

Riemann Integral of Functions from \mathbb{R} into n -dimensional Real Normed Space

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Summary. In this article, we define the Riemann integral on functions \mathbb{R} into n -dimensional real normed space and prove the linearity of this operator. As a result, the Riemann integration can be applied to the wider range. Our method refers to the [21].

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The terminology and notation used in this paper have been introduced in the following papers: [23], [24], [6], [2], [25], [8], [7], [1], [4], [3], [5], [20], [10], [14], [12], [13], [18], [22], [19], [26], [9], [11], [15], [17], and [16].

1. ON THE FUNCTIONS FROM \mathbb{R} INTO n -DIMENSIONAL REAL SPACE

For simplicity, we adopt the following convention: X denotes a set, n denotes an element of \mathbb{N} , a, b, c, d, e, r, x_0 denote real numbers, A denotes a non empty closed-interval subset of \mathbb{R} , f, g, h denote partial functions from \mathbb{R} to \mathcal{R}^n , and E denotes an element of \mathcal{R}^n . We now state a number of propositions:

- (1) If $a \leq c \leq b$, then $c \in [a, b]$ and $[a, c] \subseteq [a, b]$ and $[c, b] \subseteq [a, b]$.

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- (2) If $a \leq c \leq d \leq b$ and $[a, b] \subseteq X$, then $[c, d] \subseteq X$.
- (3) If $a \leq b$ and $c, d \in [a, b]$ and $[a, b] \subseteq X$, then $[\min(c, d), \max(c, d)] \subseteq X$.
- (4) If $a \leq c \leq d \leq b$ and $[a, b] \subseteq \text{dom } f$ and $[a, b] \subseteq \text{dom } g$, then $[c, d] \subseteq \text{dom}(f + g)$.
- (5) If $a \leq c \leq d \leq b$ and $[a, b] \subseteq \text{dom } f$ and $[a, b] \subseteq \text{dom } g$, then $[c, d] \subseteq \text{dom}(f - g)$.
- (6) Let f be a partial function from \mathbb{R} to \mathbb{R} . Suppose $a \leq c \leq d \leq b$ and f is integrable on $[a, b]$ and $f|_{[a, b]}$ is bounded and $[a, b] \subseteq \text{dom } f$. Then $r \cdot f$ is integrable on $[c, d]$ and $(r \cdot f)|_{[c, d]}$ is bounded.
- (7) Let f, g be partial functions from \mathbb{R} to \mathbb{R} . Suppose that $a \leq c \leq d \leq b$ and f is integrable on $[a, b]$ and g is integrable on $[a, b]$ and $f|_{[a, b]}$ is bounded and $g|_{[a, b]}$ is bounded and $[a, b] \subseteq \text{dom } f$ and $[a, b] \subseteq \text{dom } g$. Then $f - g$ is integrable on $[c, d]$ and $(f - g)|_{[c, d]}$ is bounded.
- (8) Suppose $a \leq b$ and f is integrable on $[a, b]$ and $f|_{[a, b]}$ is bounded and $[a, b] \subseteq \text{dom } f$ and $c \in [a, b]$. Then f is integrable on $[a, c]$ and f is integrable on $[c, b]$ and $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$.
- (9) Suppose $a \leq c \leq d \leq b$ and f is integrable on $[a, b]$ and $f|_{[a, b]}$ is bounded and $[a, b] \subseteq \text{dom } f$. Then f is integrable on $[c, d]$ and $f|_{[c, d]}$ is bounded.
- (10) Suppose that $a \leq c \leq d \leq b$ and f is integrable on $[a, b]$ and g is integrable on $[a, b]$ and $f|_{[a, b]}$ is bounded and $g|_{[a, b]}$ is bounded and $[a, b] \subseteq \text{dom } f$ and $[a, b] \subseteq \text{dom } g$. Then $f + g$ is integrable on $[c, d]$ and $(f + g)|_{[c, d]}$ is bounded.
- (11) Suppose $a \leq c \leq d \leq b$ and f is integrable on $[a, b]$ and $f|_{[a, b]}$ is bounded and $[a, b] \subseteq \text{dom } f$. Then $r \cdot f$ is integrable on $[c, d]$ and $(r \cdot f)|_{[c, d]}$ is bounded.
- (12) Suppose $a \leq c \leq d \leq b$ and f is integrable on $[a, b]$ and $f|_{[a, b]}$ is bounded and $[a, b] \subseteq \text{dom } f$. Then $-f$ is integrable on $[c, d]$ and $(-f)|_{[c, d]}$ is bounded.
- (13) Suppose that $a \leq c \leq d \leq b$ and f is integrable on $[a, b]$ and g is integrable on $[a, b]$ and $f|_{[a, b]}$ is bounded and $g|_{[a, b]}$ is bounded and $[a, b] \subseteq \text{dom } f$ and $[a, b] \subseteq \text{dom } g$. Then $f - g$ is integrable on $[c, d]$ and $(f - g)|_{[c, d]}$ is bounded.
- (14) Let n be a non empty element of \mathbb{N} and f be a function from A into \mathcal{R}^n . Then f is bounded if and only if $|f|$ is bounded.
- (15) If f is bounded and $A \subseteq \text{dom } f$, then $f|_A$ is bounded.
- (16) Let f be a partial function from \mathbb{R} to \mathcal{R}^n and g be a function from A into \mathcal{R}^n . If f is bounded and $f = g$, then g is bounded.
- (17) For every partial function f from \mathbb{R} to \mathcal{R}^n and for every function g from

