

Higher-Order Partial Differentiation¹

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Summary. In this article, we shall extend the formalization of [10] to discuss higher-order partial differentiation of real valued functions. The linearity of this operator is also proved (refer to [10], [12] and [13] for partial differentiation).

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The terminology and notation used here have been introduced in the following articles: [3], [8], [2], [4], [5], [15], [21], [17], [16], [20], [1], [6], [10], [12], [13], [18], [11], [9], [23], [7], [19], [14], and [22].

1. PRELIMINARIES

We use the following convention: m, n denote non empty elements of \mathbb{N} , i, j denote elements of \mathbb{N} , and Z denotes a set.

One can prove the following propositions:

- (1) Let S, T be real normed spaces, f be a point of the real norm space of bounded linear operators from S into T , and r be a real number. Suppose $0 \leq r$ and for every point x of S such that $\|x\| \leq 1$ holds $\|f(x)\| \leq r \cdot \|x\|$. Then $\|f\| \leq r$.
- (2) Let S be a real normed space and f be a partial function from S to \mathbb{R} . Then f is continuous on Z if and only if the following conditions are satisfied:

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- (i) $Z \subseteq \text{dom } f$, and
- (ii) for every sequence s_1 of S such that $\text{rng } s_1 \subseteq Z$ and s_1 is convergent and $\lim s_1 \in Z$ holds f_*s_1 is convergent and $f_{\lim s_1} = \lim(f_*s_1)$.
- (3) For every partial function f from \mathcal{R}^i to \mathbb{R} holds $\text{dom}\langle f \rangle = \text{dom } f$.
- (4) For every partial function f from \mathcal{R}^i to \mathbb{R} such that $Z \subseteq \text{dom } f$ holds $\text{dom}(\langle f \rangle \upharpoonright Z) = Z$.
- (5) For every partial function f from \mathcal{R}^i to \mathbb{R} holds $\langle f \upharpoonright Z \rangle = \langle f \rangle \upharpoonright Z$.
- (6) Let f be a partial function from \mathcal{R}^i to \mathbb{R} and x be an element of \mathcal{R}^i . If $x \in \text{dom } f$, then $\langle f \rangle(x) = \langle f(x) \rangle$ and $\langle f \rangle_x = \langle f_x \rangle$.
- (7) For all partial functions f, g from \mathcal{R}^i to \mathbb{R} holds $\langle f + g \rangle = \langle f \rangle + \langle g \rangle$ and $\langle f - g \rangle = \langle f \rangle - \langle g \rangle$.
- (8) For every partial function f from \mathcal{R}^i to \mathbb{R} and for every real number r holds $\langle r \cdot f \rangle = r \cdot \langle f \rangle$.
- (9) Let f be a partial function from \mathcal{R}^i to \mathbb{R} and g be a partial function from \mathcal{R}^i to \mathcal{R}^1 . If $\langle f \rangle = g$, then $|f| = |g|$.
- (10) For every subset X of \mathcal{R}^m and for every subset Y of $\langle \mathcal{E}^m, \|\cdot\| \rangle$ such that $X = Y$ holds X is open iff Y is open.
- (11) For every element q of \mathbb{R} such that $1 \leq i \leq j$ holds $|(\text{reproj}(i, \underbrace{\langle 0, \dots, 0 \rangle}_j))(q)| = |q|$.
- (12) For every element x of \mathcal{R}^j holds $x = (\text{reproj}(i, x))((\text{proj}(i, j))(x))$.

2. CONTINUITY AND DIFFERENTIABILITY

The following two propositions are true:

- (13) Let X be a subset of \mathcal{R}^m and f be a partial function from \mathcal{R}^m to \mathcal{R}^n . If f is differentiable on X , then X is open.
- (14) Let X be a subset of \mathcal{R}^m and f be a partial function from \mathcal{R}^m to \mathcal{R}^n . Suppose X is open. Then f is differentiable on X if and only if the following conditions are satisfied:
 - (i) $X \subseteq \text{dom } f$, and
 - (ii) for every element x of \mathcal{R}^m such that $x \in X$ holds f is differentiable in x .

Let m, n be non empty elements of \mathbb{N} , let Z be a set, and let f be a partial function from \mathcal{R}^m to \mathcal{R}^n . Let us assume that $Z \subseteq \text{dom } f$. The functor $f'_{\upharpoonright Z}$ yields a partial function from \mathcal{R}^m to $(\mathcal{R}^n)^{\mathcal{R}^m}$ and is defined by:

- (Def. 1) $\text{dom}(f'_{\upharpoonright Z}) = Z$ and for every element x of \mathcal{R}^m such that $x \in Z$ holds $(f'_{\upharpoonright Z})_x = f'(x)$.

