

# Fundamental Group of $n$ -sphere for $n \geq 2$

Marco Riccardi  
Via del Pero 102  
54038 Montignoso  
Italy

Artur Korniłowicz<sup>1</sup>  
Institute of Informatics  
University of Białystok  
Sosnowa 64, 15-887 Białystok  
Poland

**Summary.** Triviality of fundamental groups of spheres of dimension greater than 1 is proven, [17].

MML identifier: TOPALG\_6, version: 7.12.02 4.176.1140

The notation and terminology used in this paper have been introduced in the following papers: [4], [11], [12], [19], [9], [3], [5], [6], [21], [22], [1], [2], [7], [18], [20], [24], [25], [23], [16], [13], [14], [10], [15], and [8].

## 1. PRELIMINARIES

In this paper  $T, U$  are non empty topological spaces,  $t$  is a point of  $T$ , and  $n$  is a natural number.

Let  $S$  be a topological space and let  $T$  be a non empty topological space. Note that every function from  $S$  into  $T$  which is constant is also continuous.

The following two propositions are true:

- (1)  $L_{01}(0, 1, 0, 1) = \text{id}_{[0, 1]_{\mathbb{T}}}$ .
- (2) For all real numbers  $r_1, r_2, r_3, r_4, r_5, r_6$  such that  $r_1 < r_2$  and  $r_3 \leq r_4$  and  $r_5 < r_6$  holds  $L_{01}(r_1, r_2, r_3, r_4) \cdot L_{01}(r_5, r_6, r_1, r_2) = L_{01}(r_5, r_6, r_3, r_4)$ .

Let  $n$  be a positive natural number. Observe that  $\mathcal{E}_{\mathbb{T}}^n$  is infinite and every non empty topological space which is  $n$ -locally Euclidean is also infinite.

The following propositions are true:

---

<sup>1</sup>This work has been supported by the Polish Ministry of Science and Higher Education project “Managing a Large Repository of Computer-verified Mathematical Knowledge” (N N519 385136).

- (3) For every point  $p$  of  $\mathcal{E}_T^n$  such that  $p \in \text{Sphere}((0_{\mathcal{E}_T^n}), 1)$  holds  $-p \in \text{Sphere}((0_{\mathcal{E}_T^n}), 1) \setminus \{p\}$ .
- (4) Let  $T$  be a non empty topological structure,  $t_1, t_2$  be points of  $T$ , and  $p$  be a path from  $t_1$  to  $t_2$ . Then  $\inf \text{dom } p = 0$  and  $\sup \text{dom } p = 1$ .
- (5) For all constant loops  $C_1, C_2$  of  $t$  holds  $C_1, C_2$  are homotopic.
- (6) Let  $S$  be a non empty subspace of  $T$ ,  $t_1, t_2$  be points of  $T$ ,  $s_1, s_2$  be points of  $S$ ,  $A, B$  be paths from  $t_1$  to  $t_2$ , and  $C, D$  be paths from  $s_1$  to  $s_2$ . Suppose  $s_1, s_2$  are connected and  $t_1, t_2$  are connected and  $A = C$  and  $B = D$  and  $C, D$  are homotopic. Then  $A, B$  are homotopic.
- (7) Let  $S$  be a non empty subspace of  $T$ ,  $t_1, t_2$  be points of  $T$ ,  $s_1, s_2$  be points of  $S$ ,  $A, B$  be paths from  $t_1$  to  $t_2$ , and  $C, D$  be paths from  $s_1$  to  $s_2$ . Suppose  $s_1, s_2$  are connected and  $t_1, t_2$  are connected and  $A = C$  and  $B = D$  and  $[C]_{\text{EqRel}(S, s_1, s_2)} = [D]_{\text{EqRel}(S, s_1, s_2)}$ . Then  $[A]_{\text{EqRel}(T, t_1, t_2)} = [B]_{\text{EqRel}(T, t_1, t_2)}$ .
- (8) Let  $T$  be a trivial non empty topological space,  $t$  be a point of  $T$ , and  $L$  be a loop of  $t$ . Then the carrier of  $\pi_1(T, t) = \{[L]_{\text{EqRel}(T, t)}\}$ .
- (9) For every point  $p$  of  $\mathcal{E}_T^n$  and for every subset  $S$  of  $\mathcal{E}_T^n$  such that  $n \geq 2$  and  $S = \Omega_{\mathcal{E}_T^n} \setminus \{p\}$  holds  $\mathcal{E}_T^n \upharpoonright S$  is pathwise connected.
- (10) Let  $S$  be a non empty subset of  $T$ . Suppose  $n \geq 2$  and  $S = \Omega_T \setminus \{t\}$  and  $\mathcal{E}_T^n$  and  $T$  are homeomorphic. Then  $T \upharpoonright S$  is pathwise connected.

