

Semantics of MML Query¹

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Summary. In the paper the semantics of MML Query queries is given. The formalization is done according to [4].

MML identifier: MMLQUERY, version: 7.12.02 4.181.1147

The notation and terminology used here have been introduced in the following papers: [1], [5], [11], [8], [10], [6], [2], [3], [15], [13], [14], [9], [12], and [7].

1. ELEMENTARY QUERIES

Let X be a set. A list of X is a subset of X . An operation of X is a binary relation on X .

Let x, y, R be sets. The predicate $x, y \in R$ is defined by:

(Def. 1) $\langle x, y \rangle \in R$.

Let x, y, R be sets. We introduce $x, y \notin R$ as an antonym of $x, y \in R$.

For simplicity, we use the following convention: X, Y, z, s denote sets, L, L_1, L_2, A denote lists of X , x denotes an element of X , O, O_2, O_3 denote operations of X , and m denotes a natural number.

The following proposition is true

- (1) For all binary relations R_1, R_2 holds $R_1 \subseteq R_2$ iff for every z holds $R_1 \circ z \subseteq R_2 \circ z$.

Let us consider X, O, x . We introduce $x O$ as a synonym of $O \circ x$.

Let us consider X, O, x . Then $x O$ is a list of X .

One can prove the following proposition

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(2) $x, y \in O$ iff $y \in x O$.

Let us consider X, O, L . We introduce $L|O$ as a synonym of $O^\circ L$.

Let us consider X, O, L . Then $L|O$ is a list of X and it can be characterized by the condition:

(Def. 2) $L|O = \bigcup\{x O : x \in L\}$.

The functor $L\&O$ yielding a list of X is defined as follows:

(Def. 3) $L\&O = \bigcap\{x O : x \in L\}$.

The functor $L\text{ where }O$ yielding a list of X is defined as follows:

(Def. 4) $L\text{ where }O = \{x : \bigvee_y (x, y \in O \wedge x \in L)\}$.

Let O_2 be an operation of X . The functor $L\text{ where }O = O_2$ yielding a list of X is defined as follows:

(Def. 5) $L\text{ where }O = O_2 = \{x : \overline{\overline{x O}} = \overline{\overline{x O_2}} \wedge x \in L\}$.

The functor $L\text{ where }O \leq O_2$ yielding a list of X is defined by:

(Def. 6) $L\text{ where }O \leq O_2 = \{x : \overline{\overline{x O}} \subseteq \overline{\overline{x O_2}} \wedge x \in L\}$.

The functor $L\text{ where }O \geq O_2$ yields a list of X and is defined by:

(Def. 7) $L\text{ where }O \geq O_2 = \{x : \overline{\overline{x O_2}} \subseteq \overline{\overline{x O}} \wedge x \in L\}$.

The functor $L\text{ where }O < O_2$ yielding a list of X is defined as follows:

(Def. 8) $L\text{ where }O < O_2 = \{x : \overline{\overline{x O}} \in \overline{\overline{x O_2}} \wedge x \in L\}$.

The functor $L\text{ where }O > O_2$ yields a list of X and is defined by:

(Def. 9) $L\text{ where }O > O_2 = \{x : \overline{\overline{x O_2}} \in \overline{\overline{x O}} \wedge x \in L\}$.

Let us consider X, L, O, n . The functor $L\text{ where }O = n$ yielding a list of X is defined as follows:

(Def. 10) $L\text{ where }O = n = \{x : \overline{\overline{x O}} = n \wedge x \in L\}$.

The functor $L\text{ where }O \leq n$ yielding a list of X is defined by:

(Def. 11) $L\text{ where }O \leq n = \{x : \overline{\overline{x O}} \subseteq n \wedge x \in L\}$.

The functor $L\text{ where }O \geq n$ yielding a list of X is defined as follows:

(Def. 12) $L\text{ where }O \geq n = \{x : n \subseteq \overline{\overline{x O}} \wedge x \in L\}$.

The functor $L\text{ where }O < n$ yields a list of X and is defined as follows:

(Def. 13) $L\text{ where }O < n = \{x : \overline{\overline{x O}} \in n \wedge x \in L\}$.