A into \mathcal{R}^n such that $f = g$ holds $|f| = |g|$.

- (18) If $A \subseteq \text{dom } h$, then $|h \upharpoonright A| = |h| \upharpoonright A$.
- (19) Let n be a non empty element of \mathbb{N} and h be a partial function from \mathbb{R} to \mathcal{R}^n . If $A \subseteq \text{dom } h$ and $h \upharpoonright A$ is bounded, then $|h| \upharpoonright A$ is bounded.
- (20) Let n be a non empty element of \mathbb{N} and h be a partial function from \mathbb{R} to \mathcal{R}^n . Suppose $A \subseteq \text{dom } h$ and $h \upharpoonright A$ is bounded and h is integrable on A and $|h|$ is integrable on A . Then $|\int_A h(x)dx| \leq \int_A |h|(x)dx$.
- (21) Let n be a non empty element of \mathbb{N} and h be a partial function from \mathbb{R} to \mathcal{R}^n . Suppose $a \leq b$ and $[a, b] \subseteq \text{dom } h$ and h is integrable on $[a, b]$ and $|h|$ is integrable on $[a, b]$ and $h \upharpoonright [a, b]$ is bounded. Then $|\int_a^b h(x)dx| \leq \int_a^b |h|(x)dx$.
- (22) Let n be a non empty element of \mathbb{N} and f be a partial function from \mathbb{R} to \mathcal{R}^n . Suppose that $a \leq b$ and f is integrable on $[a, b]$ and $|f|$ is integrable on $[a, b]$ and $f \upharpoonright [a, b]$ is bounded and $[a, b] \subseteq \text{dom } f$ and $c, d \in [a, b]$. Then $|f|$ is integrable on $[\min(c, d), \max(c, d)]$ and $|f| \upharpoonright [\min(c, d), \max(c, d)]$ is bounded and $|\int_c^d f(x)dx| \leq \int_{\min(c, d)}^{\max(c, d)} |f|(x)dx$.
- (23) Let n be a non empty element of \mathbb{N} and f be a partial function from \mathbb{R} to \mathcal{R}^n . Suppose that $a \leq b$ and $c \leq d$ and f is integrable on $[a, b]$ and $|f|$ is integrable on $[a, b]$ and $f \upharpoonright [a, b]$ is bounded and $[a, b] \subseteq \text{dom } f$ and $c, d \in [a, b]$. Then $|f|$ is integrable on $[c, d]$ and $|f| \upharpoonright [c, d]$ is bounded and $|\int_c^d f(x)dx| \leq \int_c^d |f|(x)dx$ and $|\int_d^c f(x)dx| \leq \int_c^d |f|(x)dx$.
- (24) Let n be a non empty element of \mathbb{N} and f be a partial function from \mathbb{R} to \mathcal{R}^n . Suppose that $a \leq b$ and $c \leq d$ and f is integrable on $[a, b]$ and $|f|$ is integrable on $[a, b]$ and $f \upharpoonright [a, b]$ is bounded and $[a, b] \subseteq \text{dom } f$ and $c, d \in [a, b]$ and for every real number x such that $x \in [c, d]$ holds $|f_x| \leq e$. Then $|\int_c^d f(x)dx| \leq e \cdot (d - c)$ and $|\int_d^c f(x)dx| \leq e \cdot (d - c)$.
- (25) If $a \leq b$ and f is integrable on $[a, b]$ and $f \upharpoonright [a, b]$ is bounded and $[a, b] \subseteq \text{dom } f$ and $c, d \in [a, b]$, then $\int_c^d (r \cdot f)(x)dx = r \cdot \int_c^d f(x)dx$.
- (26) If $a \leq b$ and f is integrable on $[a, b]$ and $f \upharpoonright [a, b]$ is bounded and $[a, b] \subseteq \text{dom } f$ and $c, d \in [a, b]$, then $\int_c^d (-f)(x)dx = -\int_c^d f(x)dx$.

