

The Borsuk-Ulam Theorem

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Summary. The Borsuk-Ulam theorem about antipodals is proven, [18, pp. 32–33].

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The notation and terminology used here have been introduced in the following papers: [33], [36], [15], [16], [2], [5], [28], [35], [13], [26], [20], [30], [4], [34], [6], [7], [8], [38], [27], [1], [3], [9], [29], [31], [19], [41], [42], [39], [11], [43], [37], [40], [25], [32], [14], [23], [24], [22], [12], [21], [17], and [10].

1. PRELIMINARIES

For simplicity, we adopt the following rules: a, b, x, y, z, X, Y, Z denote sets, n denotes a natural number, i denotes an integer, r, r_1, r_2, r_3, s denote real numbers, c, c_1, c_2 denote complex numbers, and p denotes a point of \mathcal{E}_T^n .

Let us observe that every element of $\mathbb{I}\mathbb{Q}$ is irrational.

Next we state a number of propositions:

- (1) If $0 \leq r$ and $0 \leq s$ and $r^2 = s^2$, then $r = s$.
- (2) If $\text{frac } r \geq \text{frac } s$, then $\text{frac}(r - s) = \text{frac } r - \text{frac } s$.
- (3) If $\text{frac } r < \text{frac } s$, then $\text{frac}(r - s) = (\text{frac } r - \text{frac } s) + 1$.

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- (4) There exists i such that $\text{frac}(r - s) = (\text{frac } r - \text{frac } s) + i$ but $i = 0$ or $i = 1$.
- (5) If $\sin r = 0$, then $r = 2 \cdot \pi \cdot \lfloor \frac{r}{2\pi} \rfloor$ or $r = \pi + 2 \cdot \pi \cdot \lfloor \frac{r}{2\pi} \rfloor$.
- (6) If $\cos r = 0$, then $r = \frac{\pi}{2} + 2 \cdot \pi \cdot \lfloor \frac{r}{2\pi} \rfloor$ or $r = \frac{3\pi}{2} + 2 \cdot \pi \cdot \lfloor \frac{r}{2\pi} \rfloor$.
- (7) If $\sin r = 0$, then there exists i such that $r = \pi \cdot i$.
- (8) If $\cos r = 0$, then there exists i such that $r = \frac{\pi}{2} + \pi \cdot i$.
- (9) If $\sin r = \sin s$, then there exists i such that $r = s + 2 \cdot \pi \cdot i$ or $r = (\pi - s) + 2 \cdot \pi \cdot i$.
- (10) If $\cos r = \cos s$, then there exists i such that $r = s + 2 \cdot \pi \cdot i$ or $r = -s + 2 \cdot \pi \cdot i$.
- (11) If $\sin r = \sin s$ and $\cos r = \cos s$, then there exists i such that $r = s + 2 \cdot \pi \cdot i$.
- (12) If $|c_1| = |c_2|$ and $\text{Arg } c_1 = \text{Arg } c_2 + 2 \cdot \pi \cdot i$, then $c_1 = c_2$.

Let f be a one-to-one complex-valued function and let us consider c . One can verify that $f + c$ is one-to-one.

Let f be a one-to-one complex-valued function and let us consider c . Note that $f - c$ is one-to-one.

One can prove the following propositions:

- (13) For every complex-valued finite sequence f holds $\text{len}(-f) = \text{len } f$.
- (14) $-\underbrace{\langle 0, \dots, 0 \rangle}_n = \underbrace{\langle 0, \dots, 0 \rangle}_n$.
- (15) For every complex-valued function f such that $f \neq \underbrace{\langle 0, \dots, 0 \rangle}_n$ holds $-f \neq \underbrace{\langle 0, \dots, 0 \rangle}_n$.
- (16) ${}^2\langle r_1, r_2, r_3 \rangle = \langle r_1^2, r_2^2, r_3^2 \rangle$.
- (17) $\sum^2 \langle r_1, r_2, r_3 \rangle = r_1^2 + r_2^2 + r_3^2$.
- (18) For every complex-valued finite sequence f holds $(c \cdot f)^2 = c^2 \cdot f^2$.
- (19) For every complex-valued finite sequence f holds $(f/c)^2 = f^2/c^2$.
- (20) For every real-valued finite sequence f such that $\sum f \neq 0$ holds $\sum(f/\sum f) = 1$.

