

Differentiable Functions on Normed Linear Spaces¹

Yasunari Shidama
Shinshu University
Nagano, Japan

Summary. In this article, we formalize differentiability of functions on normed linear spaces. Partial derivative, mean value theorem for vector-valued functions, continuous differentiability, etc. are formalized. As it is well known, there is no exact analog of the mean value theorem for vector-valued functions. However a certain type of generalization of the mean value theorem for vector-valued functions is obtained as follows: If $\|f'(x+t \cdot h)\|$ is bounded for t between 0 and 1 by some constant M , then $\|f(x+t \cdot h) - f(x)\| \leq M \cdot \|h\|$. This theorem is called the mean value theorem for vector-valued functions. By this theorem, the relation between the (total) derivative and the partial derivatives of a function is derived [23].

MML identifier: NDIFF_5, version: 7.12.01 4.167.1133

The notation and terminology used here have been introduced in the following papers: [28], [29], [9], [4], [30], [12], [10], [25], [11], [1], [2], [26], [7], [3], [5], [8], [17], [22], [20], [27], [21], [31], [14], [24], [18], [16], [15], [19], [13], and [6].

1. PRELIMINARIES

In this paper r is a real number and S, T are non trivial real normed spaces. Next we state several propositions:

- (1) Let R be a function from \mathbb{R} into S . Then R is rest-like if and only if for every real number r such that $r > 0$ there exists a real number d such that $d > 0$ and for every real number z such that $z \neq 0$ and $|z| < d$ holds $|z|^{-1} \cdot \|R_z\| < r$.

¹This work was supported by JSPS KAKENHI 22300285.

- (2) Let R be a rest of S . Suppose $R_0 = 0_S$. Let e be a real number. Suppose $e > 0$. Then there exists a real number d such that $d > 0$ and for every real number h such that $|h| < d$ holds $\|R_h\| \leq e \cdot |h|$.
- (3) For every rest R of S and for every bounded linear operator L from S into T holds $L \cdot R$ is a rest of T .
- (4) Let R_1 be a rest of S . Suppose $(R_1)_0 = 0_S$. Let R_2 be a rest of S, T . If $(R_2)_{0_S} = 0_T$, then for every linear L of S holds $R_2 \cdot (L + R_1)$ is a rest of T .
- (5) Let R_1 be a rest of S . Suppose $(R_1)_0 = 0_S$. Let R_2 be a rest of S, T . Suppose $(R_2)_{0_S} = 0_T$. Let L_1 be a linear of S and L_2 be a bounded linear operator from S into T . Then $L_2 \cdot R_1 + R_2 \cdot (L_1 + R_1)$ is a rest of T .
- (6) Let x_0 be an element of \mathbb{R} and g be a partial function from \mathbb{R} to the carrier of S . Suppose g is differentiable in x_0 . Let f be a partial function from the carrier of S to the carrier of T . Suppose f is differentiable in g_{x_0} . Then $f \cdot g$ is differentiable in x_0 and $(f \cdot g)'(x_0) = f'(g_{x_0})(g'(x_0))$.
- (7) Let S be a real normed space, x_1 be a finite sequence of elements of S , and y_1 be a finite sequence of elements of \mathbb{R} . Suppose $\text{len } x_1 = \text{len } y_1$ and for every element i of \mathbb{N} such that $i \in \text{dom } x_1$ holds $y_1(i) = \|(x_1)_i\|$. Then $\|\sum x_1\| \leq \sum y_1$.
- (8) Let S be a real normed space, x be a point of S , and N_1, N_2 be neighbourhoods of x . Then $N_1 \cap N_2$ is a neighbourhood of x .
- (9) For every non-empty finite sequence X and for every set x such that $x \in \prod X$ holds x is a finite sequence.

Let G be a real norm space sequence. One can verify that $\prod G$ is constituted finite sequences.

Let G be a real linear space sequence, let z be an element of $\prod \overline{G}$, and let j be an element of $\text{dom } G$. Then $z(j)$ is an element of $G(j)$.