We now state a number of propositions:

- (15) Let X be a subset of \mathcal{R}^m and f, g be partial functions from \mathcal{R}^m to \mathcal{R}^n . Suppose f is differentiable on X and g is differentiable on X . Then $f + g$ is differentiable on X and for every element x of \mathcal{R}^m such that $x \in X$ holds $((f + g)'|_X)_x = f'(x) + g'(x)$.
- (16) Let X be a subset of \mathcal{R}^m and f, g be partial functions from \mathcal{R}^m to \mathcal{R}^n . Suppose f is differentiable on X and g is differentiable on X . Then $f - g$ is differentiable on X and for every element x of \mathcal{R}^m such that $x \in X$ holds $((f - g)'|_X)_x = f'(x) - g'(x)$.
- (17) Let X be a subset of \mathcal{R}^m , f be a partial function from \mathcal{R}^m to \mathcal{R}^n , and r be a real number. Suppose f is differentiable on X . Then $r \cdot f$ is differentiable on X and for every element x of \mathcal{R}^m such that $x \in X$ holds $((r \cdot f)'|_X)_x = r \cdot f'(x)$.
- (18) Let f be a point of the real norm space of bounded linear operators from $\langle \mathcal{E}^1, \|\cdot\| \rangle$ into $\langle \mathcal{E}^j, \|\cdot\| \rangle$. Then there exists a point p of $\langle \mathcal{E}^j, \|\cdot\| \rangle$ such that
- $p = f(\langle 1 \rangle)$,
 - for every real number r and for every point x of $\langle \mathcal{E}^1, \|\cdot\| \rangle$ such that $x = \langle r \rangle$ holds $f(x) = r \cdot p$, and
 - for every point x of $\langle \mathcal{E}^1, \|\cdot\| \rangle$ holds $\|f(x)\| = \|p\| \cdot \|x\|$.
- (19) Let f be a point of the real norm space of bounded linear operators from $\langle \mathcal{E}^1, \|\cdot\| \rangle$ into $\langle \mathcal{E}^j, \|\cdot\| \rangle$. Then there exists a point p of $\langle \mathcal{E}^j, \|\cdot\| \rangle$ such that $p = f(\langle 1 \rangle)$ and $\|p\| = \|f\|$.
- (20) Let f be a point of the real norm space of bounded linear operators from $\langle \mathcal{E}^1, \|\cdot\| \rangle$ into $\langle \mathcal{E}^j, \|\cdot\| \rangle$ and x be a point of $\langle \mathcal{E}^1, \|\cdot\| \rangle$. Then $\|f(x)\| = \|f\| \cdot \|x\|$.
- (21) Let f be a partial function from \mathcal{R}^m to \mathcal{R}^n , g be a partial function from $\langle \mathcal{E}^m, \|\cdot\| \rangle$ to $\langle \mathcal{E}^n, \|\cdot\| \rangle$, X be a subset of \mathcal{R}^m , and Y be a subset of $\langle \mathcal{E}^m, \|\cdot\| \rangle$. Suppose $1 \leq i \leq m$ and X is open and $g = f$ and $X = Y$ and f is partially differentiable on X w.r.t. i . Let x be an element of \mathcal{R}^m and y be a point of $\langle \mathcal{E}^m, \|\cdot\| \rangle$. If $x \in X$ and $x = y$, then $\text{partdiff}(f, x, i) = (\text{partdiff}(g, y, i))(\langle 1 \rangle)$.
- (22) Let f be a partial function from \mathcal{R}^m to \mathcal{R}^n , g be a partial function from $\langle \mathcal{E}^m, \|\cdot\| \rangle$ to $\langle \mathcal{E}^n, \|\cdot\| \rangle$, X be a subset of \mathcal{R}^m , and Y be a subset of $\langle \mathcal{E}^m, \|\cdot\| \rangle$. Suppose $1 \leq i \leq m$ and X is open and $g = f$ and $X = Y$ and f is partially differentiable on X w.r.t. i . Let x_0, x_1 be elements of \mathcal{R}^m and y_0, y_1 be points of $\langle \mathcal{E}^m, \|\cdot\| \rangle$. If $x_0 = y_0$ and $x_1 = y_1$ and $x_0, x_1 \in X$, then $\|(f|^{iX})_{x_1} - (f|^{iX})_{x_0}\| = \|(g|^{iY})_{y_1} - (g|^{iY})_{y_0}\|$.
- (23) Let f be a partial function from \mathcal{R}^m to \mathcal{R}^n , g be a partial function from $\langle \mathcal{E}^m, \|\cdot\| \rangle$ to $\langle \mathcal{E}^n, \|\cdot\| \rangle$, X be a subset of \mathcal{R}^m , and Y be a subset of $\langle \mathcal{E}^m, \|\cdot\| \rangle$. Suppose $1 \leq i \leq m$ and X is open and $g = f$ and $X = Y$. Then the following statements are equivalent
- f is partially differentiable on X w.r.t. i and $f|^{iX}$ is continuous on X ,