Let a, b, c, x, y, z be sets. The functor $[a \mapsto x, b \mapsto y, c \mapsto z]$ is defined by:

(Def. 1) $[a \mapsto x, b \mapsto y, c \mapsto z] = [a \mapsto x, b \mapsto y] + \cdot (c \mapsto z)$.

Let a, b, c, x, y, z be sets. One can check that $[a \mapsto x, b \mapsto y, c \mapsto z]$ is function-like and relation-like.

The following propositions are true:

- (21) $\text{dom}([a \mapsto x, b \mapsto y, c \mapsto z]) = \{a, b, c\}$.
- (22) $\text{rng}([a \mapsto x, b \mapsto y, c \mapsto z]) \subseteq \{x, y, z\}$.
- (23) $[a \mapsto x, a \mapsto y, a \mapsto z] = a \mapsto z$.
- (24) $[a \mapsto x, a \mapsto y, b \mapsto z] = [a \mapsto y, b \mapsto z]$.
- (25) If $a \neq b$, then $[a \mapsto x, b \mapsto y, a \mapsto z] = [a \mapsto z, b \mapsto y]$.

- (26) $[a \mapsto x, b \mapsto y, b \mapsto z] = [a \mapsto x, b \mapsto z]$.
- (27) If $a \neq b$ and $a \neq c$, then $([a \mapsto x, b \mapsto y, c \mapsto z])(a) = x$.
- (28) If a, b, c are mutually different, then $([a \mapsto x, b \mapsto y, c \mapsto z])(a) = x$ and $([a \mapsto x, b \mapsto y, c \mapsto z])(b) = y$ and $([a \mapsto x, b \mapsto y, c \mapsto z])(c) = z$.
- (29) For every function f such that $\text{dom } f = \{a, b, c\}$ and $f(a) = x$ and $f(b) = y$ and $f(c) = z$ holds $f = [a \mapsto x, b \mapsto y, c \mapsto z]$.
- (30) $\langle a, b, c \rangle = [1 \mapsto a, 2 \mapsto b, 3 \mapsto c]$.
- (31) If a, b, c are mutually different, then $\prod([a \mapsto \{x\}, b \mapsto \{y\}, c \mapsto \{z\}]) = \{[a \mapsto x, b \mapsto y, c \mapsto z]\}$.
- (32) For all sets A, B, C, D, E, F such that $A \subseteq B$ and $C \subseteq D$ and $E \subseteq F$ holds $\prod([a \mapsto A, b \mapsto C, c \mapsto E]) \subseteq \prod([a \mapsto B, b \mapsto D, c \mapsto F])$.
- (33) If a, b, c are mutually different and $x \in X$ and $y \in Y$ and $z \in Z$, then $[a \mapsto x, b \mapsto y, c \mapsto z] \in \prod([a \mapsto X, b \mapsto Y, c \mapsto Z])$.

Let f be a function. We say that f is odd if and only if:

- (Def. 2) For all complex-valued functions x, y such that $x, -x \in \text{dom } f$ and $y = f(x)$ holds $f(-x) = -y$.

Let us mention that \emptyset is odd.

Let us observe that there exists a function which is odd and complex-functions-valued.

The following propositions are true:

- (34) For every point p of \mathcal{E}_T^3 holds ${}^2p = \langle (p_1)^2, (p_2)^2, (p_3)^2 \rangle$.
- (35) For every point p of \mathcal{E}_T^3 holds $\sum^2 p = (p_1)^2 + (p_2)^2 + (p_3)^2$.

