

# The Borsuk-Ulam Theorem

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**Summary.** The Borsuk-Ulam theorem about antipodals is proven, [18, pp. 32–33].

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The notation and terminology used here have been introduced in the following papers: [33], [36], [15], [16], [2], [5], [28], [35], [13], [26], [20], [30], [4], [34], [6], [7], [8], [38], [27], [1], [3], [9], [29], [31], [19], [41], [42], [39], [11], [43], [37], [40], [25], [32], [14], [23], [24], [22], [12], [21], [17], and [10].

## 1. PRELIMINARIES

For simplicity, we adopt the following rules:  $a, b, x, y, z, X, Y, Z$  denote sets,  $n$  denotes a natural number,  $i$  denotes an integer,  $r, r_1, r_2, r_3, s$  denote real numbers,  $c, c_1, c_2$  denote complex numbers, and  $p$  denotes a point of  $\mathcal{E}_T^n$ .

Let us observe that every element of  $\mathbb{I}\mathbb{Q}$  is irrational.

Next we state a number of propositions:

- (1) If  $0 \leq r$  and  $0 \leq s$  and  $r^2 = s^2$ , then  $r = s$ .
- (2) If  $\text{frac } r \geq \text{frac } s$ , then  $\text{frac}(r - s) = \text{frac } r - \text{frac } s$ .
- (3) If  $\text{frac } r < \text{frac } s$ , then  $\text{frac}(r - s) = (\text{frac } r - \text{frac } s) + 1$ .

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- (4) There exists  $i$  such that  $\text{frac}(r - s) = (\text{frac } r - \text{frac } s) + i$  but  $i = 0$  or  $i = 1$ .
- (5) If  $\sin r = 0$ , then  $r = 2 \cdot \pi \cdot \lfloor \frac{r}{2 \cdot \pi} \rfloor$  or  $r = \pi + 2 \cdot \pi \cdot \lfloor \frac{r}{2 \cdot \pi} \rfloor$ .
- (6) If  $\cos r = 0$ , then  $r = \frac{\pi}{2} + 2 \cdot \pi \cdot \lfloor \frac{r}{2 \cdot \pi} \rfloor$  or  $r = \frac{3\pi}{2} + 2 \cdot \pi \cdot \lfloor \frac{r}{2 \cdot \pi} \rfloor$ .
- (7) If  $\sin r = 0$ , then there exists  $i$  such that  $r = \pi \cdot i$ .
- (8) If  $\cos r = 0$ , then there exists  $i$  such that  $r = \frac{\pi}{2} + \pi \cdot i$ .
- (9) If  $\sin r = \sin s$ , then there exists  $i$  such that  $r = s + 2 \cdot \pi \cdot i$  or  $r = (\pi - s) + 2 \cdot \pi \cdot i$ .
- (10) If  $\cos r = \cos s$ , then there exists  $i$  such that  $r = s + 2 \cdot \pi \cdot i$  or  $r = -s + 2 \cdot \pi \cdot i$ .
- (11) If  $\sin r = \sin s$  and  $\cos r = \cos s$ , then there exists  $i$  such that  $r = s + 2 \cdot \pi \cdot i$ .
- (12) If  $|c_1| = |c_2|$  and  $\text{Arg } c_1 = \text{Arg } c_2 + 2 \cdot \pi \cdot i$ , then  $c_1 = c_2$ .

Let  $f$  be a one-to-one complex-valued function and let us consider  $c$ . One can verify that  $f + c$  is one-to-one.

Let  $f$  be a one-to-one complex-valued function and let us consider  $c$ . Note that  $f - c$  is one-to-one.

