FORMALIZED MATHEMATICS
Vol. 20, No. 1, Pages 79–86, 2012
DOI: 10.2478/v10037-012-0011-3

VERSITA
versita.com/fm/

Riemann Integral of Functions from $\mathbb{R}$ into n-dimensional Real Normed Space

Keiichi Miyajima
Ibaraki University
Faculty of Engineering
Hitachi, Japan

Artur Korniłowicz
Institute of Informatics
University of Białystok
Sosnowa 64, 15-887 Białystok
Poland

Yasunari Shidama[1]
Shinshu University
Nagano, Japan

Summary. In this article, we define the Riemann integral on functions $\mathbb{R}$ into n-dimensional real normed space and prove the linearity of this operator. As a result, the Riemann integration can be applied to the wider range. Our method refers to the [21].

MML identifier: `INTEGR19`, version: `7.12.02 4.175.1137`

The terminology and notation used in this paper have been introduced in the following papers: [23], [24], [6], [2], [25], [8], [7], [1], [4], [3], [5], [20], [10], [14], [12], [13], [18], [22], [19], [26], [9], [11], [15], [17], and [16].

1. On the Functions from $\mathbb{R}$ into n-dimensional Real Space

For simplicity, we adopt the following convention: X denotes a set, n denotes an element of $\mathbb{N}$, a, b, c, d, e, r, x_0 denote real numbers, A denotes a non empty closed-interval subset of $\mathbb{R}$, f, g, h denote partial functions from $\mathbb{R}$ to $\mathcal{R}^n$, and E denotes an element of $\mathcal{R}^n$. We now state a number of propositions:

(1) If $a \leq c \leq b$, then $c \in [a, b]$ and $[a, c] \subseteq [a, b]$ and $[c, b] \subseteq [a, b]$.

[1]This work was supported by JSPS KAKENHI 22300285.

ISSN 1426–2630(p), 1898-9934(e)

(2) If $a \leq c \leq d \leq b$ and $[a, b] \subseteq X$, then $[c, d] \subseteq X$.

(3) If $a \leq b$ and c, $d \in [a, b]$ and $[a, b] \subseteq X$, then $[\min(c, d), \max(c, d)] \subseteq X$.

(4) If $a \leq c \leq d \leq b$ and $[a, b] \subseteq \operatorname{dom} f$ and $[a, b] \subseteq \operatorname{dom} g$, then $[c, d] \subseteq \operatorname{dom}(f + g)$.

(5) If $a \leq c \leq d \leq b$ and $[a, b] \subseteq \operatorname{dom} f$ and $[a, b] \subseteq \operatorname{dom} g$, then $[c, d] \subseteq \operatorname{dom}(f - g)$.

(6) Let f be a partial function from $\mathbb{R}$ to $\mathbb{R}$. Suppose $a \leq c \leq d \leq b$ and f is integrable on $[a, b]$ and $f \restriction [a, b]$ is bounded and $[a, b] \subseteq \operatorname{dom} f$. Then $r \cdot f$ is integrable on $[c, d]$ and $(r \cdot f) \restriction [c, d]$ is bounded.

(7) Let f, g be partial functions from $\mathbb{R}$ to $\mathbb{R}$. Suppose that $a \leq c \leq d \leq b$ and f is integrable on $[a, b]$ and g is integrable on $[a, b]$ and $f \restriction [a, b]$ is bounded and $g \restriction [a, b]$ is bounded and $[a, b] \subseteq \operatorname{dom} f$ and $[a, b] \subseteq \operatorname{dom} g$. Then $f - g$ is integrable on $[c, d]$ and $(f - g) \restriction [c, d]$ is bounded.

(8) Suppose $a \leq b$ and f is integrable on $[a, b]$ and $f \restriction [a, b]$ is bounded and $[a, b] \subseteq \operatorname{dom} f$ and $c \in [a, b]$. Then f is integrable on $[a, c]$ and f is integrable on $[c, b]$ and $\int\limits_a^b f(x)dx = \int\limits_a^c f(x)dx + \int\limits_c^b f(x)dx$.