The functor $L\text{ where }O > n$ yields a list of X and is defined by:

(Def. 14) $L\text{ where }O > n = \{x : n \in \overline{\overline{x O}} \wedge x \in L\}$.

One can prove the following propositions:

(3) $x \in L\text{ where }O$ iff $x \in L$ and $x O \neq \emptyset$.

(4) $L\text{ where }O \subseteq L$.

(5) If $L \subseteq \text{dom }O$, then $L\text{ where }O = L$.

(6) If $n \neq 0$ and $L_1 \subseteq L_2$, then $L_1\text{ where }O \geq n \subseteq L_2\text{ where }O$.

(7) $L\text{ where }O \geq 1 = L\text{ where }O$.

- (8) If $L_1 \subseteq L_2$, then $L_1 \text{ where } O > n \subseteq L_2 \text{ where } O$.
- (9) $L \text{ where } O > 0 = L \text{ where } O$.
- (10) If $n \neq 0$ and $L_1 \subseteq L_2$, then $L_1 \text{ where } O = n \subseteq L_2 \text{ where } O$.
- (11) $L \text{ where } O \geq n + 1 = L \text{ where } O > n$.
- (12) $L \text{ where } O \leq n = L \text{ where } O < n + 1$.
- (13) If $n \leq m$ and $L_1 \subseteq L_2$ and $O_1 \subseteq O_2$, then $L_1 \text{ where } O_1 \geq m \subseteq L_2 \text{ where } O_2 \geq n$.
- (14) If $n \leq m$ and $L_1 \subseteq L_2$ and $O_1 \subseteq O_2$, then $L_1 \text{ where } O_1 > m \subseteq L_2 \text{ where } O_2 > n$.
- (15) If $n \leq m$ and $L_1 \subseteq L_2$ and $O_1 \subseteq O_2$, then $L_1 \text{ where } O_2 \leq n \subseteq L_2 \text{ where } O_1 \leq m$.
- (16) If $n \leq m$ and $L_1 \subseteq L_2$ and $O_1 \subseteq O_2$, then $L_1 \text{ where } O_2 < n \subseteq L_2 \text{ where } O_1 < m$.
- (17) If $O_1 \subseteq O_2$ and $L_1 \subseteq L_2$ and $O \subseteq O_3$, then $L_1 \text{ where } O \geq O_2 \subseteq L_2 \text{ where } O_3 \geq O_1$.
- (18) If $O_1 \subseteq O_2$ and $L_1 \subseteq L_2$ and $O \subseteq O_3$, then $L_1 \text{ where } O > O_2 \subseteq L_2 \text{ where } O_3 > O_1$.
- (19) If $O_1 \subseteq O_2$ and $L_1 \subseteq L_2$ and $O \subseteq O_3$, then $L_1 \text{ where } O_3 \leq O_1 \subseteq L_2 \text{ where } O \leq O_2$.
- (20) If $O_1 \subseteq O_2$ and $L_1 \subseteq L_2$ and $O \subseteq O_3$, then $L_1 \text{ where } O_3 < O_1 \subseteq L_2 \text{ where } O < O_2$.
- (21) $L \text{ where } O > O_1 \subseteq L \text{ where } O$.
- (22) If $O_1 \subseteq O_2$ and $L_1 \subseteq L_2$, then $L_1 \text{ where } O_1 \subseteq L_2 \text{ where } O_2$.
- (23) $a \in L|O$ iff there exists b such that $a \in b O$ and $b \in L$.

Let us consider X, A, B . We introduce $A \text{ and } B$ as a synonym of $A \cap B$. We introduce $A \text{ or } B$ as a synonym of $A \cup B$. We introduce $A \text{ butnot } B$ as a synonym of $A \setminus B$.

Let us consider X, A, B . Then $A \text{ and } B$ is a list of X . Then $A \text{ or } B$ is a list of X . Then $A \text{ butnot } B$ is a list of X .