One can prove the following propositions:

- (10) The carrier of $\prod G = \prod \overline{G}$.
- (11) Let i be an element of $\text{dom } G$, r be a set, and x be a function. If $r \in$ the carrier of $G(i)$ and $x \in \prod \overline{G}$, then $x + \cdot (i, r) \in$ the carrier of $\prod G$.

Let G be a real norm space sequence. We say that G is nontrivial if and only if:

- (Def. 1) For every element j of $\text{dom } G$ holds $G(j)$ is non trivial.

Let us mention that there exists a real norm space sequence which is non-trivial.

Let G be a nontrivial real norm space sequence and let i be an element of $\text{dom } G$. Note that $G(i)$ is non trivial.

Let G be a nontrivial real norm space sequence. Note that $\prod G$ is non trivial. The following propositions are true:

- (12) Let G be a real norm space sequence, p, q be points of $\prod G$, and r_0, p_0, q_0 be elements of $\prod \overline{G}$. Suppose $p = p_0$ and $q = q_0$. Then $p + q = r_0$ if and only if for every element i of $\text{dom } G$ holds $r_0(i) = p_0(i) + q_0(i)$.
- (13) Let G be a real norm space sequence, p be a point of $\prod G$, r be a real number, and r_0, p_0 be elements of $\prod \overline{G}$. Suppose $p = p_0$. Then $r \cdot p = r_0$ if and only if for every element i of $\text{dom } G$ holds $r_0(i) = r \cdot p_0(i)$.
- (14) Let G be a real norm space sequence and p_0 be an element of $\prod \overline{G}$. Then $0_{\prod G} = p_0$ if and only if for every element i of $\text{dom } G$ holds $p_0(i) = 0_{G(i)}$.
- (15) Let G be a real norm space sequence, p, q be points of $\prod G$, and r_0, p_0, q_0 be elements of $\prod \overline{G}$. Suppose $p = p_0$ and $q = q_0$. Then $p - q = r_0$ if and only if for every element i of $\text{dom } G$ holds $r_0(i) = p_0(i) - q_0(i)$.

2. MEAN VALUE THEOREM FOR VECTOR-VALUED FUNCTIONS

Let S be a real linear space and let p, q be points of S . The functor $]p, q[$ yielding a subset of S is defined as follows:

(Def. 2) $]p, q[= \{p + t \cdot (q - p); t \text{ ranges over real numbers: } 0 < t \wedge t < 1\}$.

Let S be a real linear space and let p, q be points of S . We introduce $[p, q]$ as a synonym of $\mathcal{L}(p, q)$.

Next we state several propositions:

- (16) For every real linear space S and for all points p, q of S holds $]p, q[\subseteq [p, q]$.
- (17) Let T be a non trivial real normed space and R be a partial function from \mathbb{R} to T . Suppose R is total. Then R is rest-like if and only if for every real number r such that $r > 0$ there exists a real number d such that $d > 0$ and for every real number z such that $z \neq 0$ and $|z| < d$ holds $\frac{\|R_z\|}{|z|} < r$.
- (18) Let R be a function from \mathbb{R} into \mathbb{R} . Then R is rest-like if and only if for every real number r such that $r > 0$ there exists a real number d such that $d > 0$ and for every real number z such that $z \neq 0$ and $|z| < d$ holds $\frac{|R(z)|}{|z|} < r$.
- (19) Let S, T be non trivial real normed spaces, f be a partial function from S to T , p, q be points of S , and M be a real number. Suppose that
- (i) $]p, q[\subseteq \text{dom } f$,
 - (ii) for every point x of S such that $x \in [p, q]$ holds f is continuous in x ,
 - (iii) for every point x of S such that $x \in]p, q[$ holds f is differentiable in x , and
 - (iv) for every point x of S such that $x \in]p, q[$ holds $\|f'(x)\| \leq M$.
- Then $\|f_q - f_p\| \leq M \cdot \|q - p\|$.