(9) Suppose $a \leq c \leq d \leq b$ and f is integrable on $[a, b]$ and $f \restriction [a, b]$ is bounded and $[a, b] \subseteq \operatorname{dom} f$. Then f is integrable on $[c, d]$ and $f \restriction [c, d]$ is bounded.

(10) Suppose that $a \leq c \leq d \leq b$ and f is integrable on $[a, b]$ and g is integrable on $[a, b]$ and $f \restriction [a, b]$ is bounded and $g \restriction [a, b]$ is bounded and $[a, b] \subseteq \operatorname{dom} f$ and $[a, b] \subseteq \operatorname{dom} g$. Then $f + g$ is integrable on $[c, d]$ and $(f + g) \restriction [c, d]$ is bounded.

(11) Suppose $a \leq c \leq d \leq b$ and f is integrable on $[a, b]$ and $f \restriction [a, b]$ is bounded and $[a, b] \subseteq \operatorname{dom} f$. Then $r \cdot f$ is integrable on $[c, d]$ and $(r \cdot f) \restriction [c, d]$ is bounded.

(12) Suppose $a \leq c \leq d \leq b$ and f is integrable on $[a, b]$ and $f \restriction [a, b]$ is bounded and $[a, b] \subseteq \operatorname{dom} f$. Then $-f$ is integrable on $[c, d]$ and $(-f) \restriction [c, d]$ is bounded.

(13) Suppose that $a \leq c \leq d \leq b$ and f is integrable on $[a, b]$ and g is integrable on $[a, b]$ and $f \restriction [a, b]$ is bounded and $g \restriction [a, b]$ is bounded and $[a, b] \subseteq \operatorname{dom} f$ and $[a, b] \subseteq \operatorname{dom} g$. Then $f - g$ is integrable on $[c, d]$ and $(f - g) \restriction [c, d]$ is bounded.

(14) Let n be a non empty element of $\mathbb{N}$ and f be a function from A into $\mathcal{R}^n$. Then f is bounded if and only if $|f|$ is bounded.

(15) If f is bounded and $A \subseteq \operatorname{dom} f$, then $f \restriction A$ is bounded.

(16) Let f be a partial function from $\mathbb{R}$ to $\mathcal{R}^n$ and g be a function from A into $\mathcal{R}^n$. If f is bounded and $f = g$, then g is bounded.

(17) For every partial function f from $\mathbb{R}$ to $\mathcal{R}^n$ and for every function g from

A into $\mathcal{R}^n$ such that $f = g$ holds $|f| = |g|$.

(18) If $A \subseteq \operatorname{dom} h$, then $|h{\restriction}A| = |h|{\restriction}A$.

(19) Let n be a non empty element of $\mathbb{N}$ and h be a partial function from $\mathbb{R}$ to $\mathcal{R}^n$. If $A \subseteq \operatorname{dom} h$ and $h{\restriction}A$ is bounded, then $|h|{\restriction}A$ is bounded.

(20) Let n be a non empty element of $\mathbb{N}$ and h be a partial function from $\mathbb{R}$ to $\mathcal{R}^n$. Suppose $A \subseteq \operatorname{dom} h$ and $h{\restriction}A$ is bounded and h is integrable on A and $|h|$ is integrable on A. Then $|\int\limits_A h(x)dx| \leq \int\limits_A |h|(x)dx$.

(21) Let n be a non empty element of $\mathbb{N}$ and h be a partial function from $\mathbb{R}$ to $\mathcal{R}^n$. Suppose $a \leq b$ and $[a, b] \subseteq \operatorname{dom} h$ and h is integrable on $[a, b]$ and $|h|$ is integrable on $[a, b]$ and $h{\restriction}[a, b]$ is bounded. Then $|\int\limits_a^b h(x)dx| \leq \int\limits_a^b |h|(x)dx$.

(22) Let n be a non empty element of $\mathbb{N}$ and f be a partial function from $\mathbb{R}$ to $\mathcal{R}^n$. Suppose that $a \leq b$ and f is integrable on $[a, b]$ and $|f|$ is integrable on $[a, b]$ and $f{\restriction}[a, b]$ is bounded and $[a, b] \subseteq \operatorname{dom} f$ and c, $d \in [a, b]$. Then $|f|$ is integrable on $[\min(c, d), \max(c, d)]$ and $|f|{\restriction}[\min(c, d), \max(c, d)]$ is bounded and $|\int\limits_c^d f(x)dx| \leq \int\limits_{\min(c,d)}^{\max(c,d)} |f|(x)dx$.