One can prove the following propositions:

- (13) For every complex-valued finite sequence  $f$  holds  $\text{len}(-f) = \text{len } f$ .
- (14)  $-\underbrace{\langle 0, \dots, 0 \rangle}_n = \underbrace{\langle 0, \dots, 0 \rangle}_n$ .
- (15) For every complex-valued function  $f$  such that  $f \neq \underbrace{\langle 0, \dots, 0 \rangle}_n$  holds  $-f \neq \underbrace{\langle 0, \dots, 0 \rangle}_n$ .
- (16)  ${}^2\langle r_1, r_2, r_3 \rangle = \langle r_1^2, r_2^2, r_3^2 \rangle$ .
- (17)  $\sum^2 \langle r_1, r_2, r_3 \rangle = r_1^2 + r_2^2 + r_3^2$ .
- (18) For every complex-valued finite sequence  $f$  holds  $(c \cdot f)^2 = c^2 \cdot f^2$ .
- (19) For every complex-valued finite sequence  $f$  holds  $(f/c)^2 = f^2/c^2$ .
- (20) For every real-valued finite sequence  $f$  such that  $\sum f \neq 0$  holds  $\sum(f/\sum f) = 1$ .

Let  $a, b, c, x, y, z$  be sets. The functor  $[a \mapsto x, b \mapsto y, c \mapsto z]$  is defined by:

(Def. 1)  $[a \mapsto x, b \mapsto y, c \mapsto z] = [a \mapsto x, b \mapsto y] + \cdot (c \mapsto z)$ .

Let  $a, b, c, x, y, z$  be sets. One can check that  $[a \mapsto x, b \mapsto y, c \mapsto z]$  is function-like and relation-like.

The following propositions are true:

- (21)  $\text{dom}([a \mapsto x, b \mapsto y, c \mapsto z]) = \{a, b, c\}$ .
- (22)  $\text{rng}([a \mapsto x, b \mapsto y, c \mapsto z]) \subseteq \{x, y, z\}$ .
- (23)  $[a \mapsto x, a \mapsto y, a \mapsto z] = a \mapsto z$ .
- (24)  $[a \mapsto x, a \mapsto y, b \mapsto z] = [a \mapsto y, b \mapsto z]$ .
- (25) If  $a \neq b$ , then  $[a \mapsto x, b \mapsto y, a \mapsto z] = [a \mapsto z, b \mapsto y]$ .

- (26)  $[a \mapsto x, b \mapsto y, b \mapsto z] = [a \mapsto x, b \mapsto z]$ .
- (27) If  $a \neq b$  and  $a \neq c$ , then  $([a \mapsto x, b \mapsto y, c \mapsto z])(a) = x$ .
- (28) If  $a, b, c$  are mutually different, then  $([a \mapsto x, b \mapsto y, c \mapsto z])(a) = x$  and  $([a \mapsto x, b \mapsto y, c \mapsto z])(b) = y$  and  $([a \mapsto x, b \mapsto y, c \mapsto z])(c) = z$ .
- (29) For every function  $f$  such that  $\text{dom } f = \{a, b, c\}$  and  $f(a) = x$  and  $f(b) = y$  and  $f(c) = z$  holds  $f = [a \mapsto x, b \mapsto y, c \mapsto z]$ .
- (30)  $\langle a, b, c \rangle = [1 \mapsto a, 2 \mapsto b, 3 \mapsto c]$ .
- (31) If  $a, b, c$  are mutually different, then  $\prod([a \mapsto \{x\}, b \mapsto \{y\}, c \mapsto \{z\}]) = \{[a \mapsto x, b \mapsto y, c \mapsto z]\}$ .
- (32) For all sets  $A, B, C, D, E, F$  such that  $A \subseteq B$  and  $C \subseteq D$  and  $E \subseteq F$  holds  $\prod([a \mapsto A, b \mapsto C, c \mapsto E]) \subseteq \prod([a \mapsto B, b \mapsto D, c \mapsto F])$ .
- (33) If  $a, b, c$  are mutually different and  $x \in X$  and  $y \in Y$  and  $z \in Z$ , then  $[a \mapsto x, b \mapsto y, c \mapsto z] \in \prod([a \mapsto X, b \mapsto Y, c \mapsto Z])$ .

Let  $f$  be a function. We say that  $f$  is odd if and only if:

- (Def. 2) For all complex-valued functions  $x, y$  such that  $x, -x \in \text{dom } f$  and  $y = f(x)$  holds  $f(-x) = -y$ .