Let  $n$  be an element of  $\mathbb{N}$  and let  $p, q$  be points of  $\mathcal{E}_T^n$ . Observe that  $\text{TPlane}(p, q)$  is convex.

## 2. FUNDAMENTAL GROUPS

Let us consider  $T$ . We say that  $T$  has trivial fundamental group if and only if:

- (Def. 1) For every point  $t$  of  $T$  holds  $\pi_1(T, t)$  is trivial.

Let us consider  $T$ . We say that  $T$  is simply connected if and only if:

- (Def. 2)  $T$  is pathwise connected and has trivial fundamental group.

One can verify that every non empty topological space which is simply connected is also pathwise connected and has trivial fundamental group and every non empty topological space which is pathwise connected and has trivial fundamental group is also simply connected.

The following proposition is true

- (11) If  $T$  has trivial fundamental group, then for every point  $t$  of  $T$  and for all loops  $P, Q$  of  $t$  holds  $P, Q$  are homotopic.

Let  $n$  be a natural number. Note that  $\mathcal{E}_T^n$  has trivial fundamental group.

Let us note that every non empty topological space which is trivial also has trivial fundamental group.

The following proposition is true

- (12)  $T$  is simply connected if and only if for all points  $t_1, t_2$  of  $T$  holds  $t_1, t_2$  are connected and for all paths  $P, Q$  from  $t_1$  to  $t_2$  holds  $[P]_{\text{EqRel}(T, t_1, t_2)} = [Q]_{\text{EqRel}(T, t_1, t_2)}$ .

Let  $T$  be a non empty topological space with trivial fundamental group and let  $t$  be a point of  $T$ . One can check that  $\pi_1(T, t)$  is trivial.

Next we state three propositions:

- (13) Let  $S, T$  be non empty topological spaces. Suppose  $S$  and  $T$  are homeomorphic. If  $S$  has trivial fundamental group, then  $T$  has trivial fundamental group.
- (14) Let  $S, T$  be non empty topological spaces. Suppose  $S$  and  $T$  are homeomorphic. If  $S$  is simply connected, then  $T$  is simply connected.
- (15) Let  $T$  be a non empty topological space with trivial fundamental group,  $t$  be a point of  $T$ , and  $P_1, P_2$  be loops of  $t$ . Then  $P_1, P_2$  are homotopic.

Let us consider  $T, t$  and let  $l$  be a loop of  $t$ . We say that  $l$  is null-homotopic if and only if:

- (Def. 3) There exists a constant loop  $c$  of  $t$  such that  $l, c$  are homotopic.

Let us consider  $T, t$ . Observe that every loop of  $t$  which is constant is also null-homotopic.

Let us consider  $T, t$ . Note that there exists a loop of  $t$  which is constant.

The following proposition is true

- (16) Let  $f$  be a loop of  $t$  and  $g$  be a continuous function from  $T$  into  $U$ . If  $f$  is null-homotopic, then  $g \cdot f$  is null-homotopic.

Let  $T, U$  be non empty topological spaces, let  $t$  be a point of  $T$ , let  $f$  be a null-homotopic loop of  $t$ , and let  $g$  be a continuous function from  $T$  into  $U$ . Note that  $g \cdot f$  is null-homotopic.

Let  $T$  be a non empty topological space with trivial fundamental group and let  $t$  be a point of  $T$ . Note that every loop of  $t$  is null-homotopic.