- (20) Let S, T be non trivial real normed spaces, f be a partial function from S to T , p, q be points of S , M be a real number, and L be a point of the real norm space of bounded linear operators from S into T . Suppose that
- (i) $[p, q] \subseteq \text{dom } f$,
 - (ii) for every point x of S such that $x \in [p, q]$ holds f is continuous in x ,
 - (iii) for every point x of S such that $x \in]p, q[$ holds f is differentiable in x ,
and
 - (iv) for every point x of S such that $x \in]p, q[$ holds $\|f'(x) - L\| \leq M$.
- Then $\|f_q - f_p - L(q - p)\| \leq M \cdot \|q - p\|$.

3. PARTIAL DERIVATIVE OF A FUNCTION OF SEVERAL VARIABLES

Let G be a real norm space sequence and let i be an element of $\text{dom } G$. The projection onto i yielding a function from $\prod G$ into $G(i)$ is defined by:

(Def. 3) For every element x of $\prod \overline{G}$ holds (the projection onto i)(x) = $x(i)$.

Let G be a real norm space sequence, let i be an element of $\text{dom } G$, and let x be an element of $\prod G$. The functor $\text{reproj}(i, x)$ yielding a function from $G(i)$ into $\prod G$ is defined by:

(Def. 4) For every element r of $G(i)$ holds $(\text{reproj}(i, x))(r) = x + \cdot (i, r)$.

Let G be a nontrivial real norm space sequence and let j be a set. Let us assume that $j \in \text{dom } G$. The functor $\text{modetrans}(G, j)$ yields an element of $\text{dom } G$ and is defined by:

(Def. 5) $\text{modetrans}(G, j) = j$.

Let G be a nontrivial real norm space sequence, let F be a non trivial real normed space, let i be a set, let f be a partial function from $\prod G$ to F , and let x be an element of $\prod G$. We say that f is partially differentiable in x w.r.t. i if and only if:

(Def. 6) $f \cdot \text{reproj}(\text{modetrans}(G, i), x)$ is differentiable in (the projection onto $\text{modetrans}(G, i)$)(x).

Let G be a nontrivial real norm space sequence, let F be a non trivial real normed space, let i be a set, let f be a partial function from $\prod G$ to F , and let x be a point of $\prod G$. The functor $\text{partdiff}(f, x, i)$ yielding a point of the real norm space of bounded linear operators from $G(\text{modetrans}(G, i))$ into F is defined as follows:

(Def. 7) $\text{partdiff}(f, x, i) = (f \cdot \text{reproj}(\text{modetrans}(G, i), x))'((\text{the projection onto } \text{modetrans}(G, i))(x))$.

4. LINEARITY OF PARTIAL DIFFERENTIAL OPERATOR

For simplicity, we adopt the following rules: G denotes a nontrivial real norm space sequence, F denotes a non trivial real normed space, i denotes an element of $\text{dom } G$, f, f_1, f_2 denote partial functions from $\prod G$ to F , x denotes a point of $\prod G$, and X denotes a set.

Let G be a nontrivial real norm space sequence, let F be a non trivial real normed space, let i be a set, let f be a partial function from $\prod G$ to F , and let X be a set. We say that f is partially differentiable on X w.r.t. i if and only if:

(Def. 8) $X \subseteq \text{dom } f$ and for every point x of $\prod G$ such that $x \in X$ holds $f|X$ is partially differentiable in x w.r.t. i .

Next we state several propositions:

