FORMALIZED MATHEMATICS
Vol. 20, No. 1, Pages 15–22, 2012
DOI: 10.2478/v10037-012-0003-3

VERSITA
versita.com/fm/

Functional Space $C(\Omega)$, $C_0(\Omega)$

Katuhiko Kanazashi
Shizuoka City
Japan

Hiroyuki Okazaki[1]
Shinshu University
Nagano, Japan

Yasunari Shidama[2]
Shinshu University
Nagano, Japan

Summary. In this article, first we give a definition of a functional space which is constructed from all complex-valued continuous functions defined on a compact topological space. We prove that this functional space is a Banach algebra. Next, we give a definition of a function space which is constructed from all complex-valued continuous functions with bounded support. We also prove that this function space is a complex normed space.

MML identifier: `CC0SP2`, version: `7.12.01 4.167.1133`

The terminology and notation used here have been introduced in the following articles: [6], [24], [25], [1], [26], [5], [4], [2], [21], [15], [3], [18], [19], [23], [22], [17], [7], [11], [12], [9], [10], [13], [8], [14], [20], and [16].

Let X be a topological structure and let f be a function from the carrier of X into $\mathbb{C}$. We say that f is continuous if and only if:

(Def. 1) For every subset Y of $\mathbb{C}$ such that Y is closed holds $f^{-1}(Y)$ is closed.

Let X be a 1-sorted structure and let y be a complex number. The functor $X \longmapsto y$ yielding a function from the carrier of X into $\mathbb{C}$ is defined by:

(Def. 2) $X \longmapsto y = (\text{the carrier of } X) \longmapsto y$.

One can prove the following proposition

(1) Let X be a non empty topological space, y be a complex number, and f be a function from the carrier of X into $\mathbb{C}$. If $f = X \longmapsto y$, then f is continuous.

Let X be a non empty topological space and let y be a complex number. Observe that $X \longmapsto y$ is continuous.

[1]The work of this author was supported by JSPS KAKENHI 22300285.
[2]The work of this author was supported by JSPS KAKENHI 22300285.

ISSN 1426–2630(p), 1898-9934(e)

Let X be a non empty topological space. One can verify that there exists a function from the carrier of X into $\mathbb{C}$ which is continuous.

The following propositions are true:

(2) Let X be a non empty topological space and f be a function from the carrier of X into $\mathbb{C}$. Then f is continuous if and only if for every subset Y of $\mathbb{C}$ such that Y is open holds $f^{-1}(Y)$ is open.

(3) Let X be a non empty topological space and f be a function from the carrier of X into $\mathbb{C}$. Then f is continuous if and only if for every point x of X and for every subset V of $\mathbb{C}$ such that $f(x) \in V$ and V is open there exists a subset W of X such that $x \in W$ and W is open and $f^{\circ}W \subseteq V$.

(4) Let X be a non empty topological space and f, g be continuous functions from the carrier of X into $\mathbb{C}$. Then $f+g$ is a continuous function from the carrier of X into $\mathbb{C}$.

(5) Let X be a non empty topological space, a be a complex number, and f be a continuous function from the carrier of X into $\mathbb{C}$. Then $a \cdot f$ is a continuous function from the carrier of X into $\mathbb{C}$.

(6) Let X be a non empty topological space and f, g be continuous functions from the carrier of X into $\mathbb{C}$. Then $f-g$ is a continuous function from the carrier of X into $\mathbb{C}$.

(7) Let X be a non empty topological space and f, g be continuous functions from the carrier of X into $\mathbb{C}$. Then $f \cdot g$ is a continuous function from the carrier of X into $\mathbb{C}$.

(8) Let X be a non empty topological space and f be a continuous function from the carrier of X into $\mathbb{C}$. Then $|f|$ is a function from the carrier of X into $\mathbb{R}$ and $|f|$ is continuous.

Let X be a non empty topological space. The $\mathbb{C}$-continuous functions of X yields a subset of $\mathbb{C}$-Algebra(the carrier of X) and is defined by:

(Def. 3) The $\mathbb{C}$-continuous functions of $X = \{f : f$ ranges over continuous functions from the carrier of X into $\mathbb{C}\}$.

