

Operations of Points on Elliptic Curve in Projective Coordinates

Yuichi Futa
Shinshu University
Nagano, Japan

Hiroyuki Okazaki¹
Shinshu University
Nagano, Japan

Daichi Mizushima
Shinshu University
Nagano, Japan

Yasunari Shidama²
Shinshu University
Nagano, Japan

Summary. In this article, we formalize operations of points on an elliptic curve over $\mathbf{GF}(\mathbf{p})$. Elliptic curve cryptography [7], whose security is based on a difficulty of discrete logarithm problem of elliptic curves, is important for information security. We prove that the two operations of points: `compellprojCo` and `addellprojCo` are unary and binary operations of a point over the elliptic curve.

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The terminology and notation used here are introduced in the following papers: [5], [17], [3], [1], [13], [4], [2], [12], [14], [10], [9], [16], [15], [8], [11], and [6].

1. ARITHMETIC IN $\mathbf{GF}(\mathbf{p})$

For simplicity, we adopt the following convention: i, j denote integers, n denotes a natural number, K denotes a field, and $a_1, a_2, a_3, a_4, a_5, a_6$ denote elements of K .

One can prove the following propositions:

- (1) If $a_1 = -a_2$, then $a_1^2 = a_2^2$.
- (2) $(1_K)^{-1} = 1_K$.

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- (3) If $a_2 \neq 0_K$ and $a_4 \neq 0_K$ and $a_1 \cdot a_2^{-1} = a_3 \cdot a_4^{-1}$, then $a_1 \cdot a_4 = a_2 \cdot a_3$.
- (4) If $a_2 \neq 0_K$ and $a_4 \neq 0_K$ and $a_1 \cdot a_4 = a_2 \cdot a_3$, then $a_1 \cdot a_2^{-1} = a_3 \cdot a_4^{-1}$.
- (5) If $a_1 = 0_K$ and $n > 1$, then $a_1^n = 0_K$.
- (6) If $a_1 = -a_2$, then $-a_1 = a_2$.
- (7) $a_1 + a_2 + a_3 + a_4 = a_4 + a_2 + a_3 + a_1$ and $a_1 + a_2 + a_3 + a_4 = a_1 + a_4 + a_3 + a_2$.
- (8) $(a_1 + a_2 + a_3) + a_4 = a_1 + (a_2 + a_3 + a_4)$ and $(a_1 + a_2 + a_3 + a_4) + a_5 = a_1 + (a_2 + a_3 + a_4 + a_5)$.
- (9) $(a_1 + a_2 + a_3 + a_4 + a_5) + a_6 = a_1 + (a_2 + a_3 + a_4 + a_5 + a_6)$.
- (10) $a_1 \cdot a_2 \cdot a_3 \cdot a_4 = a_4 \cdot a_2 \cdot a_3 \cdot a_1$ and $a_1 \cdot a_2 \cdot a_3 \cdot a_4 = a_1 \cdot a_4 \cdot a_3 \cdot a_2$.
- (11) $(a_1 \cdot a_2 \cdot a_3) \cdot a_4 = a_1 \cdot (a_2 \cdot a_3 \cdot a_4)$ and $(a_1 \cdot a_2 \cdot a_3 \cdot a_4) \cdot a_5 = a_1 \cdot (a_2 \cdot a_3 \cdot a_4 \cdot a_5)$.
- (12) $(a_1 \cdot a_2 \cdot a_3 \cdot a_4 \cdot a_5) \cdot a_6 = a_1 \cdot (a_2 \cdot a_3 \cdot a_4 \cdot a_5 \cdot a_6)$ and $a_1 \cdot a_2 \cdot a_3 \cdot a_4 \cdot a_5 \cdot a_6 = a_1 \cdot (a_2 \cdot a_3 \cdot a_4) \cdot a_5 \cdot a_6$.
- (13) $(a_1 \cdot a_2 \cdot a_3)^n = a_1^n \cdot a_2^n \cdot a_3^n$.
- (14) $a_1 \cdot (a_2 + a_3 + a_4) = a_1 \cdot a_2 + a_1 \cdot a_3 + a_1 \cdot a_4$ and $a_1 \cdot ((a_2 + a_3) - a_4) = (a_1 \cdot a_2 + a_1 \cdot a_3) - a_1 \cdot a_4$ and $a_1 \cdot ((a_2 - a_3) + a_4) = (a_1 \cdot a_2 - a_1 \cdot a_3) + a_1 \cdot a_4$ and $a_1 \cdot (a_2 - a_3 - a_4) = a_1 \cdot a_2 - a_1 \cdot a_3 - a_1 \cdot a_4$ and $a_1 \cdot (-a_2 + a_3 + a_4) = -a_1 \cdot a_2 + a_1 \cdot a_3 + a_1 \cdot a_4$ and $a_1 \cdot ((-a_2 + a_3) - a_4) = (-a_1 \cdot a_2 + a_1 \cdot a_3) - a_1 \cdot a_4$ and $a_1 \cdot ((-a_2 - a_3) + a_4) = (-a_1 \cdot a_2 - a_1 \cdot a_3) + a_1 \cdot a_4$ and $a_1 \cdot (-a_2 - a_3 - a_4) = -a_1 \cdot a_2 - a_1 \cdot a_3 - a_1 \cdot a_4$.
- (15) $(a_1 + a_2) \cdot (a_1 - a_2) = a_1^2 - a_2^2$.
- (16) $(a_1 + a_2) \cdot ((a_1^2 - a_1 \cdot a_2) + a_2^2) = a_1^3 + a_2^3$.
- (17) $(a_1 - a_2) \cdot (a_1^2 + a_1 \cdot a_2 + a_2^2) = a_1^3 - a_2^3$.