- (21) For every element x_2 of $G(i)$ holds $\|(\text{reproj}(i, 0_{\prod G}))(x_2)\| = \|x_2\|$.
- (22) Let G be a nontrivial real norm space sequence, i be an element of $\text{dom } G$, x be a point of $\prod G$, and r be a point of $G(i)$. Then $(\text{reproj}(i, x))(r) - x = (\text{reproj}(i, 0_{\prod G}))(r - (\text{the projection onto } i)(x))$ and $x - (\text{reproj}(i, x))(r) = (\text{reproj}(i, 0_{\prod G}))((\text{the projection onto } i)(x) - r)$.
- (23) Let G be a nontrivial real norm space sequence, i be an element of $\text{dom } G$, x be a point of $\prod G$, and Z be a subset of $\prod G$. Suppose Z is open and $x \in Z$. Then there exists a neighbourhood N of $(\text{the projection onto } i)(x)$ such that for every point z of $G(i)$ if $z \in N$, then $(\text{reproj}(i, x))(z) \in Z$.
- (24) Let G be a nontrivial real norm space sequence, T be a non trivial real normed space, i be a set, f be a partial function from $\prod G$ to T , and Z be a subset of $\prod G$. Suppose Z is open. Then f is partially differentiable on Z w.r.t. i if and only if $Z \subseteq \text{dom } f$ and for every point x of $\prod G$ such that $x \in Z$ holds f is partially differentiable in x w.r.t. i .
- (25) For every set i such that $i \in \text{dom } G$ and f is partially differentiable on X w.r.t. i holds X is a subset of $\prod G$.

Let G be a nontrivial real norm space sequence, let S be a non trivial real normed space, and let i be a set. Let us assume that $i \in \text{dom } G$. Let f be a partial function from $\prod G$ to S and let X be a set. Let us assume that f is partially differentiable on X w.r.t. i . The functor $f|X$ yields a partial function from $\prod G$ to the real norm space of bounded linear operators from $G(\text{modetrans}(G, i))$ into S and is defined by:

(Def. 9) $\text{dom}(f|X) = X$ and for every point x of $\prod G$ such that $x \in X$ holds $(f|X)_x = \text{partdiff}(f, x, i)$.

One can prove the following propositions:

- (26) For every set i such that $i \in \text{dom } G$ holds $(f_1 + f_2) \cdot \text{reproj}(\text{modetrans}(G, i), x) = f_1 \cdot \text{reproj}(\text{modetrans}(G, i), x) + f_2 \cdot \text{reproj}(\text{modetrans}(G, i), x)$.

- $\text{reproj}(\text{modetrans}(G, i), x)$ and $(f_1 - f_2) \cdot \text{reproj}(\text{modetrans}(G, i), x) = f_1 \cdot \text{reproj}(\text{modetrans}(G, i), x) - f_2 \cdot \text{reproj}(\text{modetrans}(G, i), x)$.
- (27) For every set i such that $i \in \text{dom } G$ holds $r \cdot (f \cdot \text{reproj}(\text{modetrans}(G, i), x)) = (r \cdot f) \cdot \text{reproj}(\text{modetrans}(G, i), x)$.
- (28) Let i be a set. Suppose $i \in \text{dom } G$ and f_1 is partially differentiable in x w.r.t. i and f_2 is partially differentiable in x w.r.t. i . Then $f_1 + f_2$ is partially differentiable in x w.r.t. i and $\text{partdiff}(f_1 + f_2, x, i) = \text{partdiff}(f_1, x, i) + \text{partdiff}(f_2, x, i)$.
- (29) Let i be a set. Suppose $i \in \text{dom } G$ and f_1 is partially differentiable in x w.r.t. i and f_2 is partially differentiable in x w.r.t. i . Then $f_1 - f_2$ is partially differentiable in x w.r.t. i and $\text{partdiff}(f_1 - f_2, x, i) = \text{partdiff}(f_1, x, i) - \text{partdiff}(f_2, x, i)$.
- (30) Let i be a set. Suppose $i \in \text{dom } G$ and f is partially differentiable in x w.r.t. i . Then $r \cdot f$ is partially differentiable in x w.r.t. i and $\text{partdiff}(r \cdot f, x, i) = r \cdot \text{partdiff}(f, x, i)$.

5. CONTINUOUS DIFFERENTIABILITY OF PARTIAL DERIVATIVE

Next we state the proposition

- (31) $\|(\text{the projection onto } i)(x)\| \leq \|x\|$.

Let G be a nontrivial real norm space sequence. One can verify that every point of $\prod G$ is $\text{len } G$ -element.