One can prove the following proposition

- (17) If for every point  $t$  of  $T$  holds every loop of  $t$  is null-homotopic, then  $T$  has trivial fundamental group.

Let  $n$  be an element of  $\mathbb{N}$  and let  $p, q$  be points of  $\mathcal{E}_T^n$ . Note that  $\text{TPlane}(p, q)$  has trivial fundamental group.

We now state the proposition

- (18) Let  $S$  be a non empty subspace of  $T$ ,  $s$  be a point of  $S$ ,  $f$  be a loop of  $t$ , and  $g$  be a loop of  $s$ . If  $t = s$  and  $f = g$  and  $g$  is null-homotopic, then  $f$  is null-homotopic.

## 3. CURVES

In the sequel  $T$  is a topological structure and  $f$  is a partial function from  $\mathbb{R}^1$  to  $T$ .

Let us consider  $T, f$ . We say that  $f$  is parametrized curve if and only if the conditions (Def. 4) are satisfied.

- (Def. 4)(i)  $\text{dom } f$  is an interval subset of  $\mathbb{R}$ , and  
(ii) there exists a subspace  $S$  of  $\mathbb{R}^1$  and there exists a function  $g$  from  $S$  into  $T$  such that  $f = g$  and  $S = \mathbb{R}^1 \upharpoonright \text{dom } f$  and  $g$  is continuous.

Let us consider  $T$ . Observe that there exists a partial function from  $\mathbb{R}^1$  to  $T$  which is parametrized curve.

One can prove the following proposition

- (19)  $\emptyset$  is a parametrized curve partial function from  $\mathbb{R}^1$  to  $T$ .

Let us consider  $T$ . The functor  $T$ -Curves yields a subset of  $\mathbb{R} \rightarrow \Omega_T$  and is defined as follows:

- (Def. 5)  $T$ -Curves =  $\{f \in \mathbb{R} \rightarrow \Omega_T : f \text{ is a parametrized curve partial function from } \mathbb{R}^1 \text{ to } T\}$ .

Let us consider  $T$ . One can check that  $T$ -Curves is non empty.

Let us consider  $T$ . A curve of  $T$  is an element of  $T$ -Curves.

In the sequel  $c$  is a curve of  $T$ .

We now state several propositions:

- (20) Every parametrized curve partial function from  $\mathbb{R}^1$  to  $T$  is a curve of  $T$ .  
(21)  $\emptyset$  is a curve of  $T$ .  
(22) Let  $t_1, t_2$  be points of  $T$  and  $p$  be a path from  $t_1$  to  $t_2$ . If  $t_1, t_2$  are connected, then  $p$  is a curve of  $T$ .  
(23)  $c$  is a parametrized curve partial function from  $\mathbb{R}^1$  to  $T$ .  
(24)  $\text{dom } c \subseteq \mathbb{R}$  and  $\text{rng } c \subseteq \Omega_T$ .

Let us consider  $T, c$ . One can verify that  $\text{dom } c$  is real-membered.

Let us consider  $T, c$ . We say that  $c$  has first point if and only if:

- (Def. 6)  $\text{dom } c$  is left-ended.

We say that  $c$  has last point if and only if:

- (Def. 7)  $\text{dom } c$  is right-ended.

Let us consider  $T, c$ . We say that  $c$  has endpoints if and only if:

- (Def. 8)  $c$  has first point and last point.

Let us consider  $T$ . One can check that every curve of  $T$  which has first point and last point also has endpoints and every curve of  $T$  which has endpoints also has first point and last point.

In the sequel  $T$  denotes a non empty topological structure.

Let us consider  $T$ . Note that there exists a curve of  $T$  which has endpoints.

Let us consider  $T$  and let  $c$  be a curve of  $T$  with first point. Note that  $\text{dom } c$  is non empty and  $\text{inf dom } c$  is real.

Let us consider  $T$  and let  $c$  be a curve of  $T$  with last point. Note that  $\text{dom } c$  is non empty and  $\text{sup dom } c$  is real.