Let us mention that  $\emptyset$  is odd.

Let us observe that there exists a function which is odd and complex-functions-valued.

The following propositions are true:

- (34) For every point  $p$  of  $\mathcal{E}_T^3$  holds  ${}^2p = \langle (p_1)^2, (p_2)^2, (p_3)^2 \rangle$ .
- (35) For every point  $p$  of  $\mathcal{E}_T^3$  holds  $\sum^2p = (p_1)^2 + (p_2)^2 + (p_3)^2$ .

The following two propositions are true:

- (36) For every subset  $S$  of  $\mathbb{R}^1$  such that  $S = \mathbb{Q}$  holds  $\mathbb{Q} \cap ]-\infty, r[$  is an open subset of  $\mathbb{R}^1 \upharpoonright S$ .
- (37) For every subset  $S$  of  $\mathbb{R}^1$  such that  $S = \mathbb{Q}$  holds  $\mathbb{Q} \cap ]r, +\infty[$  is an open subset of  $\mathbb{R}^1 \upharpoonright S$ .

Let  $X$  be a connected non empty topological space, let  $Y$  be a non empty topological space, and let  $f$  be a continuous function from  $X$  into  $Y$ . Note that  $\text{Im } f$  is connected.

Next we state two propositions:

- (38) Let  $S$  be a subset of  $\mathbb{R}^1$ . Suppose  $S = \mathbb{Q}$ . Let  $T$  be a connected topological space and  $f$  be a function from  $T$  into  $\mathbb{R}^1 \upharpoonright S$ . If  $f$  is continuous, then  $f$  is constant.
- (39) Let  $a, b$  be real numbers,  $f$  be a continuous function from  $[a, b]_T$  into  $\mathbb{R}^1$ , and  $g$  be a partial function from  $\mathbb{R}$  to  $\mathbb{R}$ . If  $a \leq b$  and  $f = g$ , then  $g$  is continuous.

Let  $s$  be a point of  $\mathbb{R}^1$  and let  $r$  be a real number. Then  $s + r$  is a point of  $\mathbb{R}^1$ .

Let  $s$  be a point of  $\mathbb{R}^1$  and let  $r$  be a real number. Then  $s - r$  is a point of  $\mathbb{R}^1$ .

Let  $X$  be a set, let  $f$  be a function from  $X$  into  $\mathbb{R}^1$ , and let us consider  $r$ . Then  $f + r$  is a function from  $X$  into  $\mathbb{R}^1$ .

Let  $X$  be a set, let  $f$  be a function from  $X$  into  $\mathbb{R}^1$ , and let us consider  $r$ . Then  $f - r$  is a function from  $X$  into  $\mathbb{R}^1$ .

Let  $s, t$  be points of  $\mathbb{R}^1$ , let  $f$  be a path from  $s$  to  $t$ , and let  $r$  be a real number. Then  $f + r$  is a path from  $s + r$  to  $t + r$ . Then  $f - r$  is a path from  $s - r$  to  $t - r$ .

The point  $c[100]$  of `TopUnitCircle3` is defined by:

$$\text{(Def. 3)} \quad c[100] = [1, 0, 0].$$

The point  $c[-100]$  of `TopUnitCircle3` is defined by:

$$\text{(Def. 4)} \quad c[-100] = [-1, 0, 0].$$

Next we state several propositions:

$$(40) \quad -c[100] = c[-100].$$

$$(41) \quad -c[-100] = c[100].$$

$$(42) \quad c[100] - c[-100] = [2, 0, 0].$$

$$(43) \quad \text{For every point } p \text{ of } \mathcal{E}_T^2 \text{ holds } p_1 = |p| \cdot \cos \text{Arg } p \text{ and } p_2 = |p| \cdot \sin \text{Arg } p.$$

$$(44) \quad \text{For every point } p \text{ of } \mathcal{E}_T^2 \text{ holds } p = \text{cpx2euc}(|p| \cdot \cos \text{Arg } p + |p| \cdot \sin \text{Arg } p \cdot i).$$

$$(45) \quad \text{For all points } p_1, p_2 \text{ of } \mathcal{E}_T^2 \text{ such that } |p_1| = |p_2| \text{ and } \text{Arg } p_1 = \text{Arg } p_2 + 2 \cdot \pi \cdot i \text{ holds } p_1 = p_2.$$