(23) Let n be a non empty element of $\mathbb{N}$ and f be a partial function from $\mathbb{R}$ to $\mathcal{R}^n$. Suppose that $a \leq b$ and $c \leq d$ and f is integrable on $[a, b]$ and $|f|$ is integrable on $[a, b]$ and $f{\restriction}[a, b]$ is bounded and $[a, b] \subseteq \operatorname{dom} f$ and c, $d \in [a, b]$. Then $|f|$ is integrable on $[c, d]$ and $|f|{\restriction}[c, d]$ is bounded and $|\int\limits_c^d f(x)dx| \leq \int\limits_c^d |f|(x)dx$ and $|\int\limits_d^c f(x)dx| \leq \int\limits_c^d |f|(x)dx$.

(24) Let n be a non empty element of $\mathbb{N}$ and f be a partial function from $\mathbb{R}$ to $\mathcal{R}^n$. Suppose that $a \leq b$ and $c \leq d$ and f is integrable on $[a, b]$ and $|f|$ is integrable on $[a, b]$ and $f{\restriction}[a, b]$ is bounded and $[a, b] \subseteq \operatorname{dom} f$ and c, $d \in [a, b]$ and for every real number x such that $x \in [c, d]$ holds $|f_x| \leq e$. Then $|\int\limits_c^d f(x)dx| \leq e \cdot (d - c)$ and $|\int\limits_d^c f(x)dx| \leq e \cdot (d - c)$.

(25) If $a \leq b$ and f is integrable on $[a, b]$ and $f{\restriction}[a, b]$ is bounded and $[a, b] \subseteq \operatorname{dom} f$ and c, $d \in [a, b]$, then $\int\limits_c^d (r \cdot f)(x)dx = r \cdot \int\limits_c^d f(x)dx$.

(26) If $a \leq b$ and f is integrable on $[a, b]$ and $f{\restriction}[a, b]$ is bounded and $[a, b] \subseteq \operatorname{dom} f$ and c, $d \in [a, b]$, then $\int\limits_c^d (-f)(x)dx = -\int\limits_c^d f(x)dx$.

(27) Suppose that $a \leq b$ and f is integrable on $[a, b]$ and g is integrable on $[a, b]$ and $f{\restriction}[a, b]$ is bounded and $g{\restriction}[a, b]$ is bounded and $[a, b] \subseteq \mathrm{dom}\, f$ and $[a, b] \subseteq \mathrm{dom}\, g$ and c, $d \in [a, b]$. Then $\int\limits_c^d (f + g)(x)dx = \int\limits_c^d f(x)dx + \int\limits_c^d g(x)dx$.

(28) Suppose that $a \leq b$ and f is integrable on $[a, b]$ and g is integrable on $[a, b]$ and $f{\restriction}[a, b]$ is bounded and $g{\restriction}[a, b]$ is bounded and $[a, b] \subseteq \mathrm{dom}\, f$ and $[a, b] \subseteq \mathrm{dom}\, g$ and c, $d \in [a, b]$. Then $\int\limits_c^d (f - g)(x)dx = \int\limits_c^d f(x)dx - \int\limits_c^d g(x)dx$.

(29) Suppose $a \leq b$ and $[a, b] \subseteq \mathrm{dom}\, f$ and for every real number x such that $x \in [a, b]$ holds $f(x) = E$. Then f is integrable on $[a, b]$ and $f{\restriction}[a, b]$ is bounded and $\int\limits_a^b f(x)dx = (b - a) \cdot E$.

(30) Suppose $a \leq b$ and for every real number x such that $x \in [a, b]$ holds $f(x) = E$ and $[a, b] \subseteq \mathrm{dom}\, f$ and c, $d \in [a, b]$. Then $\int\limits_c^d f(x)dx = (d - c) \cdot E$.