- (ii) g is partially differentiable on Y w.r.t. i and $g|_Y^i$ is continuous on Y .
- (24) Let f be a partial function from \mathcal{R}^m to \mathcal{R}^n , g be a partial function from $\langle \mathcal{E}^m, \|\cdot\| \rangle$ to $\langle \mathcal{E}^n, \|\cdot\| \rangle$, X be a subset of \mathcal{R}^m , and Y be a subset of $\langle \mathcal{E}^m, \|\cdot\| \rangle$. Suppose $X = Y$ and X is open and $f = g$. Then for every i such that $1 \leq i \leq m$ holds f is partially differentiable on X w.r.t. i and $f|_X^i$ is continuous on X if and only if g is differentiable on Y and $g|_Y'$ is continuous on Y .
- (25) Let f be a partial function from \mathcal{R}^m to \mathcal{R}^n , g be a partial function from $\langle \mathcal{E}^m, \|\cdot\| \rangle$ to $\langle \mathcal{E}^n, \|\cdot\| \rangle$, X be a subset of \mathcal{R}^m , and Y be a subset of $\langle \mathcal{E}^m, \|\cdot\| \rangle$. Suppose X is open and $X \subseteq \text{dom } f$ and $g = f$ and $X = Y$. Then g is differentiable on Y and $g|_Y'$ is continuous on Y if and only if the following conditions are satisfied:
- (i) f is differentiable on X , and
- (ii) for every element x_0 of \mathcal{R}^m and for every real number r such that $x_0 \in X$ and $0 < r$ there exists a real number s such that $0 < s$ and for every element x_1 of \mathcal{R}^m such that $x_1 \in X$ and $|x_1 - x_0| < s$ and for every element v of \mathcal{R}^m holds $|f'(x_1)(v) - f'(x_0)(v)| \leq r \cdot |v|$.
- (26) Let X be a subset of \mathcal{R}^m and f be a partial function from \mathcal{R}^m to \mathcal{R}^n . Suppose X is open and $X \subseteq \text{dom } f$. Then the following statements are equivalent
- (i) for every element i of \mathbb{N} such that $1 \leq i \leq m$ holds f is partially differentiable on X w.r.t. i and $f|_X^i$ is continuous on X ,
- (ii) f is differentiable on X and for every element x_0 of \mathcal{R}^m and for every real number r such that $x_0 \in X$ and $0 < r$ there exists a real number s such that $0 < s$ and for every element x_1 of \mathcal{R}^m such that $x_1 \in X$ and $|x_1 - x_0| < s$ and for every element v of \mathcal{R}^m holds $|f'(x_1)(v) - f'(x_0)(v)| \leq r \cdot |v|$.
- (27) Let f be a partial function from \mathcal{R}^m to \mathcal{R}^n and g be a partial function from $\langle \mathcal{E}^m, \|\cdot\| \rangle$ to $\langle \mathcal{E}^n, \|\cdot\| \rangle$. If $f = g$ and f is differentiable on Z , then $f|_Z' = g|_Z'$.
- (28) Let f be a partial function from \mathcal{R}^m to \mathcal{R}^n , g be a partial function from $\langle \mathcal{E}^m, \|\cdot\| \rangle$ to $\langle \mathcal{E}^n, \|\cdot\| \rangle$, X be a subset of \mathcal{R}^m , and Y be a subset of $\langle \mathcal{E}^m, \|\cdot\| \rangle$. Suppose $X = Y$ and X is open and $f = g$. Then for every element i of \mathbb{N} such that $1 \leq i \leq m$ holds f is partially differentiable on X w.r.t. i and $f|_X^i$ is continuous on X if and only if f is differentiable on X and $g|_Y'$ is continuous on Y .
- (29) Let f, g be partial functions from \mathcal{R}^m to \mathcal{R}^n and x be an element of \mathcal{R}^m . Suppose f is continuous in x and g is continuous in x . Then $f + g$ is continuous in x and $f - g$ is continuous in x .
- (30) Let f be a partial function from \mathcal{R}^m to \mathcal{R}^n , x be an element of \mathcal{R}^m , and r be a real number. If f is continuous in x , then $r \cdot f$ is continuous in x .