Let us consider  $T$ . Observe that every curve of  $T$  which has first point is also non empty and every curve of  $T$  which has last point is also non empty.

Let us consider  $T$  and let  $c$  be a curve of  $T$  with first point. The first point of  $c$  yielding a point of  $T$  is defined by:

(Def. 9) The first point of  $c = c(\text{inf dom } c)$ .

Let us consider  $T$  and let  $c$  be a curve of  $T$  with last point. The last point of  $c$  yielding a point of  $T$  is defined by:

(Def. 10) The last point of  $c = c(\text{sup dom } c)$ .

The following propositions are true:

- (25) Let  $t_1, t_2$  be points of  $T$  and  $p$  be a path from  $t_1$  to  $t_2$ . If  $t_1, t_2$  are connected, then  $p$  is a curve of  $T$  with endpoints.
- (26) For every curve  $c$  of  $T$  and for all real numbers  $r_1, r_2$  holds  $c \upharpoonright [r_1, r_2]$  is a curve of  $T$ .
- (27) For every curve  $c$  of  $T$  with endpoints holds  $\text{dom } c = [\text{inf dom } c, \text{sup dom } c]$ .
- (28) Let  $c$  be a curve of  $T$  with endpoints. Suppose  $\text{dom } c = [0, 1]$ . Then  $c$  is a path from the first point of  $c$  to the last point of  $c$ .
- (29) Let  $c$  be a curve of  $T$  with endpoints. Then  $c \cdot L_{01}(0, 1, \text{inf dom } c, \text{sup dom } c)$  is a path from the first point of  $c$  to the last point of  $c$ .
- (30) Let  $c$  be a curve of  $T$  with endpoints and  $t_1, t_2$  be points of  $T$ . Suppose  $c \cdot L_{01}(0, 1, \text{inf dom } c, \text{sup dom } c)$  is a path from  $t_1$  to  $t_2$  and  $t_1, t_2$  are connected. Then  $t_1 =$  the first point of  $c$  and  $t_2 =$  the last point of  $c$ .
- (31) For every curve  $c$  of  $T$  with endpoints holds the first point of  $c \in \text{rng } c$  and the last point of  $c \in \text{rng } c$ .
- (32) Let  $r_1, r_2$  be real numbers,  $t_1, t_2$  be points of  $T$ , and  $p_1$  be a path from  $t_1$  to  $t_2$ . Suppose  $t_1, t_2$  are connected and  $r_1 < r_2$ . Then  $p_1 \cdot L_{01}(r_1, r_2, 0, 1)$  is a curve of  $T$  with endpoints.
- (33) For every curve  $c$  of  $T$  with endpoints holds the first point of  $c$ , the last point of  $c$  are connected.

Let  $T$  be a non empty topological structure and let  $c_1, c_2$  be curves of  $T$  with endpoints. We say that  $c_1, c_2$  are homotopic if and only if the condition (Def. 11) is satisfied.

(Def. 11) There exist points  $a, b$  of  $T$  and there exist paths  $p_1, p_2$  from  $a$  to  $b$  such that  $p_1 = c_1 \cdot L_{01}(0, 1, \text{inf dom } c_1, \text{sup dom } c_1)$  and  $p_2 = c_2 \cdot L_{01}(0, 1, \text{inf dom } c_2, \text{sup dom } c_2)$  and  $p_1, p_2$  are homotopic.

Let us note that the predicate  $c_1, c_2$  are homotopic is symmetric.

Let  $T$  be a non empty topological space and let  $c_1, c_2$  be curves of  $T$  with endpoints. Let us notice that the predicate  $c_1, c_2$  are homotopic is reflexive and symmetric.