We now state a number of propositions:

- (32) Let G be a nontrivial real norm space sequence, T be a non trivial real normed space, i be a set, Z be a subset of $\prod G$, and f be a partial function from $\prod G$ to T . Suppose Z is open. Then f is partially differentiable on Z w.r.t. i if and only if $Z \subseteq \text{dom } f$ and for every point x of $\prod G$ such that $x \in Z$ holds f is partially differentiable in x w.r.t. i .
- (33) Let i, j be elements of $\text{dom } G$, x be a point of $G(i)$, and z be an element of $\prod \overline{G}$ such that $z = (\text{reproj}(i, 0_{\prod G}))(x)$. Then
- (i) if $i = j$, then $z(j) = x$, and
 - (ii) if $i \neq j$, then $z(j) = 0_{G(j)}$.
- (34) For all points x, y of $G(i)$ holds $(\text{reproj}(i, 0_{\prod G}))(x + y) = (\text{reproj}(i, 0_{\prod G}))(x) + (\text{reproj}(i, 0_{\prod G}))(y)$.
- (35) Let x, y be points of $\prod G$. Then $(\text{the projection onto } i)(x + y) = (\text{the projection onto } i)(x) + (\text{the projection onto } i)(y)$.
- (36) For all points x, y of $G(i)$ holds $(\text{reproj}(i, 0_{\prod G}))(x - y) = (\text{reproj}(i, 0_{\prod G}))(x) - (\text{reproj}(i, 0_{\prod G}))(y)$.

- (37) Let x, y be points of $\prod G$. Then (the projection onto i)($x - y$) = (the projection onto i)(x) - (the projection onto i)(y).
- (38) For every point x of $G(i)$ such that $x \neq 0_{G(i)}$ holds $(\text{reproj}(i, 0_{\prod G}))(x) \neq 0_{\prod G}$.
- (39) For every point x of $G(i)$ and for every element a of \mathbb{R} holds $(\text{reproj}(i, 0_{\prod G}))(a \cdot x) = a \cdot (\text{reproj}(i, 0_{\prod G}))(x)$.
- (40) Let x be a point of $\prod G$ and a be an element of \mathbb{R} . Then (the projection onto i)($a \cdot x$) = $a \cdot$ (the projection onto i)(x).
- (41) Let G be a nontrivial real norm space sequence, S be a non trivial real normed space, f be a partial function from $\prod G$ to S , x be a point of $\prod G$, and i be a set. Suppose f is differentiable in x . Then f is partially differentiable in x w.r.t. i and $\text{partdiff}(f, x, i) = f'(x) \cdot \text{reproj}(\text{modetrans}(G, i), 0_{\prod G})$.
- (42) Let S be a real normed space and h, g be finite sequences of elements of S . Suppose $\text{len } h = \text{len } g + 1$ and for every natural number i such that $i \in \text{dom } g$ holds $g_i = h_i - h_{i+1}$. Then $h_1 - h_{\text{len } h} = \sum g$.
- (43) Let G be a nontrivial real norm space sequence, x, y be elements of $\prod \overline{G}$, and Z be a set. Then $x + \cdot y \upharpoonright Z$ is an element of $\prod \overline{G}$.
- (44) Let G be a nontrivial real norm space sequence, x, y be points of $\prod G$, Z, x_0 be elements of $\prod \overline{G}$, and X be a set. If $Z = 0_{\prod G}$ and $x_0 = x$ and $y = Z + \cdot x_0 \upharpoonright X$, then $\|y\| \leq \|x\|$.
- (45) Let G be a nontrivial real norm space sequence, S be a non trivial real normed space, f be a partial function from $\prod G$ to S , and x, y be points of $\prod G$. Then there exists a finite sequence h of elements of $\prod G$ and there exists a finite sequence g of elements of S and there exist elements Z, y_0 of $\prod \overline{G}$ such that $y_0 = y$ and $Z = 0_{\prod G}$ and $\text{len } h = \text{len } G + 1$ and $\text{len } g = \text{len } G$ and for every natural number i such that $i \in \text{dom } h$ holds $h_i = Z + \cdot y_0 \upharpoonright \text{Seg}((\text{len } G + 1) - i)$ and for every natural number i such that $i \in \text{dom } g$ holds $g_i = f_{x+h_i} - f_{x+h_{i+1}}$ and for every natural number i and for every point h_1 of $\prod G$ such that $i \in \text{dom } h$ and $h_i = h_1$ holds $\|h_1\| \leq \|y\|$ and $f_{x+y} - f_x = \sum g$.
- (46) Let G be a nontrivial real norm space sequence, i be an element of $\text{dom } G$, x, y be points of $\prod G$, and x_2 be a point of $G(i)$. If $y = (\text{reproj}(i, x))(x_2)$, then (the projection onto i)(y) = x_2 .
- (47) Let G be a nontrivial real norm space sequence, i be an element of $\text{dom } G$, y be a point of $\prod G$, and q be a point of $G(i)$. If $q =$ (the projection onto i)(y), then $y = (\text{reproj}(i, y))(q)$.
- (48) Let G be a nontrivial real norm space sequence, i be an element of $\text{dom } G$, x, y be points of $\prod G$, and x_2 be a point of $G(i)$. If $y = (\text{reproj}(i, x))(x_2)$, then $\text{reproj}(i, x) = \text{reproj}(i, y)$.