One can prove the following propositions:

$$(46) \quad \text{For every point } p \text{ of } \mathcal{E}_T^2 \text{ such that } p = \text{CircleMap}(r) \text{ holds } \text{Arg } p = 2 \cdot \pi \cdot \text{frac } r.$$

$$(47) \quad \text{Let } p_1, p_2 \text{ be points of } \mathcal{E}_T^3 \text{ and } u_1, u_2 \text{ be points of } \mathcal{E}^3. \text{ If } u_1 = p_1 \text{ and } u_2 = p_2, \text{ then } \rho^3(u_1, u_2) = \sqrt{((p_1)_1 - (p_2)_1)^2 + ((p_1)_2 - (p_2)_2)^2 + ((p_1)_3 - (p_2)_3)^2}.$$

$$(48) \quad \text{Let } p \text{ be a point of } \mathcal{E}_T^3 \text{ and } e \text{ be a point of } \mathcal{E}^3. \text{ If } p = e \text{ and } p_3 = 0, \text{ then } \prod([1 \mapsto |p_1 - \frac{r}{\sqrt{2}}, p_1 + \frac{r}{\sqrt{2}}], [2 \mapsto |p_2 - \frac{r}{\sqrt{2}}, p_2 + \frac{r}{\sqrt{2}}], [3 \mapsto \{0\}]) \subseteq \text{Ball}(e, r).$$

$$(49) \quad \text{For every real number } s \text{ holds } c \circlearrowleft s = c \circlearrowleft s + 2 \cdot \pi \cdot i.$$

$$(50) \quad \text{For every real number } s \text{ holds } \text{Rotate } s = \text{Rotate}(s + 2 \cdot \pi \cdot i).$$

$$(51) \quad \text{For every real number } s \text{ and for every point } p \text{ of } \mathcal{E}_T^2 \text{ holds } |(\text{Rotate } s)(p)| = |p|.$$

$$(52) \quad \text{For every real number } s \text{ and for every point } p \text{ of } \mathcal{E}_T^2 \text{ holds } \text{Arg}(\text{Rotate } s)(p) = \text{Arg}(\text{euc2cpx}(p) \circlearrowleft s).$$

$$(53) \quad \text{For every real number } s \text{ and for every point } p \text{ of } \mathcal{E}_T^2 \text{ such that } p \neq 0_{\mathcal{E}_T^2} \text{ there exists } i \text{ such that } \text{Arg}(\text{Rotate } s)(p) = s + \text{Arg } p + 2 \cdot \pi \cdot i.$$

$$(54) \quad \text{For every real number } s \text{ holds } (\text{Rotate } s)(0_{\mathcal{E}_T^2}) = 0_{\mathcal{E}_T^2}.$$

- (55) For every real number  $s$  and for every point  $p$  of  $\mathcal{E}_T^2$  such that  $(\text{Rotate } s)(p) = 0_{\mathcal{E}_T^2}$  holds  $p = 0_{\mathcal{E}_T^2}$ .
- (56) For every real number  $s$  and for every point  $p$  of  $\mathcal{E}_T^2$  holds  $(\text{Rotate } s)((\text{Rotate }(-s))(p)) = p$ .
- (57) For every real number  $s$  holds  $\text{Rotate } s \cdot \text{Rotate }(-s) = \text{id}_{\mathcal{E}_T^2}$ .
- (58) For every real number  $s$  and for every point  $p$  of  $\mathcal{E}_T^2$  holds  $p \in \text{Sphere}((0_{\mathcal{E}_T^2}), r)$  iff  $(\text{Rotate } s)(p) \in \text{Sphere}((0_{\mathcal{E}_T^2}), r)$ .
- (59) For every non negative real number  $r$  and for every real number  $s$  holds  $(\text{Rotate } s)^\circ \text{Sphere}((0_{\mathcal{E}_T^2}), r) = \text{Sphere}((0_{\mathcal{E}_T^2}), r)$ .