(27) Suppose that $a \leq b$ and f is integrable on $[a, b]$ and g is integrable on $[a, b]$ and $f \upharpoonright [a, b]$ is bounded and $g \upharpoonright [a, b]$ is bounded and $[a, b] \subseteq \text{dom } f$ and $[a, b] \subseteq \text{dom } g$ and $c, d \in [a, b]$. Then $\int_c^d (f + g)(x)dx = \int_c^d f(x)dx + \int_c^d g(x)dx$.

(28) Suppose that $a \leq b$ and f is integrable on $[a, b]$ and g is integrable on $[a, b]$ and $f \upharpoonright [a, b]$ is bounded and $g \upharpoonright [a, b]$ is bounded and $[a, b] \subseteq \text{dom } f$ and $[a, b] \subseteq \text{dom } g$ and $c, d \in [a, b]$. Then $\int_c^d (f - g)(x)dx = \int_c^d f(x)dx - \int_c^d g(x)dx$.

(29) Suppose $a \leq b$ and $[a, b] \subseteq \text{dom } f$ and for every real number x such that $x \in [a, b]$ holds $f(x) = E$. Then f is integrable on $[a, b]$ and $f \upharpoonright [a, b]$ is bounded and $\int_a^b f(x)dx = (b - a) \cdot E$.

(30) Suppose $a \leq b$ and for every real number x such that $x \in [a, b]$ holds $f(x) = E$ and $[a, b] \subseteq \text{dom } f$ and $c, d \in [a, b]$. Then $\int_c^d f(x)dx = (d - c) \cdot E$.

(31) If $a \leq b$ and f is integrable on $[a, b]$ and $f \upharpoonright [a, b]$ is bounded and $[a, b] \subseteq \text{dom } f$ and $c, d \in [a, b]$, then $\int_a^d f(x)dx = \int_a^c f(x)dx + \int_c^d f(x)dx$.