- (31) Let f be a partial function from \mathcal{R}^m to \mathcal{R}^n and x be an element of \mathcal{R}^m . If f is continuous in x , then $-f$ is continuous in x .
- (32) Let f be a partial function from \mathcal{R}^m to \mathcal{R}^n and x be an element of \mathcal{R}^m . If f is continuous in x , then $|f|$ is continuous in x .
- (33) Let Z be a set and f, g be partial functions from \mathcal{R}^m to \mathcal{R}^n . Suppose f is continuous on Z and g is continuous on Z . Then $f + g$ is continuous on Z and $f - g$ is continuous on Z .
- (34) Let r be a real number and f, g be partial functions from \mathcal{R}^m to \mathcal{R}^n . If f is continuous on Z , then $r \cdot f$ is continuous on Z .
- (35) For all partial functions f, g from \mathcal{R}^m to \mathcal{R}^n such that f is continuous on Z holds $-f$ is continuous on Z .
- (36) Let f be a partial function from \mathcal{R}^i to \mathbb{R} and x_0 be an element of \mathcal{R}^i . Then f is continuous in x_0 if and only if the following conditions are satisfied:
- (i) $x_0 \in \text{dom } f$, and
 - (ii) for every real number r such that $0 < r$ there exists a real number s such that $0 < s$ and for every element x of \mathcal{R}^i such that $x \in \text{dom } f$ and $|x - x_0| < s$ holds $|f_x - f_{x_0}| < r$.
- (37) Let f be a partial function from \mathcal{R}^m to \mathbb{R} and x_0 be an element of \mathcal{R}^m . Then f is continuous in x_0 if and only if $\langle f \rangle$ is continuous in x_0 .
- (38) Let f, g be partial functions from \mathcal{R}^m to \mathbb{R} and x_0 be an element of \mathcal{R}^m . Suppose f is continuous in x_0 and g is continuous in x_0 . Then $f + g$ is continuous in x_0 and $f - g$ is continuous in x_0 .
- (39) Let f be a partial function from \mathcal{R}^m to \mathbb{R} , x_0 be an element of \mathcal{R}^m , and r be a real number. If f is continuous in x_0 , then $r \cdot f$ is continuous in x_0 .
- (40) Let f be a partial function from \mathcal{R}^m to \mathbb{R} and x_0 be an element of \mathcal{R}^m . If f is continuous in x_0 , then $|f|$ is continuous in x_0 .
- (41) Let f, g be partial functions from \mathcal{R}^i to \mathbb{R} and x be an element of \mathcal{R}^i . If f is continuous in x and g is continuous in x , then $f \cdot g$ is continuous in x .

Let m be a non empty element of \mathbb{N} , let Z be a set, and let f be a partial function from \mathcal{R}^m to \mathbb{R} . We say that f is continuous on Z if and only if:

- (Def. 2) For every element x_0 of \mathcal{R}^m such that $x_0 \in Z$ holds $f|_Z$ is continuous in x_0 .

We now state a number of propositions:

- (42) Let f be a partial function from \mathcal{R}^m to \mathbb{R} and g be a partial function from $\langle \mathcal{E}^m, \|\cdot\| \rangle$ to \mathbb{R} . Suppose $f = g$. Then $Z \subseteq \text{dom } f$ and f is continuous on Z if and only if g is continuous on Z .
- (43) Let f be a partial function from \mathcal{R}^m to \mathbb{R} and g be a partial function from $\langle \mathcal{E}^m, \|\cdot\| \rangle$ to \mathbb{R} . Suppose $f = g$ and $Z \subseteq \text{dom } f$. Then f is continuous on Z

if and only if for every sequence s of $\langle \mathcal{E}^m, \|\cdot\| \rangle$ such that $\text{rng } s \subseteq Z$ and s is convergent and $\lim s \in Z$ holds g_*s is convergent and $g_{\lim s} = \lim(g_*s)$.

- (44) Let f be a partial function from \mathcal{R}^m to \mathbb{R} and g be a partial function from \mathcal{R}^m to \mathcal{R}^1 . Suppose $\langle f \rangle = g$. Then $Z \subseteq \text{dom } f$ and f is continuous on Z if and only if g is continuous on Z .
- (45) Let f be a partial function from \mathcal{R}^m to \mathbb{R} . Suppose $Z \subseteq \text{dom } f$. Then f is continuous on Z if and only if for every element x_0 of \mathcal{R}^m and for every real number r such that $x_0 \in Z$ and $0 < r$ there exists a real number s such that $0 < s$ and for every element x_1 of \mathcal{R}^m such that $x_1 \in Z$ and $|x_1 - x_0| < s$ holds $|f_{x_1} - f_{x_0}| < r$.
- (46) Let f, g be partial functions from \mathcal{R}^m to \mathbb{R} . Suppose f is continuous on Z and g is continuous on Z and $Z \subseteq \text{dom } f$ and $Z \subseteq \text{dom } g$. Then $f + g$ is continuous on Z and $f - g$ is continuous on Z .
- (47) Let f be a partial function from \mathcal{R}^m to \mathbb{R} and r be a real number. If $Z \subseteq \text{dom } f$ and f is continuous on Z , then $r \cdot f$ is continuous on Z .
- (48) Let f, g be partial functions from \mathcal{R}^m to \mathbb{R} . Suppose f is continuous on Z and g is continuous on Z and $Z \subseteq \text{dom } f$ and $Z \subseteq \text{dom } g$. Then $f \cdot g$ is continuous on Z .
- (49) Let f be a partial function from \mathcal{R}^m to \mathbb{R} and g be a partial function from $\langle \mathcal{E}^m, \|\cdot\| \rangle$ to \mathbb{R} . Suppose $f = g$. Then $Z \subseteq \text{dom } f$ and f is continuous on Z if and only if g is continuous on Z .
- (50) For all partial functions f, g from \mathcal{R}^m to \mathcal{R}^n such that f is continuous on Z holds $|f|$ is continuous on Z .
- (51) Let f, g be partial functions from \mathcal{R}^m to \mathbb{R} and x be an element of \mathcal{R}^m . Suppose f is differentiable in x and g is differentiable in x . Then $f + g$ is differentiable in x and $(f + g)'(x) = f'(x) + g'(x)$ and $f - g$ is differentiable in x and $(f - g)'(x) = f'(x) - g'(x)$.
- (52) Let f be a partial function from \mathcal{R}^m to \mathbb{R} , r be a real number, and x be an element of \mathcal{R}^m . Suppose f is differentiable in x . Then $r \cdot f$ is differentiable in x and $(r \cdot f)'(x) = r \cdot f'(x)$.