The following three propositions are true:

- (34) Let  $T$  be a non empty topological structure,  $c_1, c_2$  be curves of  $T$  with endpoints,  $a, b$  be points of  $T$ , and  $p_1, p_2$  be paths from  $a$  to  $b$ . Suppose  $c_1 = p_1$  and  $c_2 = p_2$  and  $a, b$  are connected. Then  $c_1, c_2$  are homotopic if and only if  $p_1, p_2$  are homotopic.
- (35) Let  $c_1, c_2$  be curves of  $T$  with endpoints. Suppose  $c_1, c_2$  are homotopic. Then the first point of  $c_1 =$  the first point of  $c_2$  and the last point of  $c_1 =$  the last point of  $c_2$ .
- (36) Let  $T$  be a non empty topological space,  $c_1, c_2$  be curves of  $T$  with endpoints, and  $S$  be a subset of  $\mathbb{R}^1$ . Suppose  $\text{dom } c_1 = \text{dom } c_2$  and  $S = \text{dom } c_1$ . Then  $c_1, c_2$  are homotopic if and only if there exists a function  $f$  from  $(\mathbb{R}^1 \upharpoonright S) \times \mathbb{I}$  into  $T$  and there exist points  $a, b$  of  $T$  such that  $f$  is continuous and for every point  $t$  of  $\mathbb{R}^1 \upharpoonright S$  holds  $f(t, 0) = c_1(t)$  and  $f(t, 1) = c_2(t)$  and for every point  $t$  of  $\mathbb{I}$  holds  $f(\inf S, t) = a$  and  $f(\sup S, t) = b$ .

Let  $T$  be a topological structure and let  $c_1, c_2$  be curves of  $T$ . The functor  $c_1 + c_2$  yielding a curve of  $T$  is defined as follows:

$$\text{(Def. 12)} \quad c_1 + c_2 = \begin{cases} c_1 \cup c_2, & \text{if } c_1 \cup c_2 \text{ is a curve of } T, \\ \emptyset, & \text{otherwise.} \end{cases}$$

One can prove the following three propositions:

- (37) Let  $c$  be a curve of  $T$  with endpoints and  $r$  be a real number. Then there exist elements  $c_1, c_2$  of  $T$ -Curves such that  $c = c_1 + c_2$  and  $c_1 = c \upharpoonright [\inf \text{dom } c, r]$  and  $c_2 = c \upharpoonright [r, \sup \text{dom } c]$ .
- (38) Let  $T$  be a non empty topological space and  $c_1, c_2$  be curves of  $T$  with endpoints. Suppose  $\sup \text{dom } c_1 = \inf \text{dom } c_2$  and the last point of  $c_1 =$  the first point of  $c_2$ . Then  $c_1 + c_2$  has endpoints and  $\text{dom}(c_1 + c_2) = [\inf \text{dom } c_1, \sup \text{dom } c_2]$  and  $(c_1 + c_2)(\inf \text{dom } c_1) =$  the first point of  $c_1$  and  $(c_1 + c_2)(\sup \text{dom } c_2) =$  the last point of  $c_2$ .
- (39) Let  $T$  be a non empty topological space and  $c_1, c_2, c_3, c_4, c_5, c_6$  be curves of  $T$  with endpoints. Suppose that  $c_1, c_2$  are homotopic and  $\text{dom } c_1 = \text{dom } c_2$  and  $c_3, c_4$  are homotopic and  $\text{dom } c_3 = \text{dom } c_4$  and  $c_5 = c_1 + c_3$  and  $c_6 = c_2 + c_4$  and the last point of  $c_1 =$  the first point of  $c_3$  and  $\sup \text{dom } c_1 = \inf \text{dom } c_3$ . Then  $c_5, c_6$  are homotopic.

Let  $T$  be a topological structure and let  $f$  be a finite sequence of elements of  $T$ -Curves. The functor  $(\sum_{\alpha=0}^{\kappa} f(\alpha))_{\kappa \in \mathbb{N}}$  yielding a finite sequence of elements of  $T$ -Curves is defined as follows:

$$\text{(Def. 13)} \quad \text{len } f = \text{len}((\sum_{\alpha=0}^{\kappa} f(\alpha))_{\kappa \in \mathbb{N}}) \text{ and } f(1) = (\sum_{\alpha=0}^{\kappa} f(\alpha))_{\kappa \in \mathbb{N}}(1) \text{ and for every natural number } i \text{ such that } 1 \leq i < \text{len } f \text{ holds } (\sum_{\alpha=0}^{\kappa} f(\alpha))_{\kappa \in \mathbb{N}}(i +$$

$$1) = ((\sum_{\alpha=0}^{\kappa} f(\alpha))_{\kappa \in \mathbb{N}})_i + f_{i+1}.$$