(32) Suppose that $a \leq b$ and f is integrable on $[a, b]$ and $f \upharpoonright [a, b]$ is bounded and $[a, b] \subseteq \text{dom } f$ and $c, d \in [a, b]$ and for every real number x such that $x \in [\min(c, d), \max(c, d)]$ holds $|f_x| \leq e$. Then $|\int_c^d f(x)dx| \leq n \cdot e \cdot |d - c|$.

$$(33) \quad \int_b^a f(x)dx = - \int_a^b f(x)dx.$$

2. ON THE FUNCTIONS FROM \mathbb{R} INTO n -DIMENSIONAL REAL NORMED SPACE

Let R be a real normed space, let X be a non empty set, and let g be a partial function from X to R . We say that g is bounded if and only if:

(Def. 1) There exists a real number r such that for every set y such that $y \in \text{dom } g$ holds $\|g_y\| < r$.

Next we state a number of propositions:

- (34) Let f be a partial function from \mathbb{R} to \mathcal{R}^n and g be a partial function from \mathbb{R} to $\langle \mathcal{E}^n, \|\cdot\| \rangle$. If $f = g$, then f is bounded iff g is bounded.
- (35) Let X, Y be sets and f_1, f_2 be partial functions from \mathbb{R} to $\langle \mathcal{E}^n, \|\cdot\| \rangle$. Suppose $f_1|_X$ is bounded and $f_2|_Y$ is bounded. Then $(f_1 + f_2)|(X \cap Y)$ is bounded and $(f_1 - f_2)|(X \cap Y)$ is bounded.
- (36) Let f be a function from A into \mathcal{R}^n , g be a function from A into $\langle \mathcal{E}^n, \|\cdot\| \rangle$, D be a Division of A , p be a finite sequence of elements of \mathcal{R}^n , and q be a finite sequence of elements of $\langle \mathcal{E}^n, \|\cdot\| \rangle$. Suppose $f = g$ and $p = q$. Then p is a middle volume of f and D if and only if q is a middle volume of g and D .
- (37) Let f be a function from A into \mathcal{R}^n , g be a function from A into $\langle \mathcal{E}^n, \|\cdot\| \rangle$, D be a Division of A , p be a middle volume of f and D , and q be a middle volume of g and D . If $f = g$ and $p = q$, then $\text{middle sum}(f, p) = \text{middle sum}(g, q)$.
- (38) Let f be a function from A into \mathcal{R}^n , g be a function from A into $\langle \mathcal{E}^n, \|\cdot\| \rangle$, T be a division sequence of A , p be a function from \mathbb{N} into $(\mathcal{R}^n)^*$, and q be a function from \mathbb{N} into $(\text{the carrier of } \langle \mathcal{E}^n, \|\cdot\| \rangle)^*$. Suppose $f = g$ and $p = q$. Then p is a middle volume sequence of f and T if and only if q is a middle volume sequence of g and T .
- (39) Let f be a function from A into \mathcal{R}^n , g be a function from A into $\langle \mathcal{E}^n, \|\cdot\| \rangle$, T be a division sequence of A , S be a middle volume sequence of f and T , and U be a middle volume sequence of g and T . If $f = g$ and $S = U$, then $\text{middle sum}(f, S) = \text{middle sum}(g, U)$.
- (40) Let f be a function from A into \mathcal{R}^n , g be a function from A into $\langle \mathcal{E}^n, \|\cdot\| \rangle$, I be an element of \mathcal{R}^n , and J be a point of $\langle \mathcal{E}^n, \|\cdot\| \rangle$. Suppose $f = g$ and $I = J$. Then the following statements are equivalent
- (i) for every division sequence T of A and for every middle volume sequence S of f and T such that δ_T is convergent and $\lim(\delta_T) = 0$ holds $\text{middle sum}(f, S)$ is convergent and $\lim \text{middle sum}(f, S) = I$,
 - (ii) for every division sequence T of A and for every middle volume sequence S of g and T such that δ_T is convergent and $\lim(\delta_T) = 0$ holds $\text{middle sum}(g, S)$ is convergent and $\lim \text{middle sum}(g, S) = J$.
- (41) Let f be a function from A into \mathcal{R}^n and g be a function from A into $\langle \mathcal{E}^n, \|\cdot\| \rangle$. Suppose $f = g$ and f is bounded. Then f is integrable if and only if g is integrable.
- (42) Let f be a function from A into \mathcal{R}^n and g be a function from A into $\langle \mathcal{E}^n, \|\cdot\| \rangle$. Suppose $f = g$ and f is bounded and integrable. Then g is integrable and $\text{integral } f = \text{integral } g$.
- (43) Let f be a partial function from \mathbb{R} to \mathcal{R}^n and g be a partial function