The following two propositions are true:

- (36) For every subset S of \mathbb{R}^1 such that $S = \mathbb{Q}$ holds $\mathbb{Q} \cap]-\infty, r[$ is an open subset of $\mathbb{R}^1 \upharpoonright S$.
- (37) For every subset S of \mathbb{R}^1 such that $S = \mathbb{Q}$ holds $\mathbb{Q} \cap]r, +\infty[$ is an open subset of $\mathbb{R}^1 \upharpoonright S$.

Let X be a connected non empty topological space, let Y be a non empty topological space, and let f be a continuous function from X into Y . Note that $\text{Im } f$ is connected.

Next we state two propositions:

- (38) Let S be a subset of \mathbb{R}^1 . Suppose $S = \mathbb{Q}$. Let T be a connected topological space and f be a function from T into $\mathbb{R}^1 \upharpoonright S$. If f is continuous, then f is constant.
- (39) Let a, b be real numbers, f be a continuous function from $[a, b]_T$ into \mathbb{R}^1 , and g be a partial function from \mathbb{R} to \mathbb{R} . If $a \leq b$ and $f = g$, then g is continuous.

Let s be a point of \mathbb{R}^1 and let r be a real number. Then $s + r$ is a point of \mathbb{R}^1 .

Let s be a point of \mathbb{R}^1 and let r be a real number. Then $s - r$ is a point of \mathbb{R}^1 .

Let X be a set, let f be a function from X into \mathbb{R}^1 , and let us consider r . Then $f + r$ is a function from X into \mathbb{R}^1 .

Let X be a set, let f be a function from X into \mathbb{R}^1 , and let us consider r . Then $f - r$ is a function from X into \mathbb{R}^1 .

Let s, t be points of \mathbb{R}^1 , let f be a path from s to t , and let r be a real number. Then $f + r$ is a path from $s + r$ to $t + r$. Then $f - r$ is a path from $s - r$ to $t - r$.

The point $c[100]$ of `TopUnitCircle3` is defined by:

$$\text{(Def. 3)} \quad c[100] = [1, 0, 0].$$

The point $c[-100]$ of `TopUnitCircle3` is defined by:

$$\text{(Def. 4)} \quad c[-100] = [-1, 0, 0].$$

Next we state several propositions:

$$(40) \quad -c[100] = c[-100].$$

$$(41) \quad -c[-100] = c[100].$$

$$(42) \quad c[100] - c[-100] = [2, 0, 0].$$

$$(43) \quad \text{For every point } p \text{ of } \mathcal{E}_T^2 \text{ holds } p_1 = |p| \cdot \cos \text{Arg } p \text{ and } p_2 = |p| \cdot \sin \text{Arg } p.$$

$$(44) \quad \text{For every point } p \text{ of } \mathcal{E}_T^2 \text{ holds } p = \text{cpx2euc}(|p| \cdot \cos \text{Arg } p + |p| \cdot \sin \text{Arg } p \cdot i).$$

$$(45) \quad \text{For all points } p_1, p_2 \text{ of } \mathcal{E}_T^2 \text{ such that } |p_1| = |p_2| \text{ and } \text{Arg } p_1 = \text{Arg } p_2 + 2 \cdot \pi \cdot i \text{ holds } p_1 = p_2.$$

One can prove the following propositions:

$$(46) \quad \text{For every point } p \text{ of } \mathcal{E}_T^2 \text{ such that } p = \text{CircleMap}(r) \text{ holds } \text{Arg } p = 2 \cdot \pi \cdot \text{frac } r.$$

$$(47) \quad \text{Let } p_1, p_2 \text{ be points of } \mathcal{E}_T^3 \text{ and } u_1, u_2 \text{ be points of } \mathcal{E}^3. \text{ If } u_1 = p_1 \text{ and } u_2 = p_2, \text{ then } \rho^3(u_1, u_2) = \sqrt{((p_1)_1 - (p_2)_1)^2 + ((p_1)_2 - (p_2)_2)^2 + ((p_1)_3 - (p_2)_3)^2}.$$