Let X be a non empty topological space. Observe that the $\mathbb{C}$-continuous functions of X is non empty.

Let X be a non empty topological space. Observe that the $\mathbb{C}$-continuous functions of X is $\mathbb{C}$-additively linearly closed and multiplicatively closed.

Let X be a non empty topological space. The $\mathbb{C}$-algebra of continuous functions of X yielding a complex algebra is defined by the condition (Def. 4).

(Def. 4) The $\mathbb{C}$-algebra of continuous functions of $X = \langle$the $\mathbb{C}$-continuous functions of X, mult(the $\mathbb{C}$-continuous functions of X, $\mathbb{C}$-Algebra(the carrier of X)), Add(the $\mathbb{C}$-continuous functions of X, $\mathbb{C}$-Algebra(the carrier of X)), Mult(the $\mathbb{C}$-continuous functions of X, $\mathbb{C}$-Algebra(the carrier of X)), One(the $\mathbb{C}$-continuous functions of X, $\mathbb{C}$-Algebra(the carrier of

X)), Zero(the $\mathbb{C}$-continuous functions of X, $\mathbb{C}$-Algebra(the carrier of X))⟩.

Next we state the proposition

(9) Let X be a non empty topological space. Then the $\mathbb{C}$-algebra of continuous functions of X is a complex subalgebra of $\mathbb{C}$-Algebra(the carrier of X).

Let X be a non empty topological space. Observe that the $\mathbb{C}$-algebra of continuous functions of X is strict and non empty.

Let X be a non empty topological space. One can check that the $\mathbb{C}$-algebra of continuous functions of X is Abelian, add-associative, right zeroed, right complementable, vector distributive, scalar distributive, scalar associative, scalar unital, commutative, associative, right unital, right distributive, vector distributive, scalar distributive, scalar associative, and vector associative.

Next we state several propositions:

(10) Let X be a non empty topological space, F, G, H be vectors of the $\mathbb{C}$-algebra of continuous functions of X, and f, g, h be functions from the carrier of X into $\mathbb{C}$. Suppose $f = F$ and $g = G$ and $h = H$. Then $H = F + G$ if and only if for every element x of the carrier of X holds $h(x) = f(x) + g(x)$.

(11) Let X be a non empty topological space, F, G be vectors of the $\mathbb{C}$-algebra of continuous functions of X, f, g be functions from the carrier of X into $\mathbb{C}$, and a be a complex number. Suppose $f = F$ and $g = G$. Then $G = a \cdot F$ if and only if for every element x of X holds $g(x) = a \cdot f(x)$.

(12) Let X be a non empty topological space, F, G, H be vectors of the $\mathbb{C}$-algebra of continuous functions of X, and f, g, h be functions from the carrier of X into $\mathbb{C}$. Suppose $f = F$ and $g = G$ and $h = H$. Then $H = F \cdot G$ if and only if for every element x of the carrier of X holds $h(x) = f(x) \cdot g(x)$.

(13) For every non empty topological space X holds $0_{\text{the } \mathbb{C}\text{-algebra of continuous functions of } X} = X \longmapsto 0_{\mathbb{C}}$.

(14) For every non empty topological space X holds $\mathbf{1}_{\text{the } \mathbb{C}\text{-algebra of continuous functions of } X} = X \longmapsto 1_{\mathbb{C}}$.

(15) Let A be a complex algebra and A_1, A_2 be complex subalgebras of A. Suppose the carrier of $A_1 \subseteq$ the carrier of A_2. Then A_1 is a complex subalgebra of A_2.

(16) Let X be a non empty compact topological space. Then the $\mathbb{C}$-algebra of continuous functions of X is a complex subalgebra of the $\mathbb{C}$-algebra of bounded functions of the carrier of X.

Let X be a non empty compact topological space. The $\mathbb{C}$-continuous functions norm of X yields a function from the $\mathbb{C}$-continuous functions of X into $\mathbb{R}$ and is defined by:

(Def. 5) The $\mathbb{C}$-continuous functions norm of $X =$ ($\mathbb{C}$-BoundedFunctionsNorm (the carrier of X))$\restriction$the $\mathbb{C}$-continuous functions of X.