from \mathbb{R} to $\langle \mathcal{E}^n, \|\cdot\| \rangle$. Suppose $f = g$ and $f \upharpoonright A$ is bounded and $A \subseteq \text{dom } f$. Then f is integrable on A if and only if g is integrable on A .

(44) Let f be a partial function from \mathbb{R} to \mathcal{R}^n and g be a partial function from \mathbb{R} to $\langle \mathcal{E}^n, \|\cdot\| \rangle$. Suppose $f = g$ and $f \upharpoonright A$ is bounded and $A \subseteq \text{dom } f$ and f is integrable on A . Then g is integrable on A and $\int_A f(x)dx = \int_A g(x)dx$.

(45) Let f be a partial function from \mathbb{R} to \mathcal{R}^n and g be a partial function from \mathbb{R} to $\langle \mathcal{E}^n, \|\cdot\| \rangle$. Suppose $f = g$ and $a \leq b$ and $f \upharpoonright [a, b]$ is bounded and $[a, b] \subseteq \text{dom } f$ and f is integrable on $[a, b]$. Then $\int_a^b f(x)dx = \int_a^b g(x)dx$.

(46) Let f, g be partial functions from \mathbb{R} to $\langle \mathcal{E}^n, \|\cdot\| \rangle$. Suppose $a \leq b$ and f is integrable on $[a, b]$ and g is integrable on $[a, b]$ and $[a, b] \subseteq \text{dom } f$ and $[a, b] \subseteq \text{dom } g$. Then $\int_a^b (f + g)(x)dx = \int_a^b f(x)dx + \int_a^b g(x)dx$ and $\int_a^b (f - g)(x)dx = \int_a^b f(x)dx - \int_a^b g(x)dx$.

(47) For every partial function f from \mathbb{R} to $\langle \mathcal{E}^n, \|\cdot\| \rangle$ such that $a \leq b$ and $[a, b] \subseteq \text{dom } f$ holds $\int_b^a f(x)dx = -\int_a^b f(x)dx$.

(48) Let f be a partial function from \mathbb{R} to $\langle \mathcal{E}^n, \|\cdot\| \rangle$ and g be a partial function from \mathbb{R} to \mathcal{R}^n . Suppose $f = g$ and $a \leq b$ and $[a, b] \subseteq \text{dom } f$ and $f \upharpoonright [a, b]$ is bounded and f is integrable on $[a, b]$ and $c, d \in [a, b]$. Then $\int_c^d f(x)dx = \int_c^d g(x)dx$.

(49) Let f, g be partial functions from \mathbb{R} to $\langle \mathcal{E}^n, \|\cdot\| \rangle$. Suppose that $a \leq b$ and f is integrable on $[a, b]$ and g is integrable on $[a, b]$ and $f \upharpoonright [a, b]$ is bounded and $g \upharpoonright [a, b]$ is bounded and $[a, b] \subseteq \text{dom } f$ and $[a, b] \subseteq \text{dom } g$ and $c, d \in [a, b]$. Then $\int_c^d (f + g)(x)dx = \int_c^d f(x)dx + \int_c^d g(x)dx$.