We now state several propositions:

- (24) If $L_1 \neq \emptyset$ and $L_2 \neq \emptyset$, then $(L_1 \text{ or } L_2) \& O = (L_1 \& O) \text{ and } (L_2 \& O)$.
- (25) If $L_1 \subseteq L_2$ and $O_1 \subseteq O_2$, then $L_1|O_1 \subseteq L_2|O_2$.
- (26) If $O_1 \subseteq O_2$, then $L \& O_1 \subseteq L \& O_2$.
- (27) $L \& (O_1 \text{ and } O_2) = (L \& O_1) \text{ and } (L \& O_2)$.
- (28) If $L_1 \neq \emptyset$ and $L_1 \subseteq L_2$, then $L_2 \& O \subseteq L_1 \& O$.

2. OPERATIONS

One can prove the following two propositions:

- (29) For all operations O_1, O_2 of X such that for every x holds $x O_1 = x O_2$ holds $O_1 = O_2$.
- (30) For all operations O_1, O_2 of X such that for every L holds $L|O_1 = L|O_2$ holds $O_1 = O_2$.

The functor **not** O yielding an operation of X is defined as follows:

- (Def. 15) For every L holds $L|\text{not } O = \bigcup\{(x O = \emptyset \rightarrow \{x\}, \emptyset) : x \in L\}$.

Let us consider X and let O_1, O_2 be operations of X . We introduce O_1 **and** O_2 as a synonym of $O_1 \cap O_2$. We introduce O_1 **or** O_2 as a synonym of $O_1 \cup O_2$. We introduce O_1 **butnot** O_2 as a synonym of $O_1 \setminus O_2$. We introduce $O_1|O_2$ as a synonym of $O_1 \cdot O_2$.

Let us consider X and let O_1, O_2 be operations of X . Then O_1 **and** O_2 is an operation of X and it can be characterized by the condition:

- (Def. 16) For every L holds $L|(O_1 \text{ and } O_2) = \bigcup\{(x O_1) \text{ and } (x O_2) : x \in L\}$.

Then O_1 **or** O_2 is an operation of X and it can be characterized by the condition:

- (Def. 17) For every L holds $L|(O_1 \text{ or } O_2) = \bigcup\{(x O_1) \text{ or } (x O_2) : x \in L\}$.

Then O_1 **butnot** O_2 is an operation of X and it can be characterized by the condition:

- (Def. 18) For every L holds $L|(O_1 \text{ butnot } O_2) = \bigcup\{(x O_1) \text{ butnot } (x O_2) : x \in L\}$.

Then $O_1|O_2$ is an operation of X and it can be characterized by the condition:

- (Def. 19) For every L holds $L|(O_1|O_2) = L|O_1|O_2$.

The functor O_1 **&** O_2 yielding an operation of X is defined as follows:

- (Def. 20) For every L holds $L|(O_1 \& O_2) = \bigcup\{(x O_1) \& O_2 : x \in L\}$.

We now state a number of propositions:

- (31) $x (O_1 \text{ and } O_2) = (x O_1) \text{ and } (x O_2)$.
- (32) $x (O_1 \text{ or } O_2) = (x O_1) \text{ or } (x O_2)$.
- (33) $x (O_1 \text{ butnot } O_2) = (x O_1) \text{ butnot } (x O_2)$.
- (34) $x (O_1|O_2) = (x O_1)|O_2$.
- (35) $x (O_1 \& O_2) = (x O_1) \& O_2$.
- (36) $z, s \in \text{not } O$ iff $z = s$ and $z \in X$ and $z \notin \text{dom } O$.
- (37) $\text{not } O = \text{id}_{X \setminus \text{dom } O}$.
- (38) $\text{dom not not } O = \text{dom } O$.
- (39) $L \text{ where not not } O = L \text{ where } O$.
- (40) $L \text{ where } O = 0 = L \text{ where not } O$.
- (41) $\text{not not not } O = \text{not } O$.
- (42) $\text{not } O_1 \text{ or not } O_2 \subseteq \text{not}(O_1 \text{ and } O_2)$.