Let n, p be natural numbers. We say that p is n or greater if and only if:

(Def. 1) $n \leq p$.

Let us note that there exists a natural number which is 5 or greater and prime.

The following propositions are true:

- (18) For all elements g_1, g_2, g_3, a of $\text{GF}(p)$ such that $g_1 = i \pmod p$ and $g_2 = j \pmod p$ and $g_3 = (i + j) \pmod p$ holds $g_1 \cdot a + g_2 \cdot a = g_3 \cdot a$.
- (19) For all elements g_1, g_2, a of $\text{GF}(p)$ such that $g_1 = i \pmod p$ and $g_2 = j \pmod p$ and $j = i + 1$ holds $g_1 \cdot a + a = g_2 \cdot a$.
- (20) For all elements g_4, a of $\text{GF}(p)$ such that $g_4 = 2 \pmod p$ holds $a + a = g_4 \cdot a$.
- (21) For all elements g_1, g_2, g_3, a of $\text{GF}(p)$ such that $g_1 = i \pmod p$ and $g_2 = j \pmod p$ and $g_3 = (i - j) \pmod p$ holds $g_1 \cdot a - g_2 \cdot a = g_3 \cdot a$.
- (22) For all elements g_1, g_2, a of $\text{GF}(p)$ such that $g_1 = i \pmod p$ and $g_2 = j \pmod p$ and $i = j + 1$ holds $g_1 \cdot a - g_2 \cdot a = a$.
- (23) For all elements g_1, g_2, a of $\text{GF}(p)$ such that $g_1 = i \pmod p$ and $g_2 = j \pmod p$ and $i = j + 1$ holds $g_1 \cdot a - a = g_2 \cdot a$.

- (24) For all elements g_4, a of $\text{GF}(p)$ such that $g_4 = 2 \pmod p$ holds $g_4 \cdot a - a = a$.
- (25) For all elements g_4, a, b of $\text{GF}(p)$ such that $g_4 = 2 \pmod p$ holds $(a + b)^2 = a^2 + g_4 \cdot a \cdot b + b^2$.
- (26) For all elements g_4, a, b of $\text{GF}(p)$ such that $g_4 = 2 \pmod p$ holds $(a - b)^2 = (a^2 - g_4 \cdot a \cdot b) + b^2$.
- (27) For all elements g_4, a, b, c, d of $\text{GF}(p)$ such that $g_4 = 2 \pmod p$ holds $(a \cdot c + b \cdot d)^2 = a^2 \cdot c^2 + g_4 \cdot a \cdot b \cdot c \cdot d + b^2 \cdot d^2$.
- (28) Let p be a prime number, n be a natural number, and g_4 be an element of $\text{GF}(p)$. If $p > 2$ and $g_4 = 2 \pmod p$, then $g_4 \neq 0_{\text{GF}(p)}$ and $g_4^n \neq 0_{\text{GF}(p)}$.
- (29) Let p be a prime number, n be a natural number, and g_4, g_5 be elements of $\text{GF}(p)$. If $p > 3$ and $g_5 = 3 \pmod p$, then $g_5 \neq 0_{\text{GF}(p)}$ and $g_5^n \neq 0_{\text{GF}(p)}$.

2. PARAMETERS OF AN ELLIPTIC CURVE

Let p be a 5 or greater prime number. The parameters of elliptic curve p yielding a subset of $(\text{the carrier of } \text{GF}(p)) \times (\text{the carrier of } \text{GF}(p))$ is defined as follows:

- (Def. 2) The parameters of elliptic curve $p = \{ \langle a, b \rangle; a \text{ ranges over elements of } \text{GF}(p), b \text{ ranges over elements of } \text{GF}(p): \text{Disc}(a) \neq 0_{\text{GF}(p)} \}$.

Let p be a 5 or greater prime number. Observe that the parameters of elliptic curve p is non empty.

Let p be a 5 or greater prime number and let z be an element of the parameters of elliptic curve p . Then z_1 is an element of $\text{GF}(p)$. Then z_2 is an element of $\text{GF}(p)$.

The following proposition is true

- (30) Let p be a 5 or greater prime number and z be an element of the parameters of elliptic curve p . Then $p > 3$ and $\text{Disc}(z_1) \neq 0_{\text{GF}(p)}$.

For simplicity, we adopt the following rules: p_1, p_2, p_3 denote sets, P_1, P_2, P_3 denote elements of $\text{GF}(p)$, P denotes an element of $\text{ProjCo}(\text{GF}(p))$, and O denotes an element of $\text{EC}_{\text{SetProjCo}}(a)$.

Let p be a prime number, let a, b be elements of $\text{GF}(p)$, and let P be an element of $\text{EC}_{\text{SetProjCo}}(a)$. The functor P_1 yields an element of $\text{GF}(p)$ and is defined as follows:

- (Def. 3) If $P = \langle p_1, p_2, p_3 \rangle$, then $P_1 = p_1$.

