

Planes and Spheres as Topological Manifolds. Stereographic Projection

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Summary. The goal of this article is to show some examples of topological manifolds: planes and spheres in Euclidean space. In doing it, the article introduces the stereographic projection [25].

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The papers [29], [34], [9], [14], [40], [41], [11], [10], [4], [2], [18], [13], [31], [20], [21], [30], [32], [16], [17], [35], [26], [1], [22], [38], [36], [24], [19], [37], [28], [6], [15], [8], [27], [39], [3], [42], [12], [23], [7], [5], and [33] provide the notation and terminology for this paper.

1. PRELIMINARIES

Let us observe that \emptyset is \emptyset -valued and \emptyset is onto.

Next we state three propositions:

- (1) For every function f and for every set Y holds $\text{dom}(Y \downarrow f) = f^{-1}(Y)$.
- (2) For every function f and for all sets Y_1, Y_2 such that $Y_2 \subseteq Y_1$ holds $(Y_1 \downarrow f)^{-1}(Y_2) = f^{-1}(Y_2)$.
- (3) Let S, T be topological structures and f be a function from S into T . If f is homeomorphism, then f^{-1} is homeomorphism.

Let S, T be topological structures. Let us note that the predicate S and T are homeomorphic is symmetric.

For simplicity, we use the following convention: T_1, T_2, T_3 denote topological spaces, A_1 denotes a subset of T_1 , A_2 denotes a subset of T_2 , and A_3 denotes a subset of T_3 .

Next we state several propositions:

- (4) Let f be a function from T_1 into T_2 . Suppose f is homeomorphism. Let g be a function from $T_1 \setminus f^{-1}(A_2)$ into $T_2 \setminus A_2$. If $g = A_2 \setminus f$, then g is homeomorphism.
- (5) For every function f from T_1 into T_2 such that f is homeomorphism holds $f^{-1}(A_2)$ and A_2 are homeomorphic.
- (6) If A_1 and A_2 are homeomorphic, then A_2 and A_1 are homeomorphic.
- (7) If A_1 and A_2 are homeomorphic, then A_1 is empty iff A_2 is empty.
- (8) If A_1 and A_2 are homeomorphic and A_2 and A_3 are homeomorphic, then A_1 and A_3 are homeomorphic.
- (9) If T_1 is second-countable and T_1 and T_2 are homeomorphic, then T_2 is second-countable.

In the sequel n, k are natural numbers and M, N are non empty topological spaces.

The following propositions are true:

- (10) If M is Hausdorff and M and N are homeomorphic, then N is Hausdorff.
- (11) If M is n -locally Euclidean and M and N are homeomorphic, then N is n -locally Euclidean.
- (12) If M is n -manifold and M and N are homeomorphic, then N is n -manifold.
- (13) Let x_1, x_2 be finite sequences of elements of \mathbb{R} and i be an element of \mathbb{N} . If $i \in \text{dom}(x_1 \bullet x_2)$, then $(x_1 \bullet x_2)(i) = (x_1)_i \cdot (x_2)_i$ and $(x_1 \bullet x_2)_i = (x_1)_i \cdot (x_2)_i$.
- (14) For all finite sequences x_1, x_2, y_1, y_2 of elements of \mathbb{R} such that $\text{len } x_1 = \text{len } x_2$ and $\text{len } y_1 = \text{len } y_2$ holds $x_1 \hat{\ } y_1 \bullet x_2 \hat{\ } y_2 = (x_1 \bullet x_2) \hat{\ } (y_1 \bullet y_2)$.
- (15) For all finite sequences x_1, x_2, y_1, y_2 of elements of \mathbb{R} such that $\text{len } x_1 = \text{len } x_2$ and $\text{len } y_1 = \text{len } y_2$ holds $|(x_1 \hat{\ } y_1, x_2 \hat{\ } y_2)| = |(x_1, x_2)| + |(y_1, y_2)|$.

In the sequel p, q, p_1 are points of \mathcal{E}_T^n and r is a real number.

One can prove the following propositions:

- (16) If $k \in \text{Seg } n$, then $(p_1 + p_2)(k) = p_1(k) + p_2(k)$.
- (17) For every set X holds X is a linear combination of $\mathbb{R}_{\mathbb{R}}^{\text{Seg } n}$ iff X is a linear combination of \mathcal{E}_T^n .
- (18) Let F be a finite sequence of elements of \mathcal{E}_T^n , f_1 be a function from \mathcal{E}_T^n into \mathbb{R} , F_1 be a finite sequence of elements of $\mathbb{R}_{\mathbb{R}}^{\text{Seg } n}$, and f_2 be a function from $\mathbb{R}_{\mathbb{R}}^{\text{Seg } n}$ into \mathbb{R} . If $f_1 = f_2$ and $F = F_1$, then $f_1 \cdot F = f_2 \cdot F_1$.
- (19) Let F be a finite sequence of elements of \mathcal{E}_T^n and F_1 be a finite sequence of elements of $\mathbb{R}_{\mathbb{R}}^{\text{Seg } n}$. If $F_1 = F$, then $\sum F = \sum F_1$.
- (20) For every linear combination L_2 of $\mathbb{R}_{\mathbb{R}}^{\text{Seg } n}$ and for every linear combination L_1 of \mathcal{E}_T^n such that $L_1 = L_2$ holds $\sum L_1 = \sum L_2$.