Let  $r$  be a non negative real number and let  $s$  be a real number. The functor  $\text{RotateCircle}(r, s)$  yields a function from  $\text{Tcircle}(0_{\mathcal{E}_T^2}, r)$  into  $\text{Tcircle}(0_{\mathcal{E}_T^2}, r)$  and is defined by:

(Def. 5)  $\text{RotateCircle}(r, s) = \text{Rotate } s \upharpoonright \text{Tcircle}(0_{\mathcal{E}_T^2}, r)$ .

Let  $r$  be a non negative real number and let  $s$  be a real number. Note that  $\text{RotateCircle}(r, s)$  is homeomorphism.

One can prove the following proposition

- (60) For every point  $p$  of  $\mathcal{E}_T^2$  such that  $p = \text{CircleMap}(r_2)$  holds  $(\text{RotateCircle}(1, (-\text{Arg } p)))(\text{CircleMap}(r_1)) = \text{CircleMap}(r_1 - r_2)$ .

## 2. ON THE ANTIPODALS

Let  $n$  be a non empty natural number, let  $p$  be a point of  $\mathcal{E}_T^n$ , and let  $r$  be a non negative real number. The functor  $\text{CircleIso}(p, r)$  yields a function from  $\text{TopUnitCircle } n$  into  $\text{Tcircle}(p, r)$  and is defined as follows:

- (Def. 6) For every point  $a$  of  $\text{TopUnitCircle } n$  and for every point  $b$  of  $\mathcal{E}_T^n$  such that  $a = b$  holds  $(\text{CircleIso}(p, r))(a) = r \cdot b + p$ .

Let  $n$  be a non empty natural number, let  $p$  be a point of  $\mathcal{E}_T^n$ , and let  $r$  be a positive real number. Note that  $\text{CircleIso}(p, r)$  is homeomorphism.

The function  $\text{SphereMap}$  from  $\mathbb{R}^1$  into  $\text{TopUnitCircle } 3$  is defined by:

- (Def. 7) For every real number  $x$  holds  $(\text{SphereMap})(x) = [\cos(2 \cdot \pi \cdot x), \sin(2 \cdot \pi \cdot x), 0]$ .

We now state the proposition

- (61)  $(\text{SphereMap})(i) = c[100]$ .

Let us note that  $\text{SphereMap}$  is continuous.

Let  $r$  be a real number. The functor  $\text{eLoop } r$  yields a function from  $\mathbb{I}$  into  $\text{TopUnitCircle } 3$  and is defined as follows:

- (Def. 8) For every point  $x$  of  $\mathbb{I}$  holds  $(\text{eLoop } r)(x) = [\cos(2 \cdot \pi \cdot r \cdot x), \sin(2 \cdot \pi \cdot r \cdot x), 0]$ .

We now state the proposition

- (62)  $\text{eLoop } r = \text{SphereMap} \cdot \text{ExtendInt } r$ .

Let us consider  $i$ . Then  $\text{eLoop } i$  is a loop of  $\mathbb{c}[100]$ .

One can check that  $\text{eLoop } i$  is null-homotopic as a loop of  $\mathbb{c}[100]$ .

One can prove the following proposition

$$(63) \quad \text{If } p \neq 0_{\mathcal{E}_T^n}, \text{ then } |p/|p|| = 1.$$

Let  $n$  be a natural number and let  $p$  be a point of  $\mathcal{E}_T^n$ . Let us assume that  $p \neq 0_{\mathcal{E}_T^n}$ . The functor  $(R^n \rightarrow S^1)p$  yields a point of  $\text{Tcircle}(0_{\mathcal{E}_T^n}, 1)$  and is defined by:

$$(\text{Def. 9}) \quad (R^n \rightarrow S^1)p = p/|p|.$$

Let  $n$  be a non zero natural number and let  $f$  be a function

from  $\text{Tcircle}(0_{\mathcal{E}_T^{n+1}}, 1)$  into  $\mathcal{E}_T^n$ . The functor  $(S^{n+1} \rightarrow S^n)f$  yielding a function from  $\text{TopUnitCircle}(n+1)$  into  $\text{TopUnitCircle } n$  is defined as follows:

$$(\text{Def. 10}) \quad \text{For all points } x, y \text{ of } \text{Tcircle}(0_{\mathcal{E}_T^{n+1}}, 1) \text{ such that } y = -x \text{ holds} \\ ((S^{n+1} \rightarrow S^n)f)(x) = (R^n \rightarrow S^1)(f(x) - f(y)).$$