The functor P_2 yielding an element of $\text{GF}(p)$ is defined as follows:

- (Def. 4) If $P = \langle p_1, p_2, p_3 \rangle$, then $P_2 = p_2$.

The functor P_3 yielding an element of $\text{GF}(p)$ is defined by:

- (Def. 5) If $P = \langle p_1, p_2, p_3 \rangle$, then $P_3 = p_3$.

We now state three propositions:

- (31) For every prime number p and for all elements a, b of $\text{GF}(p)$ and for every element P of $\text{EC}_{\text{SetProjCo}}(a)$ holds $P = \langle P_1, P_2, P_3 \rangle$.
- (32) Let p be a prime number, a, b be elements of $\text{GF}(p)$, P be an element of $\text{EC}_{\text{SetProjCo}}(a)$, and Q be an element of $\text{ProjCo}(\text{GF}(p))$. Then $P = Q$ if and only if the following conditions are satisfied:
- (i) $P_1 = Q_1$,
 - (ii) $P_2 = Q_2$, and
 - (iii) $P_3 = Q_3$.
- (33) Let p be a prime number, a, b, P_1, P_2, P_3 be elements of $\text{GF}(p)$, and P be an element of $\text{EC}_{\text{SetProjCo}}(a)$. If $P = \langle P_1, P_2, P_3 \rangle$, then $P_1 = P_1$ and $P_2 = P_2$ and $P_3 = P_3$.

Let p be a prime number, let P be an element of $\text{ProjCo}(\text{GF}(p))$, and let C_1 be a function from $(\text{the carrier of } \text{GF}(p)) \times (\text{the carrier of } \text{GF}(p)) \times (\text{the carrier of } \text{GF}(p))$ into $\text{GF}(p)$. We say that P is on curve defined by an equation C_1 if and only if:

(Def. 6) $C_1(P) = 0_{\text{GF}(p)}$.

The following two propositions are true:

- (34) P is on curve defined by an equation $\text{EC}_{\text{WEqProjCo}}(a)$ iff P is an element of $\text{EC}_{\text{SetProjCo}}(a)$.
- (35) Let p be a prime number, a, b be elements of $\text{GF}(p)$, and P be an element of $\text{EC}_{\text{SetProjCo}}(a)$. Then $(P_2)^2 \cdot P_3 - ((P_1)^3 + a \cdot P_1 \cdot (P_3)^2 + b \cdot (P_3)^3) = 0_{\text{GF}(p)}$.

Let p be a prime number and let P be an element of $\text{ProjCo}(\text{GF}(p))$. The represent point of P yields an element of $\text{ProjCo}(\text{GF}(p))$ and is defined by:

- (Def. 7)(i) The represent point of $P = \langle P_1 \cdot (P_3)^{-1}, P_2 \cdot (P_3)^{-1}, 1 \rangle$ if $P_3 \neq 0$,
- (ii) the represent point of $P = \langle 0, 1, 0 \rangle$ if $P_3 = 0$,
 - (iii) $P_3 = 0$, otherwise.

The following propositions are true:

- (36) Let p be a 5 or greater prime number, z be an element of the parameters of elliptic curve p , and P be an element of $\text{EC}_{\text{SetProjCo}}(z_1)$. Then the represent point of $P \equiv P$ and the represent point of $P \in \text{EC}_{\text{SetProjCo}}(z_1)$.
- (37) Let p be a prime number, a, b be elements of $\text{GF}(p)$, and P be an element of $\text{ProjCo}(\text{GF}(p))$. Suppose $(\text{the represent point of } P)_3 = 0$. Then the represent point of $P = \langle 0, 1, 0 \rangle$ and $P_3 = 0$.
- (38) Let p be a prime number, a, b be elements of $\text{GF}(p)$, and P be an element of $\text{ProjCo}(\text{GF}(p))$. Suppose $(\text{the represent point of } P)_3 \neq 0$. Then the represent point of $P = \langle P_1 \cdot (P_3)^{-1}, P_2 \cdot (P_3)^{-1}, 1 \rangle$ and $P_3 \neq 0$.
- (39) Let p be a 5 or greater prime number, z be an element of the parameters of elliptic curve p , and P, Q be elements of $\text{EC}_{\text{SetProjCo}}(z_1)$. Then $P \equiv Q$ if and only if the represent point of $P =$ the represent point of Q .

3. OPERATIONS OF POINTS ON AN ELLIPTIC CURVE OVER $\mathbf{GF}(p)$

Let p be a 5 or greater prime number and let z be an element of the parameters of elliptic curve p . The functor $\text{compell}_{\text{ProjCo}}(z, p)$ yields a function from $\text{EC}_{\text{SetProjCo}}(z_1)$ into $\text{EC}_{\text{SetProjCo}}(z_1)$ and is defined as follows:

(Def. 8) For every element P of $\text{EC}_{\text{SetProjCo}}(z_1)$ holds $(\text{compell}_{\text{ProjCo}}(z, p))(P) = \langle P_1, -P_2, P_3 \rangle$.