Let Z be a set, let m be a non empty element of \mathbb{N} , and let f be a partial function from \mathcal{R}^m to \mathbb{R} . We say that f is differentiable on Z if and only if:

- (Def. 3) For every element x of \mathcal{R}^m such that $x \in Z$ holds $f \upharpoonright Z$ is differentiable in x .

Next we state three propositions:

- (53) Let f be a partial function from \mathcal{R}^m to \mathbb{R} and g be a partial function from \mathcal{R}^m to \mathcal{R}^1 . Suppose $\langle f \rangle = g$. Then $Z \subseteq \text{dom } f$ and f is differentiable on Z if and only if g is differentiable on Z .
- (54) Let X be a subset of \mathcal{R}^m and f be a partial function from \mathcal{R}^m to \mathbb{R} . Suppose $X \subseteq \text{dom } f$ and X is open. Then f is differentiable on X if and

only if for every element x of \mathcal{R}^m such that $x \in X$ holds f is differentiable in x .

- (55) Let X be a subset of \mathcal{R}^m and f be a partial function from \mathcal{R}^m to \mathbb{R} . If $X \subseteq \text{dom } f$ and f is differentiable on X , then X is open.

Let m be a non empty element of \mathbb{N} , let Z be a set, and let f be a partial function from \mathcal{R}^m to \mathbb{R} . Let us assume that $Z \subseteq \text{dom } f$. The functor $f'_{\upharpoonright Z}$ yields a partial function from \mathcal{R}^m to $\mathbb{R}^{\mathcal{R}^m}$ and is defined by:

- (Def. 4) $\text{dom}(f'_{\upharpoonright Z}) = Z$ and for every element x of \mathcal{R}^m such that $x \in Z$ holds $(f'_{\upharpoonright Z})_x = f'(x)$.

One can prove the following four propositions:

- (56) Let X be a subset of \mathcal{R}^m , f be a partial function from \mathcal{R}^m to \mathbb{R} , and g be a partial function from \mathcal{R}^m to \mathcal{R}^1 . Suppose $\langle f \rangle = g$ and $X \subseteq \text{dom } f$ and f is differentiable on X . Then g is differentiable on X and for every element x of \mathcal{R}^m such that $x \in X$ holds $(f'_{\upharpoonright X})_x = \text{proj}(1, 1) \cdot (g'_{\upharpoonright X})_x$.
- (57) Let X be a subset of \mathcal{R}^m and f, g be partial functions from \mathcal{R}^m to \mathbb{R} . Suppose $X \subseteq \text{dom } f$ and $X \subseteq \text{dom } g$ and f is differentiable on X and g is differentiable on X . Then $f + g$ is differentiable on X and for every element x of \mathcal{R}^m such that $x \in X$ holds $((f + g)'_{\upharpoonright X})_x = (f'_{\upharpoonright X})_x + (g'_{\upharpoonright X})_x$.
- (58) Let X be a subset of \mathcal{R}^m and f, g be partial functions from \mathcal{R}^m to \mathbb{R} . Suppose $X \subseteq \text{dom } f$ and $X \subseteq \text{dom } g$ and f is differentiable on X and g is differentiable on X . Then $f - g$ is differentiable on X and for every element x of \mathcal{R}^m such that $x \in X$ holds $((f - g)'_{\upharpoonright X})_x = (f'_{\upharpoonright X})_x - (g'_{\upharpoonright X})_x$.
- (59) Let X be a subset of \mathcal{R}^m , f be a partial function from \mathcal{R}^m to \mathbb{R} , and r be a real number. Suppose $X \subseteq \text{dom } f$ and f is differentiable on X . Then $r \cdot f$ is differentiable on X and for every element x of \mathcal{R}^m such that $x \in X$ holds $((r \cdot f)'_{\upharpoonright X})_x = r \cdot (f'_{\upharpoonright X})_x$.

Let m be a non empty element of \mathbb{N} , let Z be a set, let i be an element of \mathbb{N} , and let f be a partial function from \mathcal{R}^m to \mathbb{R} . We say that f is partially differentiable on Z w.r.t. i if and only if:

- (Def. 5) $Z \subseteq \text{dom } f$ and for every element x of \mathcal{R}^m such that $x \in Z$ holds $f|Z$ is partially differentiable in x w.r.t. i .