Let  $x_0, y_0$  be points of  $\text{TopUnitCircle } 2$ , let  $x_1$  be a set, and let  $f$  be a path from  $x_0$  to  $y_0$ . Let us assume that  $x_1 \in \text{CircleMap}^{-1}(\{x_0\})$ . The functor  $\text{liftPath}(f, x_1)$  yielding a function from  $\mathbb{I}$  into  $\mathbb{R}^1$  is defined by the conditions (Def. 11).

$$(\text{Def. 11})(i) \quad (\text{liftPath}(f, x_1))(0) = x_1, \\ (ii) \quad f = \text{CircleMap} \cdot \text{liftPath}(f, x_1), \\ (iii) \quad \text{liftPath}(f, x_1) \text{ is continuous, and} \\ (iv) \quad \text{for every function } f_1 \text{ from } \mathbb{I} \text{ into } \mathbb{R}^1 \text{ such that } f_1 \text{ is continuous and} \\ f = \text{CircleMap} \cdot f_1 \text{ and } f_1(0) = x_1 \text{ holds } \text{liftPath}(f, x_1) = f_1.$$

Let  $n$  be a natural number, let  $p, x, y$  be points of  $\mathcal{E}_T^n$ , and let  $r$  be a real number. We say that  $x$  and  $y$  are antipodals of  $p$  and  $r$  if and only if:

$$(\text{Def. 12}) \quad x \text{ is a point of } \text{Tcircle}(p, r) \text{ and } y \text{ is a point of } \text{Tcircle}(p, r) \text{ and } p \in \mathcal{L}(x, y).$$

Let  $n$  be a natural number, let  $p, x, y$  be points of  $\mathcal{E}_T^n$ , let  $r$  be a real number, and let  $f$  be a function. We say that  $x$  and  $y$  are antipodals of  $p, r$  and  $f$  if and only if:

$$(\text{Def. 13}) \quad x \text{ and } y \text{ are antipodals of } p \text{ and } r \text{ and } x, y \in \text{dom } f \text{ and } f(x) = f(y).$$

Let  $m, n$  be natural numbers, let  $p$  be a point of  $\mathcal{E}_T^m$ , let  $r$  be a real number, and let  $f$  be a function from  $\text{Tcircle}(p, r)$  into  $\mathcal{E}_T^n$ . We say that  $f$  has antipodals if and only if:

$$(\text{Def. 14}) \quad \text{There exist points } x, y \text{ of } \mathcal{E}_T^m \text{ such that } x \text{ and } y \text{ are antipodals of } p, r \text{ and } f.$$

Let  $m, n$  be natural numbers, let  $p$  be a point of  $\mathcal{E}_T^m$ , let  $r$  be a real number, and let  $f$  be a function from  $\text{Tcircle}(p, r)$  into  $\mathcal{E}_T^n$ . We introduce  $f$  is without antipodals as an antonym of  $f$  has antipodals.