Let p be a 5 or greater prime number, let z be an element of the parameters of elliptic curve p , let F be a function from $\text{EC}_{\text{SetProjCo}}(z_1)$ into $\text{EC}_{\text{SetProjCo}}(z_1)$, and let P be an element of $\text{EC}_{\text{SetProjCo}}(z_1)$. Then $F(P)$ is an element of $\text{EC}_{\text{SetProjCo}}(z_1)$.

We now state a number of propositions:

- (40) Let p be a 5 or greater prime number, z be an element of the parameters of elliptic curve p , and O be an element of $\text{EC}_{\text{SetProjCo}}(z_1)$. If $O = \langle 0, 1, 0 \rangle$, then $(\text{compell}_{\text{ProjCo}}(z, p))(O) \equiv O$.
- (41) Let p be a 5 or greater prime number, z be an element of the parameters of elliptic curve p , and P be an element of $\text{EC}_{\text{SetProjCo}}(z_1)$. Then $(\text{compell}_{\text{ProjCo}}(z, p))((\text{compell}_{\text{ProjCo}}(z, p))(P)) = P$.
- (42) Let p be a 5 or greater prime number, z be an element of the parameters of elliptic curve p , and P be an element of $\text{EC}_{\text{SetProjCo}}(z_1)$. Suppose $P_3 \neq 0$. Then the represent point of $(\text{compell}_{\text{ProjCo}}(z, p))(P) = (\text{compell}_{\text{ProjCo}}(z, p))(\text{the represent point of } P)$.
- (43) Let p be a 5 or greater prime number, z be an element of the parameters of elliptic curve p , and P, Q be elements of $\text{EC}_{\text{SetProjCo}}(z_1)$. Then $P = Q$ if and only if $(\text{compell}_{\text{ProjCo}}(z, p))(P) = (\text{compell}_{\text{ProjCo}}(z, p))(Q)$.
- (44) Let p be a 5 or greater prime number, z be an element of the parameters of elliptic curve p , and P be an element of $\text{EC}_{\text{SetProjCo}}(z_1)$. If $P_3 \neq 0$, then $P \equiv (\text{compell}_{\text{ProjCo}}(z, p))(P)$ iff $P_2 = 0$.
- (45) Let p be a 5 or greater prime number, z be an element of the parameters of elliptic curve p , and P, Q be elements of $\text{EC}_{\text{SetProjCo}}(z_1)$. If $P_3 \neq 0$, then $P_1 = Q_1$ and $P_3 = Q_3$ iff $P = Q$ or $P = (\text{compell}_{\text{ProjCo}}(z, p))(Q)$.
- (46) Let p be a 5 or greater prime number, z be an element of the parameters of elliptic curve p , and P, Q be elements of $\text{EC}_{\text{SetProjCo}}(z_1)$. Then $P \equiv Q$ if and only if $(\text{compell}_{\text{ProjCo}}(z, p))(P) \equiv (\text{compell}_{\text{ProjCo}}(z, p))(Q)$.
- (47) Let p be a 5 or greater prime number, z be an element of the parameters of elliptic curve p , and P, Q be elements of $\text{EC}_{\text{SetProjCo}}(z_1)$. Then $P \equiv (\text{compell}_{\text{ProjCo}}(z, p))(Q)$ if and only if $(\text{compell}_{\text{ProjCo}}(z, p))(P) \equiv Q$.
- (48) Let p be a 5 or greater prime number, z be an element of the parameters of elliptic curve p , and P, Q be elements of $\text{EC}_{\text{SetProjCo}}(z_1)$. Suppose $P_3 \neq 0$ and $Q_3 \neq 0$. Then the represent point of $P = (\text{compell}_{\text{ProjCo}}(z, p))(\text{the$

represent point of Q) if and only if $P \equiv (\text{compell}_{\text{ProjCo}}(z, p))(Q)$.