Let X be a non empty compact topological space. The $\mathbb{C}$-normed algebra of continuous functions of X yields a normed complex algebra structure and is defined by the condition (Def. 6).

(Def. 6) The $\mathbb{C}$-normed algebra of continuous functions of $X = \langle$the $\mathbb{C}$-continuous functions of X, mult(the $\mathbb{C}$-continuous functions of X, $\mathbb{C}$-Algebra(the carrier of X)), Add(the $\mathbb{C}$-continuous functions of X, $\mathbb{C}$-Algebra(the carrier of X)), Mult(the $\mathbb{C}$-continuous functions of X, $\mathbb{C}$-Algebra(the carrier of X)), One(the $\mathbb{C}$-continuous functions of X, $\mathbb{C}$-Algebra(the carrier of X)), Zero(the $\mathbb{C}$-continuous functions of X, $\mathbb{C}$-Algebra(the carrier of X)), the $\mathbb{C}$-continuous functions norm of $X\rangle$.

Let X be a non empty compact topological space. Note that the $\mathbb{C}$-normed algebra of continuous functions of X is non empty and strict.

Let X be a non empty compact topological space. Observe that the $\mathbb{C}$-normed algebra of continuous functions of X is unital.

Next we state the proposition

(17) Let X be a non empty compact topological space. Then the $\mathbb{C}$-normed algebra of continuous functions of X is a complex algebra.

Let X be a non empty compact topological space. One can check that the $\mathbb{C}$-normed algebra of continuous functions of X is right complementable, Abelian, add-associative, right zeroed, vector distributive, scalar distributive, scalar associative, associative, commutative, right distributive, right unital, and vector associative.

One can prove the following proposition

(18) Let X be a non empty compact topological space and F be a point of the $\mathbb{C}$-normed algebra of continuous functions of X. Then (Mult(the $\mathbb{C}$-continuous functions of X, $\mathbb{C}$-Algebra(the carrier of X)))$(1_{\mathbb{C}}, F) = F$.

Let X be a non empty compact topological space. Observe that the $\mathbb{C}$-normed algebra of continuous functions of X is vector distributive, scalar distributive, scalar associative, and scalar unital.

We now state a number of propositions:

(19) Let X be a non empty compact topological space. Then the $\mathbb{C}$-normed algebra of continuous functions of X is a complex linear space.

(20) Let X be a non empty compact topological space. Then $X \longmapsto 0 = 0_{\text{the } \mathbb{C}\text{-normed algebra of continuous functions of } X}$.

(21) Let X be a non empty compact topological space and F be a point of the $\mathbb{C}$-normed algebra of continuous functions of X. Then $0 \leq \|F\|$.

(22) Let X be a non empty compact topological space, f, g, h be functions from the carrier of X into $\mathbb{C}$, and F, G, H be points of the $\mathbb{C}$-normed

algebra of continuous functions of X. Suppose $f = F$ and $g = G$ and $h = H$. Then $H = F + G$ if and only if for every element x of X holds $h(x) = f(x) + g(x)$.

(23) Let a be a complex number, X be a non empty compact topological space, f, g be functions from the carrier of X into $\mathbb{C}$, and F, G be points of the $\mathbb{C}$-normed algebra of continuous functions of X. Suppose $f = F$ and $g = G$. Then $G = a \cdot F$ if and only if for every element x of X holds $g(x) = a \cdot f(x)$.

(24) Let X be a non empty compact topological space, f, g, h be functions from the carrier of X into $\mathbb{C}$, and F, G, H be points of the $\mathbb{C}$-normed algebra of continuous functions of X. Suppose $f = F$ and $g = G$ and $h = H$. Then $H = F \cdot G$ if and only if for every element x of X holds $h(x) = f(x) \cdot g(x)$.

(25) Let X be a non empty compact topological space. Then $\|0_{\text{the } \mathbb{C}\text{-normed algebra of continuous functions of } X}\| = 0$.

(26) Let X be a non empty compact topological space and F be a point of the $\mathbb{C}$-normed algebra of continuous functions of X. Suppose $\|F\| = 0$. Then $F = 0_{\text{the } \mathbb{C}\text{-normed algebra of continuous functions of } X}$.