$$(48) \quad \text{Let } p \text{ be a point of } \mathcal{E}_T^3 \text{ and } e \text{ be a point of } \mathcal{E}^3. \text{ If } p = e \text{ and } p_3 = 0, \text{ then } \prod([1 \mapsto]p_1 - \frac{r}{\sqrt{2}}, p_1 + \frac{r}{\sqrt{2}}, 2 \mapsto]p_2 - \frac{r}{\sqrt{2}}, p_2 + \frac{r}{\sqrt{2}}, 3 \mapsto \{0\}) \subseteq \text{Ball}(e, r).$$

$$(49) \quad \text{For every real number } s \text{ holds } c \circ s = c \circ s + 2 \cdot \pi \cdot i.$$

$$(50) \quad \text{For every real number } s \text{ holds } \text{Rotate } s = \text{Rotate}(s + 2 \cdot \pi \cdot i).$$

$$(51) \quad \text{For every real number } s \text{ and for every point } p \text{ of } \mathcal{E}_T^2 \text{ holds } |(\text{Rotate } s)(p)| = |p|.$$

$$(52) \quad \text{For every real number } s \text{ and for every point } p \text{ of } \mathcal{E}_T^2 \text{ holds } \text{Arg}(\text{Rotate } s)(p) = \text{Arg}(\text{euc2cpx}(p) \circ s).$$

$$(53) \quad \text{For every real number } s \text{ and for every point } p \text{ of } \mathcal{E}_T^2 \text{ such that } p \neq 0_{\mathcal{E}_T^2} \text{ there exists } i \text{ such that } \text{Arg}(\text{Rotate } s)(p) = s + \text{Arg } p + 2 \cdot \pi \cdot i.$$

$$(54) \quad \text{For every real number } s \text{ holds } (\text{Rotate } s)(0_{\mathcal{E}_T^2}) = 0_{\mathcal{E}_T^2}.$$

- (55) For every real number s and for every point p of \mathcal{E}_T^2 such that $(\text{Rotate } s)(p) = 0_{\mathcal{E}_T^2}$ holds $p = 0_{\mathcal{E}_T^2}$.
- (56) For every real number s and for every point p of \mathcal{E}_T^2 holds $(\text{Rotate } s)((\text{Rotate }(-s))(p)) = p$.
- (57) For every real number s holds $\text{Rotate } s \cdot \text{Rotate }(-s) = \text{id}_{\mathcal{E}_T^2}$.
- (58) For every real number s and for every point p of \mathcal{E}_T^2 holds $p \in \text{Sphere}((0_{\mathcal{E}_T^2}), r)$ iff $(\text{Rotate } s)(p) \in \text{Sphere}((0_{\mathcal{E}_T^2}), r)$.
- (59) For every non negative real number r and for every real number s holds $(\text{Rotate } s)^\circ \text{Sphere}((0_{\mathcal{E}_T^2}), r) = \text{Sphere}((0_{\mathcal{E}_T^2}), r)$.

Let r be a non negative real number and let s be a real number. The functor $\text{RotateCircle}(r, s)$ yields a function from $\text{Tcircle}(0_{\mathcal{E}_T^2}, r)$ into $\text{Tcircle}(0_{\mathcal{E}_T^2}, r)$ and is defined by:

(Def. 5) $\text{RotateCircle}(r, s) = \text{Rotate } s \upharpoonright \text{Tcircle}(0_{\mathcal{E}_T^2}, r)$.

Let r be a non negative real number and let s be a real number. Note that $\text{RotateCircle}(r, s)$ is homeomorphism.

One can prove the following proposition

- (60) For every point p of \mathcal{E}_T^2 such that $p = \text{CircleMap}(r_2)$ holds $(\text{RotateCircle}(1, (-\text{Arg } p)))(\text{CircleMap}(r_1)) = \text{CircleMap}(r_1 - r_2)$.