- (49) Let p be a 5 or greater prime number, z be an element of the parameters of elliptic curve p , and P, Q be elements of $\text{EC}_{\text{SetProjCo}}(z_1)$. If $P \equiv Q$, then $P_2 \cdot Q_3 = Q_2 \cdot P_3$.
- (50) Let p be a 5 or greater prime number, z be an element of the parameters of elliptic curve p , and P, Q be elements of $\text{EC}_{\text{SetProjCo}}(z_1)$. Suppose $P_3 \neq 0$ and $Q_3 \neq 0$. Then $P \equiv Q$ or $P \equiv (\text{compell}_{\text{ProjCo}}(z, p))(Q)$ if and only if $P_1 \cdot Q_3 = Q_1 \cdot P_3$.
- (51) Let p be a 5 or greater prime number, z be an element of the parameters of elliptic curve p , and P, Q be elements of $\text{EC}_{\text{SetProjCo}}(z_1)$. If $P_3 \neq 0$ and $Q_3 \neq 0$ and $P_2 \neq 0$, then if $P \equiv (\text{compell}_{\text{ProjCo}}(z, p))(Q)$, then $P_2 \cdot Q_3 \neq Q_2 \cdot P_3$.
- (52) Let p be a 5 or greater prime number, z be an element of the parameters of elliptic curve p , and P, Q be elements of $\text{EC}_{\text{SetProjCo}}(z_1)$. If $P \not\equiv Q$ and $P \equiv (\text{compell}_{\text{ProjCo}}(z, p))(Q)$, then $P_2 \cdot Q_3 \neq Q_2 \cdot P_3$.
- (53) Let p be a 5 or greater prime number, z be an element of the parameters of elliptic curve p , g_5 be an element of $\text{GF}(p)$, and P be an element of $\text{EC}_{\text{SetProjCo}}(z_1)$. If $g_5 = 3 \pmod{p}$ and $P_2 = 0$ and $P_3 \neq 0$, then $z_1 \cdot (P_3)^2 + g_5 \cdot (P_1)^2 \neq 0$.
- (54) Let p be a 5 or greater prime number, z be an element of the parameters of elliptic curve p , g_4, g_6, g_7, g_8 be elements of $\text{GF}(p)$, P, Q be elements of $\text{EC}_{\text{SetProjCo}}(z_1)$, and R be an element of $(\text{the carrier of } \text{GF}(p)) \times (\text{the carrier of } \text{GF}(p)) \times (\text{the carrier of } \text{GF}(p))$. Suppose that
- (i) $g_4 = 2 \pmod{p}$,
 - (ii) $g_6 = Q_2 \cdot P_3 - P_2 \cdot Q_3$,
 - (iii) $g_7 = Q_1 \cdot P_3 - P_1 \cdot Q_3$,
 - (iv) $g_8 = g_6^2 \cdot P_3 \cdot Q_3 - g_7^3 - g_4 \cdot g_7^2 \cdot P_1 \cdot Q_3$, and
 - (v) $R = \langle g_7 \cdot g_8, g_6 \cdot (g_7^2 \cdot P_1 \cdot Q_3 - g_8) - g_7^3 \cdot P_2 \cdot Q_3, g_7^3 \cdot P_3 \cdot Q_3 \rangle$.
- Then $g_7 \cdot P_3 \cdot R_2 = -(g_6 \cdot (R_1 \cdot P_3 - P_1 \cdot R_3) + g_7 \cdot P_2 \cdot R_3)$.
- (55) Let p be a 5 or greater prime number, z be an element of the parameters of elliptic curve p , g_4, g_6, g_7, g_8 be elements of $\text{GF}(p)$, P, Q be elements of $\text{EC}_{\text{SetProjCo}}(z_1)$, and R be an element of $(\text{the carrier of } \text{GF}(p)) \times (\text{the carrier of } \text{GF}(p)) \times (\text{the carrier of } \text{GF}(p))$. Suppose that
- (i) $g_4 = 2 \pmod{p}$,
 - (ii) $g_6 = Q_2 \cdot P_3 - P_2 \cdot Q_3$,
 - (iii) $g_7 = Q_1 \cdot P_3 - P_1 \cdot Q_3$,
 - (iv) $g_8 = g_6^2 \cdot P_3 \cdot Q_3 - g_7^3 - g_4 \cdot g_7^2 \cdot P_1 \cdot Q_3$, and
 - (v) $R = \langle g_7 \cdot g_8, g_6 \cdot (g_7^2 \cdot P_1 \cdot Q_3 - g_8) - g_7^3 \cdot P_2 \cdot Q_3, g_7^3 \cdot P_3 \cdot Q_3 \rangle$.
- Then $-g_7^2 \cdot (P_3 \cdot Q_3 \cdot R_1 + P_3 \cdot Q_1 \cdot R_3 + P_1 \cdot Q_3 \cdot R_3) + P_3 \cdot Q_3 \cdot R_3 \cdot g_6^2 = 0_{\text{GF}(p)}$.

(56) Let p be a 5 or greater prime number, z be an element of the parameters of elliptic curve p , g_4, g_6, g_7, g_8 be elements of $\text{GF}(p)$, P, Q be elements of $\text{EC}_{\text{SetProjCo}}(z_1)$, and R be an element of $(\text{the carrier of } \text{GF}(p)) \times (\text{the carrier of } \text{GF}(p)) \times (\text{the carrier of } \text{GF}(p))$. Suppose that

(i) $g_4 = 2 \pmod p$,

(ii) $g_6 = Q_2 \cdot P_3 - P_2 \cdot Q_3$,

(iii) $g_7 = Q_1 \cdot P_3 - P_1 \cdot Q_3$,

(iv) $g_8 = g_6^2 \cdot P_3 \cdot Q_3 - g_7^3 - g_4 \cdot g_7^2 \cdot P_1 \cdot Q_3$, and

(v) $R = \langle g_7 \cdot g_8, g_6 \cdot (g_7^2 \cdot P_1 \cdot Q_3 - g_8) - g_7^3 \cdot P_2 \cdot Q_3, g_7^3 \cdot P_3 \cdot Q_3 \rangle$.

Then $z_2 \cdot g_7^2 \cdot (P_3)^2 \cdot Q_3 \cdot R_3 = -g_7^2 \cdot P_3 \cdot P_1 \cdot Q_1 \cdot R_1 + (g_7 \cdot P_2 - g_6 \cdot P_1)^2 \cdot Q_3 \cdot R_3$.