(27) Let a be a complex number, X be a non empty compact topological space, and F be a point of the $\mathbb{C}$-normed algebra of continuous functions of X. Then $\|a \cdot F\| = |a| \cdot \|F\|$.

(28) Let X be a non empty compact topological space and F, G be points of the $\mathbb{C}$-normed algebra of continuous functions of X. Then $\|F + G\| \leq \|F\| + \|G\|$.

Let X be a non empty compact topological space. Observe that the $\mathbb{C}$-normed algebra of continuous functions of X is discernible, reflexive, and complex normed space-like.

The following propositions are true:

(29) Let X be a non empty compact topological space, f, g, h be functions from the carrier of X into $\mathbb{C}$, and F, G, H be points of the $\mathbb{C}$-normed algebra of continuous functions of X. Suppose $f = F$ and $g = G$ and $h = H$. Then $H = F - G$ if and only if for every element x of X holds $h(x) = f(x) - g(x)$.

(30) Let X be a complex Banach space, Y be a subset of X, and s_1 be a sequence of X. Suppose Y is closed and $\operatorname{rng} s_1 \subseteq Y$ and s_1 is $\mathbb{C}$-Cauchy. Then s_1 is convergent and $\lim s_1 \in Y$.

(31) Let X be a non empty compact topological space and Y be a subset of the $\mathbb{C}$-normed algebra of bounded functions of the carrier of X. If $Y =$ the $\mathbb{C}$-continuous functions of X, then Y is closed.

(32) Let X be a non empty compact topological space and s_1 be a sequence

of the $\mathbb{C}$-normed algebra of continuous functions of X. If s_1 is $\mathbb{C}$-Cauchy, then s_1 is convergent.

Let X be a non empty compact topological space. One can verify that the $\mathbb{C}$-normed algebra of continuous functions of X is complete.

Let X be a non empty compact topological space. Observe that the $\mathbb{C}$-normed algebra of continuous functions of X is Banach Algebra-like.

Next we state three propositions:

(33) For every non empty topological space X and for all functions f, g from the carrier of X into $\mathbb{C}$ holds $\mathrm{support}(f+g) \subseteq \mathrm{support}\, f \cup \mathrm{support}\, g$.

(34) Let X be a non empty topological space, a be a complex number, and f be a function from the carrier of X into $\mathbb{C}$. Then $\mathrm{support}(a \cdot f) \subseteq \mathrm{support}\, f$.

(35) For every non empty topological space X and for all functions f, g from the carrier of X into $\mathbb{C}$ holds $\mathrm{support}(f \cdot g) \subseteq \mathrm{support}\, f \cup \mathrm{support}\, g$.

Let X be a non empty topological space. The $\boldsymbol{CC_0}$-functions of X yielding a non empty subset of the $\mathbb{C}$-vector space of the carrier of X is defined by the condition (Def. 7).

(Def. 7) The $\boldsymbol{CC_0}$-functions of $X = \{f; f$ ranges over functions from the carrier of X into $\mathbb{C}$: f is continuous $\wedge \bigvee_{Y:\ \text{non empty subset of}\ X} (Y$ is compact $\wedge \bigwedge_{A:\,\text{subset of}\ X} (A = \mathrm{support}\, f \Rightarrow \overline{A}$ is a subset of $Y))\}$.

The following propositions are true:

(36) Let X be a non empty topological space. Then the $\boldsymbol{CC_0}$-functions of X is a non empty subset of $\mathbb{C}$-Algebra(the carrier of X).

(37) Let X be a non empty topological space and W be a non empty subset of $\mathbb{C}$-Algebra(the carrier of X). Suppose $W =$ the $\boldsymbol{CC_0}$-functions of X. Then W is $\mathbb{C}$-additively linearly closed.

(38) For every non empty topological space X holds the $\boldsymbol{CC_0}$-functions of X is add closed.

(39) For every non empty topological space X holds the $\boldsymbol{CC_0}$-functions of X is linearly closed.

Let X be a non empty topological space. Observe that the $\boldsymbol{CC_0}$-functions of X is non empty and linearly closed.