Let  $T$  be a topological structure and let  $f$  be a finite sequence of elements of  $T$ -Curves. The functor  $\sum f$  yields a curve of  $T$  and is defined as follows:

$$\text{(Def. 14)} \quad \sum f = \begin{cases} (\sum_{\alpha=0}^{\kappa} f(\alpha))_{\kappa \in \mathbb{N}}(\text{len } f), & \text{if } \text{len } f > 0, \\ \emptyset, & \text{otherwise.} \end{cases}$$

Next we state several propositions:

- (40) For every curve  $c$  of  $T$  holds  $\sum \langle c \rangle = c$ .
- (41) For every curve  $c$  of  $T$  and for every finite sequence  $f$  of elements of  $T$ -Curves holds  $\sum(f \wedge \langle c \rangle) = \sum f + c$ .
- (42) Let  $X$  be a set and  $f$  be a finite sequence of elements of  $T$ -Curves. Suppose that for every natural number  $i$  such that  $1 \leq i \leq \text{len } f$  holds  $\text{rng}(f_i) \subseteq X$ . Then  $\text{rng } \sum f \subseteq X$ .
- (43) Let  $T$  be a non empty topological space and  $f$  be a finite sequence of elements of  $T$ -Curves. Suppose that
  - (i)  $\text{len } f > 0$ ,
  - (ii) for every natural number  $i$  such that  $1 \leq i < \text{len } f$  holds  $f_i(\text{sup dom}(f_i)) = f_{i+1}(\text{inf dom}(f_{i+1}))$  and  $\text{sup dom}(f_i) = \text{inf dom}(f_{i+1})$ , and
  - (iii) for every natural number  $i$  such that  $1 \leq i \leq \text{len } f$  holds  $f_i$  has endpoints.

Then there exists a curve  $c$  of  $T$  with endpoints such that  $\sum f = c$  and  $\text{dom } c = [\text{inf dom}(f_1), \text{sup dom}(f_{\text{len } f})]$  and the first point of  $c = f_1(\text{inf dom}(f_1))$  and the last point of  $c = f_{\text{len } f}(\text{sup dom}(f_{\text{len } f}))$ .

- (44) Let  $T$  be a non empty topological space,  $f_1, f_2$  be finite sequences of elements of  $T$ -Curves, and  $c_1, c_2$  be curves of  $T$  with endpoints. Suppose that  $\text{len } f_1 > 0$  and  $\text{len } f_1 = \text{len } f_2$  and  $\sum f_1 = c_1$  and  $\sum f_2 = c_2$  and for every natural number  $i$  such that  $1 \leq i < \text{len } f_1$  holds  $(f_1)_i(\text{sup dom}((f_1)_i)) = (f_1)_{i+1}(\text{inf dom}((f_1)_{i+1}))$  and  $\text{sup dom}((f_1)_i) = \text{inf dom}((f_1)_{i+1})$  and for every natural number  $i$  such that  $1 \leq i < \text{len } f_2$  holds  $(f_2)_i(\text{sup dom}((f_2)_i)) = (f_2)_{i+1}(\text{inf dom}((f_2)_{i+1}))$  and  $\text{sup dom}((f_2)_i) = \text{inf dom}((f_2)_{i+1})$  and for every natural number  $i$  such that  $1 \leq i \leq \text{len } f_1$  there exist curves  $c_3, c_4$  of  $T$  with endpoints such that  $c_3 = (f_1)_i$  and  $c_4 = (f_2)_i$  and  $c_3, c_4$  are homotopic and  $\text{dom } c_3 = \text{dom } c_4$ . Then  $c_1, c_2$  are homotopic.
- (45) Let  $c$  be a curve of  $T$  with endpoints and  $h$  be a finite sequence of elements of  $\mathbb{R}$ . Suppose  $\text{len } h \geq 2$  and  $h(1) = \text{inf dom } c$  and  $h(\text{len } h) = \text{sup dom } c$  and  $h$  is increasing. Then there exists a finite sequence  $f$  of elements of  $T$ -Curves such that  $\text{len } f = \text{len } h - 1$  and  $c = \sum f$  and for every natural number  $i$  such that  $1 \leq i \leq \text{len } f$  holds  $f_i = c|[h_i, h_{i+1}]$ .
- (46) If  $n \geq 2$ , then  $\mathbb{S}^n$  has trivial fundamental group.