(57) Let p be a 5 or greater prime number, z be an element of the parameters of elliptic curve p , g_4, g_6, g_7, g_8 be elements of $\text{GF}(p)$, P, Q be elements of $\text{EC}_{\text{SetProjCo}}(z_1)$, and R be an element of $(\text{the carrier of } \text{GF}(p)) \times (\text{the carrier of } \text{GF}(p)) \times (\text{the carrier of } \text{GF}(p))$. Suppose that

(i) $g_4 = 2 \pmod p$,

(ii) $g_6 = Q_2 \cdot P_3 - P_2 \cdot Q_3$,

(iii) $g_7 = Q_1 \cdot P_3 - P_1 \cdot Q_3$,

(iv) $g_8 = g_6^2 \cdot P_3 \cdot Q_3 - g_7^3 - g_4 \cdot g_7^2 \cdot P_1 \cdot Q_3$, and

(v) $R = \langle g_7 \cdot g_8, g_6 \cdot (g_7^2 \cdot P_1 \cdot Q_3 - g_8) - g_7^3 \cdot P_2 \cdot Q_3, g_7^3 \cdot P_3 \cdot Q_3 \rangle$.

Then $z_1 \cdot g_7^2 \cdot P_3 \cdot Q_3 \cdot R_3 = g_7^2 \cdot (P_1 \cdot Q_1 \cdot R_3 + P_3 \cdot Q_1 \cdot R_1 + P_1 \cdot Q_3 \cdot R_1) + g_4 \cdot g_6 \cdot Q_3 \cdot R_3 \cdot (g_7 \cdot P_2 - g_6 \cdot P_1)$.

(58) Let p be a 5 or greater prime number, z be an element of the parameters of elliptic curve p , g_4, g_6, g_7, g_8 be elements of $\text{GF}(p)$, P, Q be elements of $\text{EC}_{\text{SetProjCo}}(z_1)$, and R be an element of $(\text{the carrier of } \text{GF}(p)) \times (\text{the carrier of } \text{GF}(p)) \times (\text{the carrier of } \text{GF}(p))$. Suppose that

(i) $g_4 = 2 \pmod p$,

(ii) $g_6 = Q_2 \cdot P_3 - P_2 \cdot Q_3$,

(iii) $g_7 = Q_1 \cdot P_3 - P_1 \cdot Q_3$,

(iv) $g_8 = g_6^2 \cdot P_3 \cdot Q_3 - g_7^3 - g_4 \cdot g_7^2 \cdot P_1 \cdot Q_3$, and

(v) $R = \langle g_7 \cdot g_8, g_6 \cdot (g_7^2 \cdot P_1 \cdot Q_3 - g_8) - g_7^3 \cdot P_2 \cdot Q_3, g_7^3 \cdot P_3 \cdot Q_3 \rangle$.

Then $g_7^2 \cdot (P_3)^2 \cdot Q_3 \cdot ((R_2)^2 \cdot R_3 - ((R_1)^3 + z_1 \cdot R_1 \cdot (R_3)^2 + z_2 \cdot (R_3)^3)) = 0_{\text{GF}(p)}$.

(59) Let p be a 5 or greater prime number, z be an element of the parameters of elliptic curve p , $g_4, g_5, g_{11}, g_9, g_6, g_7, g_8, g_{10}$ be elements of $\text{GF}(p)$, P be an element of $\text{EC}_{\text{SetProjCo}}(z_1)$, and R be an element of $(\text{the carrier of } \text{GF}(p)) \times (\text{the carrier of } \text{GF}(p)) \times (\text{the carrier of } \text{GF}(p))$. Suppose that $g_4 = 2 \pmod p$ and $g_5 = 3 \pmod p$ and $g_{11} = 4 \pmod p$ and $g_9 = 8 \pmod p$ and $g_6 = z_1 \cdot (P_3)^2 + g_5 \cdot (P_1)^2$ and $g_7 = P_2 \cdot P_3$ and $g_8 = P_1 \cdot P_2 \cdot g_7$ and $g_{10} = g_6^2 - g_9 \cdot g_8$ and $R = \langle g_4 \cdot g_{10} \cdot g_7, g_6 \cdot (g_{11} \cdot g_8 - g_{10}) - g_9 \cdot (P_2)^2 \cdot g_7^2, g_9 \cdot g_7^3 \rangle$. Then $g_4 \cdot g_7 \cdot P_3 \cdot R_2 = -(g_6 \cdot (P_3 \cdot R_1 - P_1 \cdot R_3) + g_4 \cdot g_7 \cdot P_2 \cdot R_3)$.