The following propositions are true:

(40) Let V be a complex linear space and V_1 be a subset of V. Suppose V_1 is linearly closed and V_1 is not empty. Then $\langle V_1, \mathrm{Zero}(V_1, V), \mathrm{Add}(V_1, V), \mathrm{Mult}(V_1, V)\rangle$ is a subspace of V.

(41) Let X be a non empty topological space. Then $\langle$the $\boldsymbol{CC_0}$-functions of X, Zero(the $\boldsymbol{CC_0}$-functions of X, the $\mathbb{C}$-vector space of the carrier of X), Add(the $\boldsymbol{CC_0}$-functions of X, the $\mathbb{C}$-vector space of the carrier of X), Mult(the $\boldsymbol{CC_0}$-functions of X, the $\mathbb{C}$-vector space of the carrier of X)$\rangle$ is a subspace of the $\mathbb{C}$-vector space of the carrier of X.

Let X be a non empty topological space. The $\mathbb{C}$-vector space of $\boldsymbol{C_0}$-functions of X yielding a complex linear space is defined by the condition (Def. 8).

(Def. 8) The $\mathbb{C}$-vector space of $\boldsymbol{C_0}$-functions of X = $\langle$the $\boldsymbol{CC_0}$-functions of X, Zero(the $\boldsymbol{CC_0}$-functions of X, the $\mathbb{C}$-vector space of the carrier of X), Add(the $\boldsymbol{CC_0}$-functions of X, the $\mathbb{C}$-vector space of the carrier of X), Mult(the $\boldsymbol{CC_0}$-functions of X, the $\mathbb{C}$-vector space of the carrier of X)$\rangle$.

Next we state the proposition

(42) Let X be a non empty topological space and x be a set. If $x \in$ the $\boldsymbol{CC_0}$-functions of X, then $x \in \mathbb{C}$-BoundedFunctions (the carrier of X).

Let X be a non empty topological space. The $\boldsymbol{CC_0}$-functions norm of X yielding a function from the $\boldsymbol{CC_0}$-functions of X into $\mathbb{R}$ is defined by:

(Def. 9) The $\boldsymbol{CC_0}$-functions norm of X = ($\mathbb{C}$-BoundedFunctionsNorm (the carrier of X))$\restriction$the $\boldsymbol{CC_0}$-functions of X.

Let X be a non empty topological space. The $\mathbb{C}$-normed space of $\boldsymbol{C_0}$-functions of X yielding a complex normed space structure is defined by the condition (Def. 10).

(Def. 10) The $\mathbb{C}$-normed space of $\boldsymbol{C_0}$-functions of X = $\langle$the $\boldsymbol{CC_0}$-functions of X, Zero(the $\boldsymbol{CC_0}$-functions of X, the $\mathbb{C}$-vector space of the carrier of X), Add(the $\boldsymbol{CC_0}$-functions of X, the $\mathbb{C}$-vector space of the carrier of X), Mult(the $\boldsymbol{CC_0}$-functions of X, the $\mathbb{C}$-vector space of the carrier of X), the $\boldsymbol{CC_0}$-functions norm of $X\rangle$.

Let X be a non empty topological space. One can check that the $\mathbb{C}$-normed space of $\boldsymbol{C_0}$-functions of X is strict and non empty.

One can prove the following propositions:

(43) Let X be a non empty topological space and x be a set. Suppose $x \in$ the $\boldsymbol{CC_0}$-functions of X. Then $x \in$ the $\mathbb{C}$-continuous functions of X.

(44) For every non empty topological space X holds $0_{\text{the } \mathbb{C}\text{-vector space of } \boldsymbol{C_0}\text{-functions of } X} = X \longmapsto 0$.

(45) For every non empty topological space X holds $0_{\text{the } \mathbb{C}\text{-normed space of } \boldsymbol{C_0}\text{-functions of } X} = X \longmapsto 0$.

(46) Let a be a complex number, X be a non empty topological space, and F, G be points of the $\mathbb{C}$-normed space of $\boldsymbol{C_0}$-functions of X. Then $\|F\| = 0$ iff $F = 0_{\text{the } \mathbb{C}\text{-normed space of } \boldsymbol{C_0}\text{-functions of } X}$ and $\|a \cdot F\| = |a| \cdot \|F\|$ and $\|F + G\| \leq \|F\| + \|G\|$.