- (47) Let  $n$  be a non empty natural number,  $r$  be a positive real number, and  $x$  be a point of  $\mathcal{E}_T^n$ . If  $n \geq 3$ , then  $\text{Tcircle}(x, r)$  has trivial fundamental group.

## REFERENCES

- [1] Grzegorz Bancerek. The fundamental properties of natural numbers. *Formalized Mathematics*, 1(1):41–46, 1990.
- [2] Grzegorz Bancerek. The ordinal numbers. *Formalized Mathematics*, 1(1):91–96, 1990.
- [3] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Formalized Mathematics*, 1(1):107–114, 1990.
- [4] Czesław Byliński. Binary operations. *Formalized Mathematics*, 1(1):175–180, 1990.
- [5] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(1):55–65, 1990.
- [6] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(1):153–164, 1990.
- [7] Czesław Byliński. Partial functions. *Formalized Mathematics*, 1(2):357–367, 1990.
- [8] Czesław Byliński. Some basic properties of sets. *Formalized Mathematics*, 1(1):47–53, 1990.
- [9] Agata Darmochwał. The Euclidean space. *Formalized Mathematics*, 2(4):599–603, 1991.
- [10] Agata Darmochwał and Yatsuka Nakamura. Metric spaces as topological spaces – fundamental concepts. *Formalized Mathematics*, 2(4):605–608, 1991.
- [11] Adam Grabowski. Introduction to the homotopy theory. *Formalized Mathematics*, 6(4):449–454, 1997.
- [12] Adam Grabowski and Artur Korniłowicz. Algebraic properties of homotopies. *Formalized Mathematics*, 12(3):251–260, 2004.
- [13] Artur Korniłowicz. The fundamental group of convex subspaces of  $\mathcal{E}_T^n$ . *Formalized Mathematics*, 12(3):295–299, 2004.
- [14] Artur Korniłowicz. On the isomorphism of fundamental groups. *Formalized Mathematics*, 12(3):391–396, 2004.
- [15] Artur Korniłowicz and Yasunari Shidama. Intersections of intervals and balls in  $\mathcal{E}_T^n$ . *Formalized Mathematics*, 12(3):301–306, 2004.
- [16] Artur Korniłowicz, Yasunari Shidama, and Adam Grabowski. The fundamental group. *Formalized Mathematics*, 12(3):261–268, 2004.
- [17] John M. Lee. *Introduction to Topological Manifolds*. Springer-Verlag, New York Berlin Heidelberg, 2000.
- [18] Beata Padlewska and Agata Darmochwał. Topological spaces and continuous functions. *Formalized Mathematics*, 1(1):223–230, 1990.
- [19] Konrad Raczkowski and Paweł Sadowski. Equivalence relations and classes of abstraction. *Formalized Mathematics*, 1(3):441–444, 1990.
- [20] Konrad Raczkowski and Paweł Sadowski. Topological properties of subsets in real numbers. *Formalized Mathematics*, 1(4):777–780, 1990.
- [21] Marco Riccardi. The definition of topological manifolds. *Formalized Mathematics*, 19(1):41–44, 2011, doi: 10.2478/v10037-011-0007-4.
- [22] Marco Riccardi. Planes and spheres as topological manifolds. Stereographic projection. *Formalized Mathematics*, 20(1):41–45, 2012, doi: 10.2478/v10037-012-0006-0.
- [23] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(1):67–71, 1990.
- [24] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1(1):73–83, 1990.
- [25] Edmund Woronowicz. Relations defined on sets. *Formalized Mathematics*, 1(1):181–186, 1990.

Received September 20, 2011