(43) $\text{not}(O_1 \text{ or } O_2) = \text{not } O_1 \text{ and not } O_2$.

(44) If $\text{dom } O_1 = X$ and $\text{dom } O_2 = X$, then $(O_1 \text{ or } O_2) \& O = (O_1 \& O) \text{ and } (O_2 \& O)$.

Let us consider X, O . We say that O is filtering if and only if:

(Def. 21) $O \subseteq \text{id}_X$.

Next we state the proposition

(45) O is filtering iff $O = \text{id}_{\text{dom } O}$.

Let us consider X, O . Note that $\text{not } O$ is filtering.

Let us consider X . Note that there exists an operation of X which is filtering.

In the sequel F_1, F_2 denote filtering operations of X .

Let us consider X, F, O . One can check the following observations:

- * $F \text{ and } O$ is filtering,
- * $O \text{ and } F$ is filtering, and
- * $F \text{ butnot } O$ is filtering.

Let us consider X, F_1, F_2 . One can verify that $F_1 \text{ or } F_2$ is filtering.

(46) If $z \in x F$, then $z = x$.

(47) $L|F = L \text{ where } F$.

(48) $\text{not not } F = F$.

(49) $\text{not}(F_1 \text{ and } F_2) = \text{not } F_1 \text{ or not } F_2$.

(50) $\text{dom}(O \text{ or not } O) = X$.

(51) $F \text{ or not } F = \text{id}_X$.

(52) $O \text{ and not } O = \emptyset$.

(53) $(O_1 \text{ or } O_2) \text{ and not } O_1 \subseteq O_2$.

3. ROUGH QUERIES

Let A be a finite sequence and let a be a set. The functor $\#\text{occurrences}(a, A)$ yielding a natural number is defined as follows:

(Def. 22) $\#\text{occurrences}(a, A) = \overline{\{i : i \in \text{dom } A \wedge a \in A(i)\}}$.

We now state two propositions:

(54) For every finite sequence A and for every set a holds $\#\text{occurrences}(a, A) \leq \text{len } A$.

(55) For every finite sequence A and for every set a holds $A \neq \emptyset$ and $\#\text{occurrences}(a, A) = \text{len } A$ iff $a \in \bigcap \text{rng } A$.

The functor $\max\# A$ yielding a natural number is defined as follows:

(Def. 23) For every set a holds $\#\text{occurrences}(a, A) \leq \max\# A$ and for every n such that for every set a holds $\#\text{occurrences}(a, A) \leq n$ holds $\max\# A \leq n$.

(56) For every finite sequence A holds $\max\# A \leq \text{len } A$.

(57) For every finite sequence A and for every set a such that $\#\text{occurrences}(a, A) = \text{len } A$ holds $\max\# A = \text{len } A$.

Let us consider X , let A be a finite sequence of elements of 2^X , and let n be a natural number. The functor $\text{rough } n(A)$ yields a list of X and is defined as follows:

(Def. 24) $\text{rough } n(A) = \{x : n \leq \#\text{occurrences}(x, A)\}$ if $X \neq \emptyset$.

Let m be a natural number. The functor $\text{rough } n-m(A)$ yields a list of X and is defined by:

(Def. 25) $\text{rough } n-m(A) = \{x : n \leq \#\text{occurrences}(x, A) \wedge \#\text{occurrences}(x, A) \leq m\}$ if $X \neq \emptyset$.

Let us consider X and let A be a finite sequence of elements of 2^X . The functor $\text{rough}(A)$ yielding a list of X is defined by:

(Def. 26) $\text{rough}(A) = \text{rough } \max\# A(A)$.

Next we state several propositions:

(58) For every finite sequence A of elements of 2^X holds $\text{rough } n-\text{len } A(A) = \text{rough } n(A)$.

(59) For every finite sequence A of elements of 2^X such that $n \leq m$ holds $\text{rough } m(A) \subseteq \text{rough } n(A)$.

(60) Let A be a finite sequence of elements of 2^X and n_1, n_2, m_1, m_2 be natural numbers. If $n_1 \leq m_1$ and $n_2 \leq m_2$, then $\text{rough } m_1-n_2(A) \subseteq \text{rough } n_1-m_2(A)$.