One can prove the following propositions:

- (64) Let  $n$  be a non empty natural number,  $r$  be a non negative real number, and  $x$  be a point of  $\mathcal{E}_T^n$ . Suppose  $x$  is a point of  $\text{Tcircle}(0_{\mathcal{E}_T^n}, r)$ . Then  $x$  and  $-x$  are antipodals of  $0_{\mathcal{E}_T^n}$  and  $r$ .
- (65) Let  $n$  be a non empty natural number,  $p, x, y, x_2, y_1$  be points of  $\mathcal{E}_T^n$ , and  $r$  be a positive real number. Suppose  $x$  and  $y$  are antipodals of  $0_{\mathcal{E}_T^n}$  and  $1$  and  $x_2 = (\text{CircleIso}(p, r))(x)$  and  $y_1 = (\text{CircleIso}(p, r))(y)$ . Then  $x_2$  and  $y_1$  are antipodals of  $p$  and  $r$ .
- (66) Let  $f$  be a function from  $\text{Tcircle}(0_{\mathcal{E}_T^{n+1}}, 1)$  into  $\mathcal{E}_T^n$  and  $x$  be a point of  $\text{Tcircle}(0_{\mathcal{E}_T^{n+1}}, 1)$ . If  $f$  is without antipodals, then  $f(x) - f(-x) \neq 0_{\mathcal{E}_T^n}$ .
- (67) For every function  $f$  from  $\text{Tcircle}(0_{\mathcal{E}_T^{n+1}}, 1)$  into  $\mathcal{E}_T^n$  such that  $f$  is without antipodals holds  $(S^{n+1} \rightarrow S^n) f$  is odd.
- (68) Let  $f$  be a function from  $\text{Tcircle}(0_{\mathcal{E}_T^{n+1}}, 1)$  into  $\mathcal{E}_T^n$  and  $g, B_1$  be functions from  $\text{Tcircle}(0_{\mathcal{E}_T^{n+1}}, 1)$  into  $\mathcal{E}_T^n$ . If  $g = f \circ -$  and  $B_1 = f - g$  and  $f$  is without antipodals, then  $(S^{n+1} \rightarrow S^n) f = B_1 / (n \text{NormF} \cdot B_1)$ .

Let us consider  $n$ , let  $r$  be a negative real number, and let  $p$  be a point of  $\mathcal{E}_T^{n+1}$ . Observe that every function from  $\text{Tcircle}(p, r)$  into  $\mathcal{E}_T^n$  is without antipodals.

Let  $r$  be a non negative real number and let  $p$  be a point of  $\mathcal{E}_T^3$ . Note that every function from  $\text{Tcircle}(p, r)$  into  $\mathcal{E}_T^2$  which is continuous also has antipodals.<sup>2</sup>

## REFERENCES

- [1] Grzegorz Bancerek. The fundamental properties of natural numbers. *Formalized Mathematics*, 1(1):41–46, 1990.
- [2] Grzegorz Bancerek. König's theorem. *Formalized Mathematics*, 1(3):589–593, 1990.
- [3] Grzegorz Bancerek. The ordinal numbers. *Formalized Mathematics*, 1(1):91–96, 1990.
- [4] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Formalized Mathematics*, 1(1):107–114, 1990.
- [5] Czesław Byliński. The complex numbers. *Formalized Mathematics*, 1(3):507–513, 1990.
- [6] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(1):55–65, 1990.
- [7] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(1):153–164, 1990.
- [8] Czesław Byliński. The modification of a function by a function and the iteration of the composition of a function. *Formalized Mathematics*, 1(3):521–527, 1990.
- [9] Czesław Byliński. Partial functions. *Formalized Mathematics*, 1(2):357–367, 1990.
- [10] Czesław Byliński. Some basic properties of sets. *Formalized Mathematics*, 1(1):47–53, 1990.
- [11] Czesław Byliński. The sum and product of finite sequences of real numbers. *Formalized Mathematics*, 1(4):661–668, 1990.
- [12] Agata Darmochwał. Families of subsets, subspaces and mappings in topological spaces. *Formalized Mathematics*, 1(2):257–261, 1990.
- [13] Agata Darmochwał. The Euclidean space. *Formalized Mathematics*, 2(4):599–603, 1991.
- [14] Agata Darmochwał and Yatsuka Nakamura. Metric spaces as topological spaces – fundamental concepts. *Formalized Mathematics*, 2(4):605–608, 1991.