- (49) Let G be a nontrivial real norm space sequence, i, j be elements of $\text{dom } G$, x, y be points of $\prod G$, and x_2 be a point of $G(i)$. Suppose $y = (\text{reproj}(i, x))(x_2)$ and $i \neq j$. Then $(\text{the projection onto } j)(x) = (\text{the projection onto } j)(y)$.
- (50) Let G be a nontrivial real norm space sequence, F be a non trivial real normed space, i be an element of $\text{dom } G$, x be a point of $\prod G$, x_2 be a point of $G(i)$, f be a partial function from $\prod G$ to F , and g be a partial function from $G(i)$ to F . If $(\text{the projection onto } i)(x) = x_2$ and $g = f \cdot \text{reproj}(i, x)$, then $g'(x_2) = \text{partdiff}(f, x, i)$.
- (51) Let G be a nontrivial real norm space sequence, F be a non trivial real normed space, f be a partial function from $\prod G$ to F , x be a point of $\prod G$, i be a set, M be a real number, L be a point of the real norm space of bounded linear operators from $G(\text{modetrans}(G, i))$ into F , and p, q be points of $G(\text{modetrans}(G, i))$. Suppose that
- (i) $i \in \text{dom } G$,
 - (ii) for every point h of $G(\text{modetrans}(G, i))$ such that $h \in]p, q[$ holds $\|\text{partdiff}(f, (\text{reproj}(\text{modetrans}(G, i), x))(h), i) - L\| \leq M$,
 - (iii) for every point h of $G(\text{modetrans}(G, i))$ such that $h \in [p, q]$ holds $(\text{reproj}(\text{modetrans}(G, i), x))(h) \in \text{dom } f$, and
 - (iv) for every point h of $G(\text{modetrans}(G, i))$ such that $h \in [p, q]$ holds f is partially differentiable in $(\text{reproj}(\text{modetrans}(G, i), x))(h)$ w.r.t. i .
- Then $\|f_{(\text{reproj}(\text{modetrans}(G, i), x))(q)} - f_{(\text{reproj}(\text{modetrans}(G, i), x))(p)} - L(q - p)\| \leq M \cdot \|q - p\|$.
- (52) Let G be a nontrivial real norm space sequence, x, y, z, w be points of $\prod G$, i be an element of $\text{dom } G$, d be a real number, and p, q, r be points of $G(i)$. Suppose $\|y - x\| < d$ and $\|z - x\| < d$ and $p = (\text{the projection onto } i)(y)$ and $z = (\text{reproj}(i, y))(q)$ and $r \in [p, q]$ and $w = (\text{reproj}(i, y))(r)$. Then $\|w - x\| < d$.
- (53) Let G be a nontrivial real norm space sequence, S be a non trivial real normed space, f be a partial function from $\prod G$ to S , X be a subset of $\prod G$, x, y, z be points of $\prod G$, i be a set, p, q be points of $G(\text{modetrans}(G, i))$, and d, r be real numbers. Suppose that $i \in \text{dom } G$ and X is open and $x \in X$ and $\|y - x\| < d$ and $\|z - x\| < d$ and $X \subseteq \text{dom } f$ and for every point x of $\prod G$ such that $x \in X$ holds f is partially differentiable in x w.r.t. i and for every point z of $\prod G$ such that $\|z - x\| < d$ holds $z \in X$ and for every point z of $\prod G$ such that $\|z - x\| < d$ holds $\|\text{partdiff}(f, z, i) - \text{partdiff}(f, x, i)\| \leq r$ and $z = (\text{reproj}(\text{modetrans}(G, i), y))(p)$ and $q = (\text{the projection onto } \text{modetrans}(G, i))(y)$. Then $\|f_z - f_y - (\text{partdiff}(f, x, i))(p - q)\| \leq \|p - q\| \cdot r$.
- (54) Let G be a nontrivial real norm space sequence, h be a finite sequence of elements of $\prod G$, y, x be points of $\prod G$, y_0, Z be elements of $\prod \overline{G}$, and j be an element of \mathbb{N} . Suppose $y = y_0$ and $Z = 0_{\prod G}$ and