- (21) Let A_4 be a subset of $\mathbb{R}_{\mathbb{R}}^{\text{Seg } n}$ and A_5 be a subset of $\mathcal{E}_{\mathbb{T}}^n$. Suppose $A_4 = A_5$. Then A_4 is linearly independent if and only if A_5 is linearly independent.
- (22) For every subset V of $\mathcal{E}_{\mathbb{T}}^n$ such that $V = \mathbb{RN}\text{-Base } n$ there exists a linear combination l of V such that $p = \sum l$.
- (23) $\mathbb{RN}\text{-Base } n$ is a basis of $\mathcal{E}_{\mathbb{T}}^n$.
- (24) Let V be a subset of $\mathcal{E}_{\mathbb{T}}^n$. Then $V \in$ the topology of $\mathcal{E}_{\mathbb{T}}^n$ if and only if for every p such that $p \in V$ there exists r such that $r > 0$ and $\text{Ball}(p, r) \subseteq V$.

Let n be a natural number and let p be a point of $\mathcal{E}_{\mathbb{T}}^n$.

The functor $\text{InnerProduct } p$ yields a function from $\mathcal{E}_{\mathbb{T}}^n$ into \mathbb{R}^1 and is defined by:

- (Def. 1) For every point q of $\mathcal{E}_{\mathbb{T}}^n$ holds $(\text{InnerProduct } p)(q) = |(p, q)|$.

Let us consider n, p . Note that $\text{InnerProduct } p$ is continuous.

2. PLANES

Let us consider n and let us consider p, q . The functor $\text{Plane}(p, q)$ yielding a subset of $\mathcal{E}_{\mathbb{T}}^n$ is defined as follows:

- (Def. 2) $\text{Plane}(p, q) = \{y; y \text{ ranges over points of } \mathcal{E}_{\mathbb{T}}^n: |(p, y - q)| = 0\}$.

The following propositions are true:

- (25) $(\text{transl}(p_1, \mathcal{E}_{\mathbb{T}}^n))^\circ \text{Plane}(p, p_2) = \text{Plane}(p, p_1 + p_2)$.
- (26) If $p \neq 0_{\mathcal{E}_{\mathbb{T}}^n}$, then there exists a linearly independent subset A of $\mathcal{E}_{\mathbb{T}}^n$ such that $\overline{A} = n - 1$ and $\Omega_{\text{Lin}(A)} = \text{Plane}(p, 0_{\mathcal{E}_{\mathbb{T}}^n})$.
- (27) If $p_1 \neq 0_{\mathcal{E}_{\mathbb{T}}^n}$ and $p_2 \neq 0_{\mathcal{E}_{\mathbb{T}}^n}$, then there exists a function R from $\mathcal{E}_{\mathbb{T}}^n$ into $\mathcal{E}_{\mathbb{T}}^n$ such that R is homeomorphism and $R^\circ \text{Plane}(p_1, 0_{\mathcal{E}_{\mathbb{T}}^n}) = \text{Plane}(p_2, 0_{\mathcal{E}_{\mathbb{T}}^n})$.

Let us consider n and let us consider p, q . The functor $\text{TPlane}(p, q)$ yields a non empty subspace of $\mathcal{E}_{\mathbb{T}}^n$ and is defined by:

- (Def. 3) $\text{TPlane}(p, q) = \mathcal{E}_{\mathbb{T}}^n \upharpoonright \text{Plane}(p, q)$.

The following three propositions are true:

- (28) The base finite sequence of $n + 1$ and $n + 1 = (0_{\mathcal{E}_{\mathbb{T}}^n}) \hat{\ } \langle 1 \rangle$.
- (29) For all points p, q of $\mathcal{E}_{\mathbb{T}}^{n+1}$ such that $p \neq 0_{\mathcal{E}_{\mathbb{T}}^{n+1}}$ holds $\mathcal{E}_{\mathbb{T}}^n$ and $\text{TPlane}(p, q)$ are homeomorphic.
- (30) For all points p, q of $\mathcal{E}_{\mathbb{T}}^{n+1}$ such that $p \neq 0_{\mathcal{E}_{\mathbb{T}}^{n+1}}$ holds $\text{TPlane}(p, q)$ is n -manifold.

3. SPHERES

Let us consider n . The functor \mathbb{S}^n yields a topological space and is defined by:

(Def. 4) $\mathbb{S}^n = \text{TopUnitCircle}(n + 1)$.

Let us consider n . Note that \mathbb{S}^n is non empty.

Let us consider n, p and let S be a subspace of $\mathcal{E}_{\mathbb{T}}^n$. Let us assume that $p \in \text{Sphere}((0_{\mathcal{E}_{\mathbb{T}}^n}), 1)$. The functor $\sigma_{S,p}$ yielding a function from S into $\text{TPlane}(p, 0_{\mathcal{E}_{\mathbb{T}}^n})$ is defined as follows:

(Def. 5) For every q such that $q \in S$ holds $(\sigma_{S,p})(q) = \frac{1}{1-|(q,p)|} \cdot (q - |(q,p)| \cdot p)$.

Next we state the proposition

(31) For every subspace S of $\mathcal{E}_{\mathbb{T}}^n$ such that $\Omega_S = \text{Sphere}((0_{\mathcal{E}_{\mathbb{T}}^n}), 1) \setminus \{p\}$ and $p \in \text{Sphere}((0_{\mathcal{E}_{\mathbb{T}}^n}), 1)$ holds $\sigma_{S,p}$ is homeomorphism.

Let us consider n . One can verify the following observations:

- * \mathbb{S}^n is second-countable,
- * \mathbb{S}^n is n -locally Euclidean, and
- * \mathbb{S}^n is n -manifold.

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