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- [15] Adam Grabowski. Introduction to the homotopy theory. *Formalized Mathematics*, 6(4):449–454, 1997.
- [16] Adam Grabowski. On the subcontinua of a real line. *Formalized Mathematics*, 11(3):313–322, 2003.
- [17] Jarosław Gryko. Injective spaces. *Formalized Mathematics*, 7(1):57–62, 1998.
- [18] Allen Hatcher. *Algebraic Topology*. Cambridge University Press, 2002.
- [19] Krzysztof Hryniewiecki. Basic properties of real numbers. *Formalized Mathematics*, 1(1):35–40, 1990.
- [20] Kanchun, Hiroshi Yamazaki, and Yatsuka Nakamura. Cross products and tripple vector products in 3-dimensional Euclidean space. *Formalized Mathematics*, 11(4):381–383, 2003.
- [21] Artur Korniłowicz. Arithmetic operations on functions from sets into functional sets. *Formalized Mathematics*, 17(1):43–60, 2009, doi:10.2478/v10037-009-0005-y.
- [22] Artur Korniłowicz. On the continuity of some functions. *Formalized Mathematics*, 18(3):175–183, 2010, doi: 10.2478/v10037-010-0020-z.
- [23] Artur Korniłowicz and Yasunari Shidama. Intersections of intervals and balls in  $\mathcal{E}_T^n$ . *Formalized Mathematics*, 12(3):301–306, 2004.
- [24] Artur Korniłowicz and Yasunari Shidama. Some properties of circles on the plane. *Formalized Mathematics*, 13(1):117–124, 2005.
- [25] Artur Korniłowicz, Yasunari Shidama, and Adam Grabowski. The fundamental group. *Formalized Mathematics*, 12(3):261–268, 2004.
- [26] Akihiro Kubo and Yatsuka Nakamura. Angle and triangle in Euclidian topological space. *Formalized Mathematics*, 11(3):281–287, 2003.
- [27] Adam Naumowicz and Grzegorz Bancerek. Homeomorphisms of Jordan curves. *Formalized Mathematics*, 13(4):477–480, 2005.
- [28] Beata Padlewska. Connected spaces. *Formalized Mathematics*, 1(1):239–244, 1990.
- [29] Beata Padlewska and Agata Darmochwał. Topological spaces and continuous functions. *Formalized Mathematics*, 1(1):223–230, 1990.
- [30] Konrad Raczkowski and Paweł Sadowski. Real function continuity. *Formalized Mathematics*, 1(4):787–791, 1990.
- [31] Konrad Raczkowski and Paweł Sadowski. Topological properties of subsets in real numbers. *Formalized Mathematics*, 1(4):777–780, 1990.
- [32] Marco Riccardi and Artur Korniłowicz. Fundamental group of  $n$ -sphere for  $n \geq 2$ . *Formalized Mathematics*, 20(2):97–104, 2012, doi: 10.2478/v10037-012-0013-1.
- [33] Piotr Rudnicki and Andrzej Trybulec. Abian’s fixed point theorem. *Formalized Mathematics*, 6(3):335–338, 1997.
- [34] Andrzej Trybulec. Binary operations applied to functions. *Formalized Mathematics*, 1(2):329–334, 1990.
- [35] Andrzej Trybulec. Enumerated sets. *Formalized Mathematics*, 1(1):25–34, 1990.
- [36] Andrzej Trybulec. A Borsuk theorem on homotopy types. *Formalized Mathematics*, 2(4):535–545, 1991.
- [37] Andrzej Trybulec and Czesław Byliński. Some properties of real numbers. *Formalized Mathematics*, 1(3):445–449, 1990.
- [38] Michał J. Trybulec. Integers. *Formalized Mathematics*, 1(3):501–505, 1990.
- [39] Wojciech A. Trybulec. Vectors in real linear space. *Formalized Mathematics*, 1(2):291–296, 1990.
- [40] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(1):67–71, 1990.
- [41] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1(1):73–83, 1990.
- [42] Edmund Woronowicz. Relations defined on sets. *Formalized Mathematics*, 1(1):181–186, 1990.
- [43] Yuguang Yang and Yasunari Shidama. Trigonometric functions and existence of circle ratio. *Formalized Mathematics*, 7(2):255–263, 1998.

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