2. ON THE ANTIPODALS

Let n be a non empty natural number, let p be a point of \mathcal{E}_T^n , and let r be a non negative real number. The functor $\text{CircleIso}(p, r)$ yields a function from $\text{TopUnitCircle } n$ into $\text{Tcircle}(p, r)$ and is defined as follows:

(Def. 6) For every point a of $\text{TopUnitCircle } n$ and for every point b of \mathcal{E}_T^n such that $a = b$ holds $(\text{CircleIso}(p, r))(a) = r \cdot b + p$.

Let n be a non empty natural number, let p be a point of \mathcal{E}_T^n , and let r be a positive real number. Note that $\text{CircleIso}(p, r)$ is homeomorphism.

The function SphereMap from \mathbb{R}^1 into $\text{TopUnitCircle } 3$ is defined by:

(Def. 7) For every real number x holds $(\text{SphereMap})(x) = [\cos(2 \cdot \pi \cdot x), \sin(2 \cdot \pi \cdot x), 0]$.

We now state the proposition

- (61) $(\text{SphereMap})(i) = c[100]$.

Let us note that SphereMap is continuous.

Let r be a real number. The functor $\text{eLoop } r$ yields a function from \mathbb{I} into $\text{TopUnitCircle } 3$ and is defined as follows:

(Def. 8) For every point x of \mathbb{I} holds $(\text{eLoop } r)(x) = [\cos(2 \cdot \pi \cdot r \cdot x), \sin(2 \cdot \pi \cdot r \cdot x), 0]$.

We now state the proposition

- (62) $\text{eLoop } r = \text{SphereMap} \cdot \text{ExtendInt } r$.

Let us consider i . Then $\text{eLoop } i$ is a loop of $\mathbb{c}[100]$.

One can check that $\text{eLoop } i$ is null-homotopic as a loop of $\mathbb{c}[100]$.

One can prove the following proposition

$$(63) \quad \text{If } p \neq 0_{\mathcal{E}_T^n}, \text{ then } |p/|p|| = 1.$$

Let n be a natural number and let p be a point of \mathcal{E}_T^n . Let us assume that $p \neq 0_{\mathcal{E}_T^n}$. The functor $(R^n \rightarrow S^1)p$ yields a point of $\text{Tcircle}(0_{\mathcal{E}_T^n}, 1)$ and is defined by:

$$(\text{Def. 9}) \quad (R^n \rightarrow S^1)p = p/|p|.$$

Let n be a non zero natural number and let f be a function

from $\text{Tcircle}(0_{\mathcal{E}_T^{n+1}}, 1)$ into \mathcal{E}_T^n . The functor $(S^{n+1} \rightarrow S^n)f$ yielding a function from $\text{TopUnitCircle}(n+1)$ into $\text{TopUnitCircle } n$ is defined as follows:

$$(\text{Def. 10}) \quad \text{For all points } x, y \text{ of } \text{Tcircle}(0_{\mathcal{E}_T^{n+1}}, 1) \text{ such that } y = -x \text{ holds} \\ ((S^{n+1} \rightarrow S^n)f)(x) = (R^n \rightarrow S^1)(f(x) - f(y)).$$

Let x_0, y_0 be points of $\text{TopUnitCircle } 2$, let x_1 be a set, and let f be a path from x_0 to y_0 . Let us assume that $x_1 \in \text{CircleMap}^{-1}(\{x_0\})$. The functor $\text{liftPath}(f, x_1)$ yielding a function from \mathbb{I} into \mathbb{R}^1 is defined by the conditions (Def. 11).

$$(\text{Def. 11})(i) \quad (\text{liftPath}(f, x_1))(0) = x_1, \\ (ii) \quad f = \text{CircleMap} \cdot \text{liftPath}(f, x_1), \\ (iii) \quad \text{liftPath}(f, x_1) \text{ is continuous, and} \\ (iv) \quad \text{for every function } f_1 \text{ from } \mathbb{I} \text{ into } \mathbb{R}^1 \text{ such that } f_1 \text{ is continuous and} \\ f = \text{CircleMap} \cdot f_1 \text{ and } f_1(0) = x_1 \text{ holds } \text{liftPath}(f, x_1) = f_1.$$