Let X be a non empty topological space. Note that the $\mathbb{C}$-normed space of $\boldsymbol{C_0}$-functions of X is reflexive, discernible, complex normed space-like, vector distributive, scalar distributive, scalar associative, scalar unital, Abelian, add-associative, right zeroed, and right complementable.

The following proposition is true

(47) Let X be a non empty topological space. Then the $\mathbb{C}$-normed space of $\boldsymbol{C_0}$-functions of X is a complex normed space.

References

[1] Grzegorz Bancerek. The ordinal numbers. *Formalized Mathematics*, 1(**1**):91–96, 1990.
[2] Czesław Byliński. Binary operations. *Formalized Mathematics*, 1(**1**):175–180, 1990.
[3] Czesław Byliński. The complex numbers. *Formalized Mathematics*, 1(**3**):507–513, 1990.
[4] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.
[5] Czesław Byliński. Partial functions. *Formalized Mathematics*, 1(**2**):357–367, 1990.
[6] Czesław Byliński. Some basic properties of sets. *Formalized Mathematics*, 1(**1**):47–53, 1990.
[7] Agata Darmochwał. Compact spaces. *Formalized Mathematics*, 1(**2**):383–386, 1990.
[8] Noboru Endou. Banach algebra of bounded complex linear operators. *Formalized Mathematics*, 12(**3**):237–242, 2004.
[9] Noboru Endou. Banach space of absolute summable complex sequences. *Formalized Mathematics*, 12(**2**):191–194, 2004.
[10] Noboru Endou. Complex Banach space of bounded linear operators. *Formalized Mathematics*, 12(**2**):201–209, 2004.
[11] Noboru Endou. Complex linear space and complex normed space. *Formalized Mathematics*, 12(**2**):93–102, 2004.
[12] Noboru Endou. Complex linear space of complex sequences. *Formalized Mathematics*, 12(**2**):109–117, 2004.
[13] Noboru Endou. Complex valued functions space. *Formalized Mathematics*, 12(**3**):231–235, 2004.
[14] Noboru Endou. Continuous functions on real and complex normed linear spaces. *Formalized Mathematics*, 12(**3**):403–419, 2004.
[15] Krzysztof Hryniewiecki. Basic properties of real numbers. *Formalized Mathematics*, 1(**1**):35–40, 1990.
[16] Katuhiko Kanazashi, Hiroyuki Okazaki, and Yasunari Shidama. Banach algebra of bounded complex-valued functionals. *Formalized Mathematics*, 19(**2**):121–126, 2011, doi: 10.2478/v10037-011-0019-0.
[17] Eugeniusz Kusak, Wojciech Leończuk, and Michał Muzalewski. Abelian groups, fields and vector spaces. *Formalized Mathematics*, 1(**2**):335–342, 1990.
[18] Chanapat Pacharapokin, Hiroshi Yamazaki, Yasunari Shidama, and Yatsuka Nakamura. Complex function differentiability. *Formalized Mathematics*, 17(**2**):67–72, 2009, doi: 10.2478/v10037-009-0007-9.
[19] Beata Padlewska and Agata Darmochwał. Topological spaces and continuous functions. *Formalized Mathematics*, 1(**1**):223–230, 1990.
[20] Yasunari Shidama, Hikofumi Suzuki, and Noboru Endou. Banach algebra of bounded functionals. *Formalized Mathematics*, 16(**2**):115–122, 2008, doi:10.2478/v10037-008-0017-z.
[21] Andrzej Trybulec. Binary operations applied to functions. *Formalized Mathematics*, 1(**2**):329–334, 1990.
[22] Wojciech A. Trybulec. Groups. *Formalized Mathematics*, 1(**5**):821–827, 1990.
[23] Wojciech A. Trybulec. Vectors in real linear space. *Formalized Mathematics*, 1(**2**):291–296, 1990.
[24] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(**1**):67–71, 1990.
[25] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1(**1**):73–83, 1990.
[26] Edmund Woronowicz. Relations defined on sets. *Formalized Mathematics*, 1(**1**):181–186, 1990.

Received May 30, 2011