- (60) Let p be a 5 or greater prime number, z be an element of the parameters of elliptic curve p , $g_4, g_5, g_{11}, g_9, g_6, g_7, g_8, g_{10}$ be elements of $\text{GF}(p)$, P be an element of $\text{EC}_{\text{SetProjCo}}(z_1)$, and R be an element of (the carrier of $\text{GF}(p)$) \times (the carrier of $\text{GF}(p)$) \times (the carrier of $\text{GF}(p)$). Suppose that $g_4 = 2 \pmod p$ and $g_5 = 3 \pmod p$ and $g_{11} = 4 \pmod p$ and $g_9 = 8 \pmod p$ and $g_6 = z_1 \cdot (P_3)^2 + g_5 \cdot (P_1)^2$ and $g_7 = P_2 \cdot P_3$ and $g_8 = P_1 \cdot P_2 \cdot g_7$ and $g_{10} = g_6^2 - g_9 \cdot g_8$ and $R = \langle g_4 \cdot g_{10} \cdot g_7, g_6 \cdot (g_{11} \cdot g_8 - g_{10}) - g_9 \cdot (P_2)^2 \cdot g_7^2, g_9 \cdot g_7^3 \rangle$. Then $g_{11} \cdot g_7^2 \cdot P_3 \cdot R_1 = R_3 \cdot (g_6^2 \cdot P_3 - g_9 \cdot g_7^2 \cdot P_1)$.
- (61) Let p be a 5 or greater prime number, z be an element of the parameters of elliptic curve p , $g_4, g_5, g_{11}, g_9, g_6, g_7, g_8, g_{10}$ be elements of $\text{GF}(p)$, P be an element of $\text{EC}_{\text{SetProjCo}}(z_1)$, and R be an element of (the carrier of $\text{GF}(p)$) \times (the carrier of $\text{GF}(p)$) \times (the carrier of $\text{GF}(p)$). Suppose that $g_4 = 2 \pmod p$ and $g_5 = 3 \pmod p$ and $g_{11} = 4 \pmod p$ and $g_9 = 8 \pmod p$ and $g_6 = z_1 \cdot (P_3)^2 + g_5 \cdot (P_1)^2$ and $g_7 = P_2 \cdot P_3$ and $g_8 = P_1 \cdot P_2 \cdot g_7$ and $g_{10} = g_6^2 - g_9 \cdot g_8$ and $R = \langle g_4 \cdot g_{10} \cdot g_7, g_6 \cdot (g_{11} \cdot g_8 - g_{10}) - g_9 \cdot (P_2)^2 \cdot g_7^2, g_9 \cdot g_7^3 \rangle$. Then $g_{11} \cdot g_7^2 \cdot (P_3)^2 \cdot (z_2 \cdot R_3) = R_3 \cdot (g_4 \cdot g_7 \cdot P_2 - g_6 \cdot P_1)^2 - g_{11} \cdot g_7^2 \cdot (P_1)^2 \cdot R_1$.
- (62) Let p be a 5 or greater prime number, z be an element of the parameters of elliptic curve p , $g_4, g_5, g_{11}, g_9, g_6, g_7, g_8, g_{10}$ be elements of $\text{GF}(p)$, P be an element of $\text{EC}_{\text{SetProjCo}}(z_1)$, and R be an element of (the carrier of $\text{GF}(p)$) \times (the carrier of $\text{GF}(p)$) \times (the carrier of $\text{GF}(p)$). Suppose that $g_4 = 2 \pmod p$ and $g_5 = 3 \pmod p$ and $g_{11} = 4 \pmod p$ and $g_9 = 8 \pmod p$ and $g_6 = z_1 \cdot (P_3)^2 + g_5 \cdot (P_1)^2$ and $g_7 = P_2 \cdot P_3$ and $g_8 = P_1 \cdot P_2 \cdot g_7$ and $g_{10} = g_6^2 - g_9 \cdot g_8$ and $R = \langle g_4 \cdot g_{10} \cdot g_7, g_6 \cdot (g_{11} \cdot g_8 - g_{10}) - g_9 \cdot (P_2)^2 \cdot g_7^2, g_9 \cdot g_7^3 \rangle$. Then $g_4 \cdot g_7^2 \cdot (P_3)^2 \cdot (z_1 \cdot R_3) = g_6 \cdot P_3 \cdot R_3 \cdot (g_4 \cdot g_7 \cdot P_2 - g_6 \cdot P_1) + g_7^2 \cdot (g_{11} \cdot P_1 \cdot P_3 \cdot R_1 + g_4 \cdot (P_1)^2 \cdot R_3)$.
- (63) Let p be a 5 or greater prime number, z be an element of the parameters of elliptic curve p , $g_4, g_5, g_{11}, g_9, g_6, g_7, g_8, g_{10}$ be elements of $\text{GF}(p)$, P be an element of $\text{EC}_{\text{SetProjCo}}(z_1)$, and R be an element of (the carrier of $\text{GF}(p)$) \times (the carrier of $\text{GF}(p)$) \times (the carrier of $\text{GF}(p)$). Suppose that $g_4 = 2 \pmod p$ and $g_5 = 3 \pmod p$ and $g_{11} = 4 \pmod p$ and $g_9 = 8 \pmod p$ and $g_6 = z_1 \cdot (P_3)^2 + g_5 \cdot (P_1)^2$ and $g_7 = P_2 \cdot P_3$ and $g_8 = P_1 \cdot P_2 \cdot g_7$ and $g_{10} = g_6^2 - g_9 \cdot g_8$ and $R = \langle g_4 \cdot g_{10} \cdot g_7, g_6 \cdot (g_{11} \cdot g_8 - g_{10}) - g_9 \cdot (P_2)^2 \cdot g_7^2, g_9 \cdot g_7^3 \rangle$. Then $g_{11} \cdot g_7^2 \cdot (P_3)^2 \cdot ((R_2)^2 \cdot R_3 - ((R_1)^3 + z_1 \cdot R_1 \cdot (R_3)^2 + z_2 \cdot (R_3)^3)) = 0_{\text{GF}(p)}$.

Let p be a 5 or greater prime number and let z be an element of the parameters of elliptic curve p . The functor $\text{addell}_{\text{ProjCo}}(z, p)$ yields a function from $\text{EC}_{\text{SetProjCo}}(z_1) \times \text{EC}_{\text{SetProjCo}}(z_1)$ into $\text{EC}_{\text{SetProjCo}}(z_1)$ and is defined by the condition (Def. 9).