len $h = \text{len } G + 1$ and $1 \leq j \leq \text{len } G$ and for every natural number i such that $i \in \text{dom } h$ holds $h_i = Z + \cdot y_0 \upharpoonright \text{Seg}((\text{len } G + 1) -' i)$. Then $x + h_j = (\text{reproj}(\text{modetrans}(G, (\text{len } G + 1) -' j), x + h_{j+1}))$ (the projection onto $\text{modetrans}(G, (\text{len } G + 1) -' j)(x + y)$).

(55) Let G be a nontrivial real norm space sequence, h be a finite sequence of elements of $\prod G$, y, x be points of $\prod G$, y_0, Z be elements of $\prod \bar{G}$, and j be an element of \mathbb{N} . Suppose $y = y_0$ and $Z = 0_{\prod G}$ and $\text{len } h = \text{len } G + 1$ and $1 \leq j \leq \text{len } G$ and for every natural number i such that $i \in \text{dom } h$ holds $h_i = Z + \cdot y_0 \upharpoonright \text{Seg}((\text{len } G + 1) -' i)$. Then (the projection onto $\text{modetrans}(G, (\text{len } G + 1) -' j)(x + y)$ - (the projection onto $\text{modetrans}(G, (\text{len } G + 1) -' j)(x + h_{j+1})$) = (the projection onto $\text{modetrans}(G, (\text{len } G + 1) -' j)(y)$).

(56) Let G be a nontrivial real norm space sequence, S be a non trivial real normed space, f be a partial function from $\prod G$ to S , X be a subset of $\prod G$, and x be a point of $\prod G$. Suppose that

- (i) X is open,
- (ii) $x \in X$, and
- (iii) for every set i such that $i \in \text{dom } G$ holds f is partially differentiable on X w.r.t. i and $f \upharpoonright^i X$ is continuous on X .

Then

- (iv) f is differentiable in x , and
- (v) for every point h of $\prod G$ there exists a finite sequence w of elements of S such that $\text{dom } w = \text{dom } G$ and for every set i such that $i \in \text{dom } G$ holds $w(i) = (\text{partdiff}(f, x, i))$ (the projection onto $\text{modetrans}(G, i)(h)$) and $f'(x)(h) = \sum w$.

(57) Let G be a nontrivial real norm space sequence, F be a non trivial real normed space, f be a partial function from $\prod G$ to F , and X be a subset of $\prod G$. Suppose X is open. Then for every set i such that $i \in \text{dom } G$ holds f is partially differentiable on X w.r.t. i and $f \upharpoonright^i X$ is continuous on X if and only if f is differentiable on X and $f' \upharpoonright_X$ is continuous on X .