(61) For every finite sequence A of elements of 2^X holds $\text{rough } n-m(A) \subseteq \text{rough } n(A)$.

(62) For every finite sequence A of elements of 2^X such that $A \neq \emptyset$ holds $\text{rough } \text{len } A(A) = \bigcap \text{rng } A$.

(63) For every finite sequence A of elements of 2^X holds $\text{rough } 1(A) = \bigcup A$.

(64) For all lists L_1, L_2 of X holds $\text{rough } 2(\langle L_1, L_2 \rangle) = L_1 \text{ and } L_2$.

(65) For all lists L_1, L_2 of X holds $\text{rough } 1(\langle L_1, L_2 \rangle) = L_1 \text{ or } L_2$.

4. CONSTRUCTOR DATABASE

We introduce constructor databases which are extensions of 1-sorted structures and are systems

$\langle \text{a carrier, constructors, a ref-operation} \rangle$,

where the carrier is a set, the constructors constitute a list of the carrier, and the ref-operation is a relation between the carrier and the constructors.

Let X be a 1-sorted structure. A list of X is a list of the carrier of X . An operation of X is an operation of the carrier of X .

Let us consider X , let S be a subset of X , and let R be a relation between X and S . The functor ${}^@R$ yields a binary relation on X and is defined by:

(Def. 27) ${}^@R = R$.

Let X be a constructor database and let a be an element of X . The functor $a \mathbf{ref}$ yielding a list of X is defined as follows:

(Def. 28) $a \mathbf{ref} = a {}^@$ the ref-operation of X .

The functor $a \mathbf{occur}$ yields a list of X and is defined as follows:

(Def. 29) $a \mathbf{occur} = a ({}^@$ the ref-operation of X) \smile .

The following proposition is true

(66) For every constructor database X and for all elements x, y of X holds $x \in y \mathbf{ref}$ iff $y \in x \mathbf{occur}$.

Let X be a constructor database. We say that X is ref-finite if and only if:

(Def. 30) For every element x of X holds $x \mathbf{ref}$ is finite.

One can verify that every constructor database which is finite is also ref-finite.

Let us note that there exists a constructor database which is finite and non empty.

Let X be a ref-finite constructor database and let x be an element of X . Observe that $x \mathbf{ref}$ is finite.

Let X be a constructor database and let A be a finite sequence of elements of the constructors of X . The functor $\mathbf{atleast}(A)$ yielding a list of X is defined by:

(Def. 31) $\mathbf{atleast}(A) = \{x \in X: \mathbf{rng} A \subseteq x \mathbf{ref}\}$ if the carrier of $X \neq \emptyset$.

The functor $\mathbf{atmost}(A)$ yielding a list of X is defined as follows:

(Def. 32) $\mathbf{atmost}(A) = \{x \in X: x \mathbf{ref} \subseteq \mathbf{rng} A\}$ if the carrier of $X \neq \emptyset$.

The functor $\mathbf{exactly}(A)$ yields a list of X and is defined by:

(Def. 33) $\mathbf{exactly}(A) = \{x \in X: x \mathbf{ref} = \mathbf{rng} A\}$ if the carrier of $X \neq \emptyset$.

Let n be a natural number. The functor $\mathbf{atleast\ minus\ }n(A)$ yields a list of X and is defined by:

(Def. 34) $\mathbf{atleast\ minus\ }n(A) = \{x \in X: \overline{\mathbf{rng} A \setminus x \mathbf{ref}} \leq n\}$ if the carrier of $X \neq \emptyset$.

Let X be a ref-finite constructor database, let A be a finite sequence of elements of the constructors of X , and let n be a natural number. The functor $\mathbf{atmost\ plus\ }n(A)$ yields a list of X and is defined by:

(Def. 35) $\mathbf{atmost\ plus\ }n(A) = \{x \in X: \overline{x \mathbf{ref} \setminus \mathbf{rng} A} \leq n\}$ if the carrier of $X \neq \emptyset$.