(31) If $a \leq b$ and f is integrable on $[a, b]$ and $f{\restriction}[a, b]$ is bounded and $[a, b] \subseteq \mathrm{dom}\, f$ and c, $d \in [a, b]$, then $\int\limits_a^d f(x)dx = \int\limits_a^c f(x)dx + \int\limits_c^d f(x)dx$.

(32) Suppose that $a \leq b$ and f is integrable on $[a, b]$ and $f{\restriction}[a, b]$ is bounded and $[a, b] \subseteq \mathrm{dom}\, f$ and c, $d \in [a, b]$ and for every real number x such that $x \in [\min(c, d), \max(c, d)]$ holds $|f_x| \leq e$. Then $|\int\limits_c^d f(x)dx| \leq n \cdot e \cdot |d - c|$.

(33) $\int\limits_b^a f(x)dx = -\int\limits_a^b f(x)dx$.

2. On the Functions from $\mathbb{R}$ into n-dimensional Real Normed Space

Let R be a real normed space, let X be a non empty set, and let g be a partial function from X to R. We say that g is bounded if and only if:

(Def. 1) There exists a real number r such that for every set y such that $y \in \mathrm{dom}\, g$ holds $\|g_y\| < r$.

Next we state a number of propositions:

(34) Let f be a partial function from $\mathbb{R}$ to $\mathcal{R}^n$ and g be a partial function from $\mathbb{R}$ to $\langle \mathcal{E}^n, \| \cdot \| \rangle$. If $f = g$, then f is bounded iff g is bounded.

(35) Let X, Y be sets and f_1, f_2 be partial functions from $\mathbb{R}$ to $\langle \mathcal{E}^n, \| \cdot \| \rangle$. Suppose $f_1 \restriction X$ is bounded and $f_2 \restriction Y$ is bounded. Then $(f_1 + f_2) \restriction (X \cap Y)$ is bounded and $(f_1 - f_2) \restriction (X \cap Y)$ is bounded.

(36) Let f be a function from A into $\mathcal{R}^n$, g be a function from A into $\langle \mathcal{E}^n, \| \cdot \| \rangle$, D be a Division of A, p be a finite sequence of elements of $\mathcal{R}^n$, and q be a finite sequence of elements of $\langle \mathcal{E}^n, \| \cdot \| \rangle$. Suppose $f = g$ and $p = q$. Then p is a middle volume of f and D if and only if q is a middle volume of g and D.

(37) Let f be a function from A into $\mathcal{R}^n$, g be a function from A into $\langle \mathcal{E}^n, \| \cdot \| \rangle$, D be a Division of A, p be a middle volume of f and D, and q be a middle volume of g and D. If $f = g$ and $p = q$, then middle sum$(f, p) =$ middle sum(g, q).

(38) Let f be a function from A into $\mathcal{R}^n$, g be a function from A into $\langle \mathcal{E}^n, \| \cdot \| \rangle$, T be a division sequence of A, p be a function from $\mathbb{N}$ into $(\mathcal{R}^n)^*$, and q be a function from $\mathbb{N}$ into (the carrier of $\langle \mathcal{E}^n, \| \cdot \| \rangle)^*$. Suppose $f = g$ and $p = q$. Then p is a middle volume sequence of f and T if and only if q is a middle volume sequence of g and T.

(39) Let f be a function from A into $\mathcal{R}^n$, g be a function from A into $\langle \mathcal{E}^n, \| \cdot \| \rangle$, T be a division sequence of A, S be a middle volume sequence of f and T, and U be a middle volume sequence of g and T. If $f = g$ and $S = U$, then middle sum$(f, S) =$ middle sum(g, U).

(40) Let f be a function from A into $\mathcal{R}^n$, g be a function from A into $\langle \mathcal{E}^n, \| \cdot \| \rangle$, I be an element of $\mathcal{R}^n$, and J be a point of $\langle \mathcal{E}^n, \| \cdot \| \rangle$. Suppose $f = g$ and $I = J$. Then the following statements are equivalent

(i) for every division sequence T of A and for every middle volume sequence S of f and T such that δ_T is convergent and $\lim(\delta_T) = 0$ holds middle sum(f, S) is convergent and lim middle sum$(f, S) = I$,

(ii) for every division sequence T of A and for every middle volume sequence S of g and T such that δ_T is convergent and $\lim(\delta_T) = 0$ holds middle sum(g, S) is convergent and lim middle sum$(g, S) = J$.