Let n be a natural number, let p, x, y be points of \mathcal{E}_T^n , and let r be a real number. We say that x and y are antipodals of p and r if and only if:

$$(\text{Def. 12}) \quad x \text{ is a point of } \text{Tcircle}(p, r) \text{ and } y \text{ is a point of } \text{Tcircle}(p, r) \text{ and } p \in \mathcal{L}(x, y).$$

Let n be a natural number, let p, x, y be points of \mathcal{E}_T^n , let r be a real number, and let f be a function. We say that x and y are antipodals of p, r and f if and only if:

$$(\text{Def. 13}) \quad x \text{ and } y \text{ are antipodals of } p \text{ and } r \text{ and } x, y \in \text{dom } f \text{ and } f(x) = f(y).$$

Let m, n be natural numbers, let p be a point of \mathcal{E}_T^m , let r be a real number, and let f be a function from $\text{Tcircle}(p, r)$ into \mathcal{E}_T^n . We say that f has antipodals if and only if:

$$(\text{Def. 14}) \quad \text{There exist points } x, y \text{ of } \mathcal{E}_T^m \text{ such that } x \text{ and } y \text{ are antipodals of } p, r \text{ and } f.$$

Let m, n be natural numbers, let p be a point of \mathcal{E}_T^m , let r be a real number, and let f be a function from $\text{Tcircle}(p, r)$ into \mathcal{E}_T^n . We introduce f is without antipodals as an antonym of f has antipodals.

One can prove the following propositions:

- (64) Let n be a non empty natural number, r be a non negative real number, and x be a point of \mathcal{E}_T^n . Suppose x is a point of $\text{Tcircle}(0_{\mathcal{E}_T^n}, r)$. Then x and $-x$ are antipodals of $0_{\mathcal{E}_T^n}$ and r .
- (65) Let n be a non empty natural number, p, x, y, x_2, y_1 be points of \mathcal{E}_T^n , and r be a positive real number. Suppose x and y are antipodals of $0_{\mathcal{E}_T^n}$ and 1 and $x_2 = (\text{CircleIso}(p, r))(x)$ and $y_1 = (\text{CircleIso}(p, r))(y)$. Then x_2 and y_1 are antipodals of p and r .
- (66) Let f be a function from $\text{Tcircle}(0_{\mathcal{E}_T^{n+1}}, 1)$ into \mathcal{E}_T^n and x be a point of $\text{Tcircle}(0_{\mathcal{E}_T^{n+1}}, 1)$. If f is without antipodals, then $f(x) - f(-x) \neq 0_{\mathcal{E}_T^n}$.
- (67) For every function f from $\text{Tcircle}(0_{\mathcal{E}_T^{n+1}}, 1)$ into \mathcal{E}_T^n such that f is without antipodals holds $(S^{n+1} \rightarrow S^n) f$ is odd.
- (68) Let f be a function from $\text{Tcircle}(0_{\mathcal{E}_T^{n+1}}, 1)$ into \mathcal{E}_T^n and g, B_1 be functions from $\text{Tcircle}(0_{\mathcal{E}_T^{n+1}}, 1)$ into \mathcal{E}_T^n . If $g = f \circ -$ and $B_1 = f - g$ and f is without antipodals, then $(S^{n+1} \rightarrow S^n) f = B_1 / (n \text{NormF} \cdot B_1)$.

Let us consider n , let r be a negative real number, and let p be a point of \mathcal{E}_T^{n+1} . Observe that every function from $\text{Tcircle}(p, r)$ into \mathcal{E}_T^n is without antipodals.

Let r be a non negative real number and let p be a point of \mathcal{E}_T^3 . Note that every function from $\text{Tcircle}(p, r)$ into \mathcal{E}_T^2 which is continuous also has antipodals.²

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