(50) Let f, g be partial functions from \mathbb{R} to $\langle \mathcal{E}^n, \|\cdot\| \rangle$. Suppose that $a \leq b$ and f is integrable on $[a, b]$ and g is integrable on $[a, b]$ and $f \upharpoonright [a, b]$ is bounded and $g \upharpoonright [a, b]$ is bounded and $[a, b] \subseteq \text{dom } f$ and $[a, b] \subseteq \text{dom } g$ and $c, d \in [a, b]$. Then $\int_c^d (f - g)(x)dx = \int_c^d f(x)dx - \int_c^d g(x)dx$.

(51) Let E be a point of $\langle \mathcal{E}^n, \|\cdot\| \rangle$ and f be a partial function from \mathbb{R} to $\langle \mathcal{E}^n, \|\cdot\| \rangle$. Suppose $a \leq b$ and $[a, b] \subseteq \text{dom } f$ and for every real number

x such that $x \in [a, b]$ holds $f(x) = E$. Then f is integrable on $[a, b]$ and $f|_{[a, b]}$ is bounded and $\int_a^b f(x)dx = (b - a) \cdot E$.

(52) Let E be a point of $\langle \mathcal{E}^n, \|\cdot\| \rangle$ and f be a partial function from \mathbb{R} to $\langle \mathcal{E}^n, \|\cdot\| \rangle$. Suppose $a \leq b$ and $[a, b] \subseteq \text{dom } f$ and for every real number x such that $x \in [a, b]$ holds $f(x) = E$ and $c, d \in [a, b]$. Then $\int_c^d f(x)dx = (d - c) \cdot E$.

(53) Let f be a partial function from \mathbb{R} to $\langle \mathcal{E}^n, \|\cdot\| \rangle$. Suppose $a \leq b$ and f is integrable on $[a, b]$ and $f|_{[a, b]}$ is bounded and $[a, b] \subseteq \text{dom } f$ and $c, d \in [a, b]$. Then $\int_a^d f(x)dx = \int_a^c f(x)dx + \int_c^d f(x)dx$.

(54) Let f be a partial function from \mathbb{R} to $\langle \mathcal{E}^n, \|\cdot\| \rangle$. Suppose that $a \leq b$ and f is integrable on $[a, b]$ and $f|_{[a, b]}$ is bounded and $[a, b] \subseteq \text{dom } f$ and $c, d \in [a, b]$ and for every real number x such that $x \in [\min(c, d), \max(c, d)]$ holds $\|f_x\| \leq e$. Then $\|\int_c^d f(x)dx\| \leq n \cdot e \cdot |d - c|$.

3. FUNDAMENTAL THEOREM OF CALCULUS

The following two propositions are true:

(55)² Let n be a non empty element of \mathbb{N} and F, f be partial functions from \mathbb{R} to $\langle \mathcal{E}^n, \|\cdot\| \rangle$. Suppose that $a \leq b$ and f is integrable on $[a, b]$ and $f|_{[a, b]}$ is bounded and $[a, b] \subseteq \text{dom } f$ and $]a, b[\subseteq \text{dom } F$ and for every real number x such that $x \in]a, b[$ holds $F(x) = \int_a^x f(x)dx$ and $x_0 \in]a, b[$ and f is continuous in x_0 . Then F is differentiable in x_0 and $F'(x_0) = f_{x_0}$.

(56) Let n be a non empty element of \mathbb{N} and f be a partial function from \mathbb{R} to $\langle \mathcal{E}^n, \|\cdot\| \rangle$. Suppose $a \leq b$ and f is integrable on $[a, b]$ and $f|_{[a, b]}$ is bounded and $[a, b] \subseteq \text{dom } f$ and $x_0 \in]a, b[$ and f is continuous in x_0 . Then there exists a partial function F from \mathbb{R} to $\langle \mathcal{E}^n, \|\cdot\| \rangle$ such that $]a, b[\subseteq \text{dom } F$ and for every real number x such that $x \in]a, b[$ holds $F(x) = \int_a^x f(x)dx$ and F is differentiable in x_0 and $F'(x_0) = f_{x_0}$.

²Fundamental Theorem of Calculus (for \mathcal{R}^n)

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