(41) Let f be a function from A into $\mathcal{R}^n$ and g be a function from A into $\langle \mathcal{E}^n, \| \cdot \| \rangle$. Suppose $f = g$ and f is bounded. Then f is integrable if and only if g is integrable.

(42) Let f be a function from A into $\mathcal{R}^n$ and g be a function from A into $\langle \mathcal{E}^n, \| \cdot \| \rangle$. Suppose $f = g$ and f is bounded and integrable. Then g is integrable and integral $f =$ integral g.

(43) Let f be a partial function from $\mathbb{R}$ to $\mathcal{R}^n$ and g be a partial function

from $\mathbb{R}$ to $\langle \mathcal{E}^n, \|\cdot\| \rangle$. Suppose $f = g$ and $f{\upharpoonright}A$ is bounded and $A \subseteq \operatorname{dom} f$. Then f is integrable on A if and only if g is integrable on A.

(44) Let f be a partial function from $\mathbb{R}$ to $\mathcal{R}^n$ and g be a partial function from $\mathbb{R}$ to $\langle \mathcal{E}^n, \|\cdot\| \rangle$. Suppose $f = g$ and $f{\upharpoonright}A$ is bounded and $A \subseteq \operatorname{dom} f$ and f is integrable on A. Then g is integrable on A and $\int\limits_A f(x)dx = \int\limits_A g(x)dx$.

(45) Let f be a partial function from $\mathbb{R}$ to $\mathcal{R}^n$ and g be a partial function from $\mathbb{R}$ to $\langle \mathcal{E}^n, \|\cdot\| \rangle$. Suppose $f = g$ and $a \leq b$ and $f{\upharpoonright}[a,b]$ is bounded and $[a,b] \subseteq \operatorname{dom} f$ and f is integrable on $[a,b]$. Then $\int\limits_a^b f(x)dx = \int\limits_a^b g(x)dx$.

(46) Let f, g be partial functions from $\mathbb{R}$ to $\langle \mathcal{E}^n, \|\cdot\| \rangle$. Suppose $a \leq b$ and f is integrable on $[a,b]$ and g is integrable on $[a,b]$ and $[a,b] \subseteq \operatorname{dom} f$ and $[a,b] \subseteq \operatorname{dom} g$. Then $\int\limits_a^b (f+g)(x)dx = \int\limits_a^b f(x)dx + \int\limits_a^b g(x)dx$ and $\int\limits_a^b (f-g)(x)dx = \int\limits_a^b f(x)dx - \int\limits_a^b g(x)dx$.

(47) For every partial function f from $\mathbb{R}$ to $\langle \mathcal{E}^n, \|\cdot\| \rangle$ such that $a \leq b$ and $[a,b] \subseteq \operatorname{dom} f$ holds $\int\limits_b^a f(x)dx = -\int\limits_a^b f(x)dx$.

(48) Let f be a partial function from $\mathbb{R}$ to $\langle \mathcal{E}^n, \|\cdot\| \rangle$ and g be a partial function from $\mathbb{R}$ to $\mathcal{R}^n$. Suppose $f = g$ and $a \leq b$ and $[a,b] \subseteq \operatorname{dom} f$ and $f{\upharpoonright}[a,b]$ is bounded and f is integrable on $[a,b]$ and c, $d \in [a,b]$. Then $\int\limits_c^d f(x)dx = \int\limits_c^d g(x)dx$.

(49) Let f, g be partial functions from $\mathbb{R}$ to $\langle \mathcal{E}^n, \|\cdot\| \rangle$. Suppose that $a \leq b$ and f is integrable on $[a,b]$ and g is integrable on $[a,b]$ and $f{\upharpoonright}[a,b]$ is bounded and $g{\upharpoonright}[a,b]$ is bounded and $[a,b] \subseteq \operatorname{dom} f$ and $[a,b] \subseteq \operatorname{dom} g$ and c, $d \in [a,b]$. Then $\int\limits_c^d (f+g)(x)dx = \int\limits_c^d f(x)dx + \int\limits_c^d g(x)dx$.