Let m be a natural number. The functor $\mathbf{exactly\ plus\ }n \mathbf{minus\ }m(A)$ yielding a list of X is defined by:

(Def. 36) $\overline{\text{exactly plus } n \text{ minus } m}(A) = \{x \in X: \overline{\overline{x \text{ ref} \setminus \text{rng } A}} \leq n \wedge \overline{\text{rng } A \setminus x \text{ ref}} \leq m\}$ if the carrier of $X \neq \emptyset$.

In the sequel X denotes a constructor database, x denotes an element of X , B denotes a finite sequence of elements of the constructors of Y , and y denotes an element of Y .

The following propositions are true:

- (67) $\text{atleast minus } 0(A) = \text{atleast}(A)$.
- (68) $\text{atmost plus } 0(B) = \text{atmost}(B)$.
- (69) $\text{exactly plus } 0 \text{ minus } 0(B) = \text{exactly}(B)$.
- (70) If $n \leq m$, then $\text{atleast minus } n(A) \subseteq \text{atleast minus } m(A)$.
- (71) If $n \leq m$, then $\text{atmost plus } n(B) \subseteq \text{atmost plus } m(B)$.
- (72) For all natural numbers n_1, n_2, m_1, m_2 such that $n_1 \leq m_1$ and $n_2 \leq m_2$ holds $\text{exactly plus } n_1 \text{ minus } n_2(B) \subseteq \text{exactly plus } m_1 \text{ minus } m_2(B)$.
- (73) $\text{atleast}(A) \subseteq \text{atleast minus } n(A)$.
- (74) $\text{atmost}(B) \subseteq \text{atmost plus } n(B)$.
- (75) $\text{exactly}(B) \subseteq \text{exactly plus } n \text{ minus } m(B)$.
- (76) $\text{exactly}(A) = \text{atleast}(A) \text{ and } \text{atmost}(A)$.
- (77) $\text{exactly plus } n \text{ minus } m(B) = \text{atleast minus } m(B) \text{ and } \text{atmost plus } n(B)$.
- (78) If $A \neq \emptyset$, then $\text{atleast}(A) = \bigcap \{x \text{ occur} : x \in \text{rng } A\}$.
- (79) For all elements c_1, c_2 of X such that $A = \langle c_1, c_2 \rangle$ holds $\text{atleast}(A) = c_1 \text{ occur and } c_2 \text{ occur}$.

REFERENCES

- [1] Grzegorz Bancerek. Cardinal numbers. *Formalized Mathematics*, 1(2):377–382, 1990.
- [2] Grzegorz Bancerek. The fundamental properties of natural numbers. *Formalized Mathematics*, 1(1):41–46, 1990.
- [3] Grzegorz Bancerek. The ordinal numbers. *Formalized Mathematics*, 1(1):91–96, 1990.
- [4] Grzegorz Bancerek. Information retrieval and rendering with MML query. *LNCS*, 4108:266–279, 2006.
- [5] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Formalized Mathematics*, 1(1):107–114, 1990.
- [6] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(1):55–65, 1990.
- [7] Czesław Byliński. Some basic properties of sets. *Formalized Mathematics*, 1(1):47–53, 1990.
- [8] Agata Darmochwał. Finite sets. *Formalized Mathematics*, 1(1):165–167, 1990.
- [9] Beata Padlewska. Families of sets. *Formalized Mathematics*, 1(1):147–152, 1990.
- [10] Andrzej Trybulec. Binary operations applied to functions. *Formalized Mathematics*, 1(2):329–334, 1990.
- [11] Wojciech A. Trybulec. Pigeon hole principle. *Formalized Mathematics*, 1(3):575–579, 1990.
- [12] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(1):67–71, 1990.
- [13] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1(1):73–83, 1990.
- [14] Edmund Woronowicz. Relations defined on sets. *Formalized Mathematics*, 1(1):181–186, 1990.

- [15] Bo Zhang, Hiroshi Yamazaki, and Yatsuka Nakamura. Set sequences and monotone class. *Formalized Mathematics*, 13(4):435–441, 2005.

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