Let m be a non empty element of \mathbb{N} , let Z be a set, let i be an element of \mathbb{N} , and let f be a partial function from \mathcal{R}^m to \mathbb{R} . Let us assume that f is partially differentiable on Z w.r.t. i . The functor $f^i|Z$ yields a partial function from \mathcal{R}^m to \mathbb{R} and is defined as follows:

- (Def. 6) $\text{dom}(f^i|Z) = Z$ and for every element x of \mathcal{R}^m such that $x \in Z$ holds $(f^i|Z)_x = \text{partdiff}(f, x, i)$.

Next we state several propositions:

- (60) Let X be a subset of \mathcal{R}^m and f be a partial function from \mathcal{R}^m to \mathbb{R} . Suppose X is open and $1 \leq i \leq m$. Then f is partially differentiable on X

w.r.t. i if and only if $X \subseteq \text{dom } f$ and for every element x of \mathcal{R}^m such that $x \in X$ holds f is partially differentiable in x w.r.t. i .

- (61) Let X be a subset of \mathcal{R}^m , f be a partial function from \mathcal{R}^m to \mathbb{R} , and g be a partial function from \mathcal{R}^m to \mathcal{R}^1 . Suppose $\langle f \rangle = g$ and X is open and $1 \leq i \leq m$. Then f is partially differentiable on X w.r.t. i if and only if g is partially differentiable on X w.r.t. i .
- (62) Let X be a subset of \mathcal{R}^m , f be a partial function from \mathcal{R}^m to \mathbb{R} , and g be a partial function from \mathcal{R}^m to \mathcal{R}^1 . Suppose $\langle f \rangle = g$ and X is open and $1 \leq i \leq m$ and f is partially differentiable on X w.r.t. i . Then $f|{}^i X$ is continuous on X if and only if $g|{}^i X$ is continuous on X .
- (63) Let X be a subset of \mathcal{R}^m and f be a partial function from \mathcal{R}^m to \mathbb{R} . Suppose X is open and $X \subseteq \text{dom } f$. Then the following statements are equivalent
- (i) for every element i of \mathbb{N} such that $1 \leq i \leq m$ holds f is partially differentiable on X w.r.t. i and $f|{}^i X$ is continuous on X ,
 - (ii) f is differentiable on X and for every element x_0 of \mathcal{R}^m and for every real number r such that $x_0 \in X$ and $0 < r$ there exists a real number s such that $0 < s$ and for every element x_1 of \mathcal{R}^m such that $x_1 \in X$ and $|x_1 - x_0| < s$ and for every element v of \mathcal{R}^m holds $|f'(x_1)(v) - f'(x_0)(v)| \leq r \cdot |v|$.
- (64) Let f, g be partial functions from \mathcal{R}^m to \mathbb{R} and x be an element of \mathcal{R}^m . Suppose f is partially differentiable in x w.r.t. i and g is partially differentiable in x w.r.t. i . Then $f \cdot g$ is partially differentiable in x w.r.t. i and $\text{partdiff}(f \cdot g, x, i) = \text{partdiff}(f, x, i) \cdot g(x) + f(x) \cdot \text{partdiff}(g, x, i)$.
- (65) Let X be a subset of \mathcal{R}^m and f, g be partial functions from \mathcal{R}^m to \mathbb{R} . Suppose that
- (i) X is open,
 - (ii) $1 \leq i$,
 - (iii) $i \leq m$,
 - (iv) f is partially differentiable on X w.r.t. i , and
 - (v) g is partially differentiable on X w.r.t. i .
- Then
- (vi) $f + g$ is partially differentiable on X w.r.t. i ,
 - (vii) $(f + g)|{}^i X = (f|{}^i X) + (g|{}^i X)$, and
 - (viii) for every element x of \mathcal{R}^m such that $x \in X$ holds $((f + g)|{}^i X)_x = \text{partdiff}(f, x, i) + \text{partdiff}(g, x, i)$.
- (66) Let X be a subset of \mathcal{R}^m and f, g be partial functions from \mathcal{R}^m to \mathbb{R} . Suppose that
- (i) X is open,
 - (ii) $1 \leq i$,
 - (iii) $i \leq m$,

- (iv) f is partially differentiable on X w.r.t. i , and
- (v) g is partially differentiable on X w.r.t. i .

Then

- (vi) $f - g$ is partially differentiable on X w.r.t. i ,
 - (vii) $(f - g)|^i X = (f|^i X) - (g|^i X)$, and
 - (viii) for every element x of \mathcal{R}^m such that $x \in X$ holds $((f - g)|^i X)_x = \text{partdiff}(f, x, i) - \text{partdiff}(g, x, i)$.
- (67) Let X be a subset of \mathcal{R}^m , r be a real number, and f be a partial function from \mathcal{R}^m to \mathbb{R} . Suppose X is open and $1 \leq i \leq m$ and f is partially differentiable on X w.r.t. i . Then
- (i) $r \cdot f$ is partially differentiable on X w.r.t. i ,
 - (ii) $r \cdot f|^i X = r \cdot (f|^i X)$, and
 - (iii) for every element x of \mathcal{R}^m such that $x \in X$ holds $(r \cdot f|^i X)_x = r \cdot \text{partdiff}(f, x, i)$.