(50) Let f, g be partial functions from $\mathbb{R}$ to $\langle \mathcal{E}^n, \|\cdot\| \rangle$. Suppose that $a \leq b$ and f is integrable on $[a,b]$ and g is integrable on $[a,b]$ and $f{\upharpoonright}[a,b]$ is bounded and $g{\upharpoonright}[a,b]$ is bounded and $[a,b] \subseteq \operatorname{dom} f$ and $[a,b] \subseteq \operatorname{dom} g$ and c, $d \in [a,b]$. Then $\int\limits_c^d (f-g)(x)dx = \int\limits_c^d f(x)dx - \int\limits_c^d g(x)dx$.

(51) Let E be a point of $\langle \mathcal{E}^n, \|\cdot\| \rangle$ and f be a partial function from $\mathbb{R}$ to $\langle \mathcal{E}^n, \|\cdot\| \rangle$. Suppose $a \leq b$ and $[a,b] \subseteq \operatorname{dom} f$ and for every real number

x such that $x \in [a,b]$ holds $f(x) = E$. Then f is integrable on $[a,b]$ and $f{\restriction}[a,b]$ is bounded and $\int\limits_a^b f(x)dx = (b-a)\cdot E$.

(52) Let E be a point of $\langle \mathcal{E}^n, \|\cdot\| \rangle$ and f be a partial function from $\mathbb{R}$ to $\langle \mathcal{E}^n, \|\cdot\| \rangle$. Suppose $a \leq b$ and $[a,b] \subseteq \operatorname{dom} f$ and for every real number x such that $x \in [a,b]$ holds $f(x) = E$ and c, $d \in [a,b]$. Then $\int\limits_c^d f(x)dx = (d-c)\cdot E$.

(53) Let f be a partial function from $\mathbb{R}$ to $\langle \mathcal{E}^n, \|\cdot\| \rangle$. Suppose $a \leq b$ and f is integrable on $[a,b]$ and $f{\restriction}[a,b]$ is bounded and $[a,b] \subseteq \operatorname{dom} f$ and c, $d \in [a,b]$. Then $\int\limits_a^d f(x)dx = \int\limits_a^c f(x)dx + \int\limits_c^d f(x)dx$.

(54) Let f be a partial function from $\mathbb{R}$ to $\langle \mathcal{E}^n, \|\cdot\| \rangle$. Suppose that $a \leq b$ and f is integrable on $[a,b]$ and $f{\restriction}[a,b]$ is bounded and $[a,b] \subseteq \operatorname{dom} f$ and c, $d \in [a,b]$ and for every real number x such that $x \in [\min(c,d), \max(c,d)]$ holds $\|f_x\| \leq e$. Then $\|\int\limits_c^d f(x)dx\| \leq n \cdot e \cdot |d-c|$.

3. Fundamental Theorem of Calculus

The following two propositions are true:

(55)[2] Let n be a non empty element of $\mathbb{N}$ and F, f be partial functions from $\mathbb{R}$ to $\langle \mathcal{E}^n, \|\cdot\| \rangle$. Suppose that $a \leq b$ and f is integrable on $[a,b]$ and $f{\restriction}[a,b]$ is bounded and $[a,b] \subseteq \operatorname{dom} f$ and $]a,b[\subseteq \operatorname{dom} F$ and for every real number x such that $x \in]a,b[$ holds $F(x) = \int\limits_a^x f(x)dx$ and $x_0 \in]a,b[$ and f is continuous in x_0. Then F is differentiable in x_0 and $F'(x_0) = f_{x_0}$.

(56) Let n be a non empty element of $\mathbb{N}$ and f be a partial function from $\mathbb{R}$ to $\langle \mathcal{E}^n, \|\cdot\| \rangle$. Suppose $a \leq b$ and f is integrable on $[a,b]$ and $f{\restriction}[a,b]$ is bounded and $[a,b] \subseteq \operatorname{dom} f$ and $x_0 \in]a,b[$ and f is continuous in x_0. Then there exists a partial function F from $\mathbb{R}$ to $\langle \mathcal{E}^n, \|\cdot\| \rangle$ such that $]a,b[\subseteq \operatorname{dom} F$ and for every real number x such that $x \in]a,b[$ holds $F(x) = \int\limits_a^x f(x)dx$ and F is differentiable in x_0 and $F'(x_0) = f_{x_0}$.