(Def. 9) Let P, Q, O be elements of $\text{EC}_{\text{SetProjCo}}(z_1)$ such that $O = \langle 0, 1, 0 \rangle$.

Then

- (i) if $P \equiv O$, then $(\text{addell}_{\text{ProjCo}}(z, p))(P, Q) = Q$,
- (ii) if $Q \equiv O$ and $P \not\equiv O$, then $(\text{addell}_{\text{ProjCo}}(z, p))(P, Q) = P$,

- (iii) if $P \neq O$ and $Q \neq O$ and $P \neq Q$, then for all elements g_4, g_6, g_7, g_8 of $\text{GF}(p)$ such that $g_4 = 2 \pmod p$ and $g_6 = Q_2 \cdot P_3 - P_2 \cdot Q_3$ and $g_7 = Q_1 \cdot P_3 - P_1 \cdot Q_3$ and $g_8 = g_6^2 \cdot P_3 \cdot Q_3 - g_7^3 - g_4 \cdot g_7^2 \cdot P_1 \cdot Q_3$ holds $(\text{addell}_{\text{ProjCo}}(z, p))(P, Q) = \langle g_7 \cdot g_8, g_6 \cdot (g_7^2 \cdot P_1 \cdot Q_3 - g_8) - g_7^3 \cdot P_2 \cdot Q_3, g_7^3 \cdot P_3 \cdot Q_3 \rangle$, and
- (iv) if $P \neq O$ and $Q \neq O$ and $P \equiv Q$, then for all elements $g_4, g_5, g_{11}, g_9, g_6, g_7, g_8, g_{10}$ of $\text{GF}(p)$ such that $g_4 = 2 \pmod p$ and $g_5 = 3 \pmod p$ and $g_{11} = 4 \pmod p$ and $g_9 = 8 \pmod p$ and $g_6 = z_1 \cdot (P_3)^2 + g_5 \cdot (P_1)^2$ and $g_7 = P_2 \cdot P_3$ and $g_8 = P_1 \cdot P_2 \cdot g_7$ and $g_{10} = g_6^2 - g_9 \cdot g_8$ holds $(\text{addell}_{\text{ProjCo}}(z, p))(P, Q) = \langle g_4 \cdot g_{10} \cdot g_7, g_6 \cdot (g_{11} \cdot g_8 - g_{10}) - g_9 \cdot (P_2)^2 \cdot g_7^2, g_9 \cdot g_7^3 \rangle$.

Let p be a 5 or greater prime number, let z be an element of the parameters of elliptic curve p , let F be a function from $\text{EC}_{\text{SetProjCo}}(z_1) \times \text{EC}_{\text{SetProjCo}}(z_1)$ into $\text{EC}_{\text{SetProjCo}}(z_1)$, and let Q, R be elements of $\text{EC}_{\text{SetProjCo}}(z_1)$. Then $F(Q, R)$ is an element of $\text{EC}_{\text{SetProjCo}}(z_1)$.

REFERENCES

- [1] Grzegorz Bancerek. The ordinal numbers. *Formalized Mathematics*, 1(1):91–96, 1990.
- [2] Czesław Byliński. Binary operations. *Formalized Mathematics*, 1(1):175–180, 1990.
- [3] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(1):55–65, 1990.
- [4] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(1):153–164, 1990.
- [5] Czesław Byliński. Some basic properties of sets. *Formalized Mathematics*, 1(1):47–53, 1990.
- [6] Yuichi Futa, Hiroyuki Okazaki, and Yasunari Shidama. Set of points on elliptic curve in projective coordinates. *Formalized Mathematics*, 19(3):131–138, 2011, doi: 10.2478/v10037-011-0021-6.
- [7] G. Seroussi I. Blake and N. Smart. *Elliptic Curves in Cryptography*. Number 265 in London Mathematical Society Lecture Note Series. Cambridge University Press, 1999.
- [8] Eugeniusz Kusak, Wojciech Leończuk, and Michał Muzalewski. Abelian groups, fields and vector spaces. *Formalized Mathematics*, 1(2):335–342, 1990.
- [9] Rafał Kwiatek. Factorial and Newton coefficients. *Formalized Mathematics*, 1(5):887–890, 1990.
- [10] Rafał Kwiatek and Grzegorz Zwara. The divisibility of integers and integer relative primes. *Formalized Mathematics*, 1(5):829–832, 1990.
- [11] Christoph Schwarzeweller. The binomial theorem for algebraic structures. *Formalized Mathematics*, 9(3):559–564, 2001.
- [12] Andrzej Trybulec. Domains and their Cartesian products. *Formalized Mathematics*, 1(1):115–122, 1990.
- [13] Andrzej Trybulec. Tuples, projections and Cartesian products. *Formalized Mathematics*, 1(1):97–105, 1990.
- [14] Michał J. Trybulec. Integers. *Formalized Mathematics*, 1(3):501–505, 1990.
- [15] Wojciech A. Trybulec. Groups. *Formalized Mathematics*, 1(5):821–827, 1990.
- [16] Wojciech A. Trybulec. Vectors in real linear space. *Formalized Mathematics*, 1(2):291–296, 1990.
- [17] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(1):67–71, 1990.

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