- (68) Let X be a subset of \mathcal{R}^m and f, g be partial functions from \mathcal{R}^m to \mathbb{R} . Suppose that

- (i) X is open,
- (ii) $1 \leq i$,
- (iii) $i \leq m$,
- (iv) f is partially differentiable on X w.r.t. i , and
- (v) g is partially differentiable on X w.r.t. i .

Then

- (vi) $f \cdot g$ is partially differentiable on X w.r.t. i ,
- (vii) $f \cdot g|^i X = (f|^i X) \cdot g + f \cdot (g|^i X)$, and
- (viii) for every element x of \mathcal{R}^m such that $x \in X$ holds $(f \cdot g|^i X)_x = \text{partdiff}(f, x, i) \cdot g(x) + f(x) \cdot \text{partdiff}(g, x, i)$.

3. HIGHER-ORDER PARTIAL DIFFERENTIATION

Let m be a non empty element of \mathbb{N} , let Z be a set, let I be a finite sequence of elements of \mathbb{N} , and let f be a partial function from \mathcal{R}^m to \mathbb{R} . The functor $\text{PartDiffSeq}(f, Z, I)$ yielding a sequence of partial functions from \mathcal{R}^m into \mathbb{R} is defined by:

- (Def. 7) $(\text{PartDiffSeq}(f, Z, I))(0) = f$ and for every natural number i holds $(\text{PartDiffSeq}(f, Z, I))(i + 1) = (\text{PartDiffSeq}(f, Z, I))(i)|^{I_{i+1}} Z$.

Let m be a non empty element of \mathbb{N} , let Z be a set, let I be a finite sequence of elements of \mathbb{N} , and let f be a partial function from \mathcal{R}^m to \mathbb{R} . We say that f is partially differentiable on Z w.r.t. I if and only if:

- (Def. 8) For every element i of \mathbb{N} such that $i \leq \text{len } I - 1$ holds $(\text{PartDiffSeq}(f, Z, I))(i)$ is partially differentiable on Z w.r.t. I_{i+1} .

Let m be a non empty element of \mathbb{N} , let Z be a set, let I be a finite sequence of elements of \mathbb{N} , and let f be a partial function from \mathcal{R}^m to \mathbb{R} . The functor $f \upharpoonright^I Z$ yielding a partial function from \mathcal{R}^m to \mathbb{R} is defined by:

(Def. 9) $f \upharpoonright^I Z = (\text{PartDiffSeq}(f, Z, I))(\text{len } I)$.

The following propositions are true:

(69) Let X be a subset of \mathcal{R}^m , I be a non empty finite sequence of elements of \mathbb{N} , and f, g be partial functions from \mathcal{R}^m to \mathbb{R} . Suppose that

- (i) X is open,
- (ii) $\text{rng } I \subseteq \text{Seg } m$,
- (iii) f is partially differentiable on X w.r.t. I , and
- (iv) g is partially differentiable on X w.r.t. I .

Let given i . Suppose $i \leq \text{len } I - 1$. Then $(\text{PartDiffSeq}(f + g, X, I))(i)$ is partially differentiable on X w.r.t. I_{i+1} and $(\text{PartDiffSeq}(f + g, X, I))(i) = (\text{PartDiffSeq}(f, X, I))(i) + (\text{PartDiffSeq}(g, X, I))(i)$.

(70) Let X be a subset of \mathcal{R}^m , I be a non empty finite sequence of elements of \mathbb{N} , and f, g be partial functions from \mathcal{R}^m to \mathbb{R} . Suppose that

- (i) X is open,
- (ii) $\text{rng } I \subseteq \text{Seg } m$,
- (iii) f is partially differentiable on X w.r.t. I , and
- (iv) g is partially differentiable on X w.r.t. I .

Then $f + g$ is partially differentiable on X w.r.t. I and $(f + g) \upharpoonright^I X = (f \upharpoonright^I X) + (g \upharpoonright^I X)$.

(71) Let X be a subset of \mathcal{R}^m , I be a non empty finite sequence of elements of \mathbb{N} , and f, g be partial functions from \mathcal{R}^m to \mathbb{R} . Suppose that

- (i) X is open,
- (ii) $\text{rng } I \subseteq \text{Seg } m$,
- (iii) f is partially differentiable on X w.r.t. I , and
- (iv) g is partially differentiable on X w.r.t. I .