[2]Fundamental Theorem of Calculus (for $\mathcal{R}^n$)

References

[1] Grzegorz Bancerek. The fundamental properties of natural numbers. *Formalized Mathematics*, 1(**1**):41–46, 1990.
[2] Grzegorz Bancerek. The ordinal numbers. *Formalized Mathematics*, 1(**1**):91–96, 1990.
[3] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Formalized Mathematics*, 1(**1**):107–114, 1990.
[4] Czesław Byliński. The complex numbers. *Formalized Mathematics*, 1(**3**):507–513, 1990.
[5] Czesław Byliński. Finite sequences and tuples of elements of a non-empty sets. *Formalized Mathematics*, 1(**3**):529–536, 1990.
[6] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.
[7] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.
[8] Czesław Byliński. Partial functions. *Formalized Mathematics*, 1(**2**):357–367, 1990.
[9] Agata Darmochwał. The Euclidean space. *Formalized Mathematics*, 2(**4**):599–603, 1991.
[10] Noboru Endou and Artur Korniłowicz. The definition of the Riemann definite integral and some related lemmas. *Formalized Mathematics*, 8(**1**):93–102, 1999.
[11] Noboru Endou and Yasunari Shidama. Completeness of the real Euclidean space. *Formalized Mathematics*, 13(**4**):577–580, 2005.
[12] Noboru Endou, Katsumi Wasaki, and Yasunari Shidama. Darboux's theorem. *Formalized Mathematics*, 9(**1**):197–200, 2001.
[13] Noboru Endou, Katsumi Wasaki, and Yasunari Shidama. Definition of integrability for partial functions from $\mathbb{R}$ to $\mathbb{R}$ and integrability for continuous functions. *Formalized Mathematics*, 9(**2**):281–284, 2001.
[14] Noboru Endou, Katsumi Wasaki, and Yasunari Shidama. Scalar multiple of Riemann definite integral. *Formalized Mathematics*, 9(**1**):191–196, 2001.
[15] Keiichi Miyajima, Takahiro Kato, and Yasunari Shidama. Riemann integral of functions from $\mathbb{R}$ into real normed space. *Formalized Mathematics*, 19(**1**):17–22, 2011, doi: 10.2478/v10037-011-0003-8.
[16] Keiichi Miyajima and Yasunari Shidama. Riemann integral of functions from $\mathbb{R}$ into $\mathcal{R}^n$. *Formalized Mathematics*, 17(**2**):179–185, 2009, doi: 10.2478/v10037-009-0021-y.
[17] Keiko Narita, Artur Kornilowicz, and Yasunari Shidama. More on the continuity of real functions. *Formalized Mathematics*, 19(**4**):233–239, 2011, doi: 10.2478/v10037-011-0032-3.
[18] Beata Padlewska and Agata Darmochwał. Topological spaces and continuous functions. *Formalized Mathematics*, 1(**1**):223–230, 1990.
[19] Jan Popiołek. Real normed space. *Formalized Mathematics*, 2(**1**):111–115, 1991.
[20] Konrad Raczkowski and Paweł Sadowski. Topological properties of subsets in real numbers. *Formalized Mathematics*, 1(**4**):777–780, 1990.
[21] Murray R. Spiegel. *Theory and Problems of Vector Analysis*. McGraw-Hill, 1974.
[22] Wojciech A. Trybulec. Vectors in real linear space. *Formalized Mathematics*, 1(**2**):291–296, 1990.
[23] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(**1**):67–71, 1990.
[24] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1(**1**):73–83, 1990.
[25] Edmund Woronowicz. Relations defined on sets. *Formalized Mathematics*, 1(**1**):181–186, 1990.
[26] Hiroshi Yamazaki and Yasunari Shidama. Algebra of vector functions. *Formalized Mathematics*, 3(**2**):171–175, 1992.

Received October 27, 2011