and
  - (ii) for every element  $y$  of  $K$  such that  $y \in S$  holds  $v(x) \leq v(y)$ .

(66) Suppose  $K$  has a valuation. Let  $I$  be a non empty subset of  $K$  and  $x$  be an element of  $\text{ValuatRing } v$ . If  $I$  is an ideal of  $\text{ValuatRing } v$  and  $x \in \min(I, v)$ , then  $I = \{x\}$ -ideal.

(67) For every non empty double loop structure  $R$  holds every add closed non empty subset of  $R$  is a set closed w.r.t. the addition of  $R$ .

Let  $R$  be a ring and let  $P$  be a right ideal of  $R$ . A submodule of  $\text{RightMod}(R)$  is called a submodule of  $P$  if:

(Def. 15) The carrier of it =  $P$ .

Let  $R$  be a ring and let  $P$  be a right ideal of  $R$ . Note that there exists a submodule of  $P$  which is strict. Next we state the proposition

(68) Let  $R$  be a ring,  $P$  be an ideal of  $R$ ,  $M$  be a submodule of  $P$ ,  $a$  be a binary operation on  $P$ ,  $z$  be an element of  $P$ , and  $m$  be a function from  $P \times$  the carrier of  $R$  into  $P$ . Suppose  $a = (\text{the addition of } R) \upharpoonright (P \times P)$  and  $m = (\text{the multiplication of } R) \upharpoonright (P \times \text{the carrier of } R)$  and  $z = \text{the zero of } R$ . Then the right module structure of  $M = \langle P, a, z, m \rangle$ .

Let  $R$  be a ring, let  $M_1, M_2$  be right modules over  $R$ , and let  $h$  be a function from  $M_1$  into  $M_2$ . We say that  $h$  is scalar linear if and only if:

(Def. 16) For every element  $x$  of  $M_1$  and for every element  $r$  of  $R$  holds  $h(x \cdot r) = h(x) \cdot r$ .

Let  $R$  be a ring, let  $M_1$  be a right module over  $R$ , and let  $M_2$  be a submodule of  $M_1$ . Observe that  $\text{incl}(M_2, M_1)$  is additive and scalar linear.

Next we state a number of propositions:

(69) If  $K$  has a valuation and  $b$  is an element of  $\text{ValuatRing } v$ , then  $v(a) \leq v(a) + v(b)$ .

(70) If  $K$  has a valuation and  $a$  is an element of  $\text{ValuatRing } v$ , then  $\text{power}_K(a, n)$  is an element of  $\text{ValuatRing } v$ .

(71) If  $K$  has a valuation, then for every element  $x$  of  $\text{ValuatRing } v$  such that  $x \neq 0_K$  holds  $\text{power}_K(x, n) \neq 0_K$ .

(72) If  $K$  has a valuation and  $v(a) = 0$ , then  $a$  is an element of  $\text{ValuatRing } v$  and  $a^{-1}$  is an element of  $\text{ValuatRing } v$ .

(73) If  $K$  has a valuation and  $a \neq 0_K$  and  $a$  is an element of  $\text{ValuatRing } v$  and  $a^{-1}$  is an element of  $\text{ValuatRing } v$ , then  $v(a) = 0$ .

(74) If  $K$  has a valuation and  $v(a) = 0$ , then for every ideal  $I$  of  $\text{ValuatRing } v$  holds  $a \in I$  iff  $I = \text{the carrier of } \text{ValuatRing } v$ .

(75) If  $K$  has a valuation, then  $\text{Pgenerator } v$  is an element of  $\text{ValuatRing } v$ .

(76) If  $K$  has a valuation, then  $\text{vp } v$  is proper.

(77) If  $K$  has a valuation, then for every element  $x$  of  $\text{ValuatRing } v$  such that  $x \notin \text{vp } v$  holds  $v(x) = 0$ .

(78) If  $K$  has a valuation, then  $\text{vp } v$  is prime.

- (79) If  $K$  has a valuation, then for every proper ideal  $I$  of  $\text{ValuatRing } v$  holds  $I \subseteq \text{vp } v$ .
- (80) If  $K$  has a valuation, then  $\text{vp } v$  is maximal.
- (81) If  $K$  has a valuation, then for every maximal ideal  $I$  of  $\text{ValuatRing } v$  holds  $I = \text{vp } v$ .
- (82) If  $K$  has a valuation, then  $\text{NonNegElements normal-valuation } v = \text{NonNegElements } v$ .
- (83) If  $K$  has a valuation, then  $\text{ValuatRing normal-valuation } v = \text{ValuatRing } v$ .

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