Let given i . Suppose $i \leq \text{len } I - 1$. Then $(\text{PartDiffSeq}(f - g, X, I))(i)$ is partially differentiable on X w.r.t. I_{i+1} and $(\text{PartDiffSeq}(f - g, X, I))(i) = (\text{PartDiffSeq}(f, X, I))(i) - (\text{PartDiffSeq}(g, X, I))(i)$.

(72) Let X be a subset of \mathcal{R}^m , I be a non empty finite sequence of elements of \mathbb{N} , and f, g be partial functions from \mathcal{R}^m to \mathbb{R} . Suppose that

- (i) X is open,
- (ii) $\text{rng } I \subseteq \text{Seg } m$,
- (iii) f is partially differentiable on X w.r.t. I , and
- (iv) g is partially differentiable on X w.r.t. I .

Then $f - g$ is partially differentiable on X w.r.t. I and $(f - g) \upharpoonright^I X = (f \upharpoonright^I X) - (g \upharpoonright^I X)$.

(73) Let X be a subset of \mathcal{R}^m , r be a real number, I be a non empty finite sequence of elements of \mathbb{N} , and f be a partial function from \mathcal{R}^m to \mathbb{R} .

Suppose X is open and $\text{rng } I \subseteq \text{Seg } m$ and f is partially differentiable on X w.r.t. I . Let given i . Suppose $i \leq \text{len } I - 1$. Then $(\text{PartDiffSeq}(r \cdot f, X, I))(i)$ is partially differentiable on X w.r.t. I_{i+1} and $(\text{PartDiffSeq}(r \cdot f, X, I))(i) = r \cdot (\text{PartDiffSeq}(f, X, I))(i)$.

- (74) Let X be a subset of \mathcal{R}^m , r be a real number, I be a non empty finite sequence of elements of \mathbb{N} , and f be a partial function from \mathcal{R}^m to \mathbb{R} . Suppose X is open and $\text{rng } I \subseteq \text{Seg } m$ and f is partially differentiable on X w.r.t. I . Then $r \cdot f$ is partially differentiable on X w.r.t. I and $r \cdot f \upharpoonright^I X = r \cdot (f \upharpoonright^I X)$.

Let m be a non empty element of \mathbb{N} , let f be a partial function from \mathcal{R}^m to \mathbb{R} , let k be an element of \mathbb{N} , and let Z be a set. We say that f is partial differentiable up to order k and Z if and only if the condition (Def. 10) is satisfied.

- (Def. 10) Let I be a non empty finite sequence of elements of \mathbb{N} . If $\text{len } I \leq k$ and $\text{rng } I \subseteq \text{Seg } m$, then f is partially differentiable on Z w.r.t. I .

The following proposition is true

- (75) Let f be a partial function from \mathcal{R}^m to \mathbb{R} and I, G be non empty finite sequences of elements of \mathbb{N} . Then f is partially differentiable on Z w.r.t. $G \cap I$ if and only if f is partially differentiable on Z w.r.t. G and $f \upharpoonright^G Z$ is partially differentiable on Z w.r.t. I .

One can prove the following propositions:

- (76) Let f be a partial function from \mathcal{R}^m to \mathbb{R} . Then f is partially differentiable on Z w.r.t. $\langle i \rangle$ if and only if f is partially differentiable on Z w.r.t. i .
- (77) For every partial function f from \mathcal{R}^m to \mathbb{R} holds $f \upharpoonright^{\langle i \rangle} Z = f \upharpoonright^i Z$.
- (78) Let f be a partial function from \mathcal{R}^m to \mathbb{R} and I be a non empty finite sequence of elements of \mathbb{N} . Suppose f is partial differentiable up to order $i + j$ and Z and $\text{rng } I \subseteq \text{Seg } m$ and $\text{len } I = j$. Then $f \upharpoonright^I Z$ is partial differentiable up to order i and Z .
- (79) Let f be a partial function from \mathcal{R}^m to \mathbb{R} . Suppose f is partial differentiable up to order i and Z and $j \leq i$. Then f is partial differentiable up to order j and Z .
- (80) Let X be a subset of \mathcal{R}^m and f, g be partial functions from \mathcal{R}^m to \mathbb{R} . Suppose that
- (i) X is open,
 - (ii) f is partial differentiable up to order i and X , and
 - (iii) g is partial differentiable up to order i and X .

Then $f + g$ is partial differentiable up to order i and X and $f - g$ is partial differentiable up to order i and X .

- (81) Let X be a subset of \mathcal{R}^m , f be a partial function from \mathcal{R}^m to \mathbb{R} , and r be a real number. Suppose X is open and f is partial differentiable up to

order i and X . Then $r \cdot f$ is partial differentiable up to order i and X .

- (82) Let X be a subset of \mathcal{R}^m . Suppose X is open. Let i be an element of \mathbb{N} and f, g be partial functions from \mathcal{R}^m to \mathbb{R} . Suppose f is partial differentiable up to order i and X and g is partial differentiable up to order i and X . Then $f \cdot g$ is partial differentiable up to order i and X .

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