REFERENCES

- [1] Grzegorz Bancerek. Cardinal numbers. *Formalized Mathematics*, 1(2):377–382, 1990.
- [2] Grzegorz Bancerek. The fundamental properties of natural numbers. *Formalized Mathematics*, 1(1):41–46, 1990.
- [3] Grzegorz Bancerek. König's theorem. *Formalized Mathematics*, 1(3):589–593, 1990.
- [4] Grzegorz Bancerek. The ordinal numbers. *Formalized Mathematics*, 1(1):91–96, 1990.
- [5] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Formalized Mathematics*, 1(1):107–114, 1990.
- [6] Grzegorz Bancerek and Andrzej Trybulec. Miscellaneous facts about functions. *Formalized Mathematics*, 5(4):485–492, 1996.
- [7] Czesław Byliński. The complex numbers. *Formalized Mathematics*, 1(3):507–513, 1990.
- [8] Czesław Byliński. Finite sequences and tuples of elements of a non-empty sets. *Formalized Mathematics*, 1(3):529–536, 1990.

- [9] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(1):55–65, 1990.
- [10] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(1):153–164, 1990.
- [11] Czesław Byliński. The modification of a function by a function and the iteration of the composition of a function. *Formalized Mathematics*, 1(3):521–527, 1990.
- [12] Czesław Byliński. Partial functions. *Formalized Mathematics*, 1(2):357–367, 1990.
- [13] Czesław Byliński. Introduction to real linear topological spaces. *Formalized Mathematics*, 13(1):99–107, 2005.
- [14] Agata Darmochwał. The Euclidean space. *Formalized Mathematics*, 2(4):599–603, 1991.
- [15] Noboru Endou, Yasunari Shidama, and Keiichi Miyajima. The product space of real normed spaces and its properties. *Formalized Mathematics*, 15(3):81–85, 2007, doi:10.2478/v10037-007-0010-y.
- [16] Hiroshi Imura, Morishige Kimura, and Yasunari Shidama. The differentiable functions on normed linear spaces. *Formalized Mathematics*, 12(3):321–327, 2004.
- [17] Jarosław Kotowicz. Real sequences and basic operations on them. *Formalized Mathematics*, 1(2):269–272, 1990.
- [18] Anna Lango and Grzegorz Bancerek. Product of families of groups and vector spaces. *Formalized Mathematics*, 3(2):235–240, 1992.
- [19] Hiroyuki Okazaki, Noboru Endou, Keiko Narita, and Yasunari Shidama. Differentiable functions into real normed spaces. *Formalized Mathematics*, 19(2):69–72, 2011, doi:10.2478/v10037-011-0012-7.
- [20] Beata Padlewska and Agata Darmochwał. Topological spaces and continuous functions. *Formalized Mathematics*, 1(1):223–230, 1990.
- [21] Jan Popiołek. Real normed space. *Formalized Mathematics*, 2(1):111–115, 1991.
- [22] Konrad Raczkowski and Paweł Sadowski. Real function differentiability. *Formalized Mathematics*, 1(4):797–801, 1990.
- [23] Laurent Schwartz. Cours d’analyse, vol. 1. *Hermann Paris*, 1967.
- [24] Yasunari Shidama. Banach space of bounded linear operators. *Formalized Mathematics*, 12(1):39–48, 2004.
- [25] Andrzej Trybulec. Binary operations applied to functions. *Formalized Mathematics*, 1(2):329–334, 1990.
- [26] Andrzej Trybulec. On the sets inhabited by numbers. *Formalized Mathematics*, 11(4):341–347, 2003.
- [27] Wojciech A. Trybulec. Vectors in real linear space. *Formalized Mathematics*, 1(2):291–296, 1990.
- [28] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(1):67–71, 1990.
- [29] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1(1):73–83, 1990.
- [30] Edmund Woronowicz. Relations defined on sets. *Formalized Mathematics*, 1(1):181–186, 1990.
- [31] Hiroshi Yamazaki and Yasunari Shidama. Algebra of vector functions. *Formalized Mathematics*, 3(2):171–175, 1992.

Received June 2, 2011
