

Routh's, Menelaus' and Generalized Ceva's Theorems

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Summary. The goal of this article is to formalize Ceva's theorem that is in the [8] on the web. Alongside with it formalizations of Routh's, Menelaus' and generalized form of Ceva's theorem itself are provided.

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The papers [1], [4], [3], [6], [5], [2], [7], and [9] provide the notation and terminology for this paper.

1. SOME PROPERTIES OF THE AREA OF TRIANGLE

We use the following convention: $A, B, C, A_1, B_1, C_1, A_2, B_2, C_2$ are points of \mathcal{E}_T^2 , l_1, m_1, n_1 are real numbers, and X, Y, Z are subsets of \mathcal{E}_T^2 .

Let us consider X, Y . We introduce X is parallel to Y as a synonym of X misses Y .

Let us consider X, Y, Z . We say that X, Y, Z are concurrent if and only if:

(Def. 1) X is parallel to Y and Y is parallel to Z and Z is parallel to X or there exists A such that $A \in X$ and $A \in Y$ and $A \in Z$.

One can prove the following propositions:

- (1) $(A + B)_1 = A_1 + B_1$ and $(A + B)_2 = A_2 + B_2$.
- (2) $(l_1 \cdot A)_1 = l_1 \cdot A_1$ and $(l_1 \cdot A)_2 = l_1 \cdot A_2$.
- (3) $(-A)_1 = -A_1$ and $(-A)_2 = -A_2$.
- (4) $(l_1 \cdot A + m_1 \cdot B)_1 = l_1 \cdot A_1 + m_1 \cdot B_1$ and $(l_1 \cdot A + m_1 \cdot B)_2 = l_1 \cdot A_2 + m_1 \cdot B_2$.

- (5) $((-l_1) \cdot A)_1 = -l_1 \cdot A_1$ and $((-l_1) \cdot A)_2 = -l_1 \cdot A_2$.
- (6) $(l_1 \cdot A - m_1 \cdot B)_1 = l_1 \cdot A_1 - m_1 \cdot B_1$ and $(l_1 \cdot A - m_1 \cdot B)_2 = l_1 \cdot A_2 - m_1 \cdot B_2$.
- (7) The area of $\Delta((1-l_1) \cdot A + l_1 \cdot A_1, B, C) = (1-l_1) \cdot \text{the area of } \Delta(A, B, C) + l_1 \cdot \text{the area of } \Delta(A_1, B, C)$.
- (8) If $\angle(A, B, C) = 0$ and A, B, C are mutually different, then $\angle(B, C, A) = \pi$ or $\angle(B, A, C) = \pi$.
- (9) A, B and C are collinear iff the area of $\Delta(A, B, C) = 0$.
- (10) The area of $\Delta(0_{\mathcal{E}_T^2}, B, C) = \frac{B_1 \cdot C_2 - C_1 \cdot B_2}{2}$.
- (11) The area of $\Delta(A + A_1, B, C) = ((\text{the area of } \Delta(A, B, C)) + (\text{the area of } \Delta(A_1, B, C))) - \text{the area of } \Delta(0_{\mathcal{E}_T^2}, B, C)$.
- (12) If $A \in \mathcal{L}(B, C)$, then $A \in \text{Line}(B, C)$.
- (13) If $B \neq C$, then A, B and C are collinear iff $A \in \text{Line}(B, C)$.
- (14) If A, B, C form a triangle and $A_1 = (1-l_1) \cdot B + l_1 \cdot C$, then $A \neq A_1$.
- (15) Suppose A, B, C form a triangle. Then
- (i) A, C, B form a triangle,
 - (ii) B, A, C form a triangle,
 - (iii) B, C, A form a triangle,
 - (iv) C, A, B form a triangle, and
 - (v) C, B, A form a triangle.
- (16) Suppose A, B, C form a triangle and $A_1 = (1-l_1) \cdot B + l_1 \cdot C$ and $B_1 = (1-m_1) \cdot C + m_1 \cdot A$ and $m_1 \neq 1$. Then $(1-m_1) + l_1 \cdot m_1 \neq 0$ if and only if $\text{Line}(A, A_1)$ is not parallel to $\text{Line}(B, B_1)$.

2. CEVA'S THEOREM AND OTHERS

The following propositions are true:

- (17) Suppose $A_1 = (1-l_1) \cdot B + l_1 \cdot C$ and $B_1 = (1-m_1) \cdot C + m_1 \cdot A$ and $C_1 = (1-n_1) \cdot A + n_1 \cdot B$. Then the area of $\Delta(A_1, B_1, C_1) = ((1-l_1) \cdot (1-m_1) \cdot (1-n_1) + l_1 \cdot m_1 \cdot n_1) \cdot \text{the area of } \Delta(A, B, C)$.
- (18) Suppose A, B, C form a triangle and $A_1 = (1-l_1) \cdot B + l_1 \cdot C$ and $B_1 = (1-m_1) \cdot C + m_1 \cdot A$ and $C_1 = (1-n_1) \cdot A + n_1 \cdot B$ and $l_1 \neq 1$ and $m_1 \neq 1$ and $n_1 \neq 1$. Then A_1, B_1 and C_1 are collinear if and only if $\frac{l_1}{1-l_1} \cdot \frac{m_1}{1-m_1} \cdot \frac{n_1}{1-n_1} = -1$.
- (19) Suppose that A, B, C form a triangle and $A_1 = (1-l_1) \cdot B + l_1 \cdot C$ and $B_1 = (1-m_1) \cdot C + m_1 \cdot A$ and $C_1 = (1-n_1) \cdot A + n_1 \cdot B$ and $l_1 \neq 1$ and $m_1 \neq 1$ and $n_1 \neq 1$ and A, A_1 and C_2 are collinear and B, B_1 and C_2 are collinear and B, B_1 and A_2 are collinear and C, C_1 and A_2 are collinear and A, A_1 and B_2 are collinear and C, C_1 and B_2 are collinear. Then

- (i) $((1 - m_1) + l_1 \cdot m_1) \cdot ((1 - l_1) + n_1 \cdot l_1) \cdot ((1 - n_1) + m_1 \cdot n_1) \neq 0$, and
- (ii) the area of $\Delta(A_2, B_2, C_2) = \frac{(m_1 \cdot n_1 \cdot l_1 - (1 - m_1) \cdot (1 - n_1) \cdot (1 - l_1))^2}{((1 - m_1) + l_1 \cdot m_1) \cdot ((1 - l_1) + n_1 \cdot l_1) \cdot ((1 - n_1) + m_1 \cdot n_1)}$ · the area of $\Delta(A, B, C)$.
- (20) Suppose that A, B, C form a triangle and $A_1 = \frac{2}{3} \cdot B + \frac{1}{3} \cdot C$ and $B_1 = \frac{2}{3} \cdot C + \frac{1}{3} \cdot A$ and $C_1 = \frac{2}{3} \cdot A + \frac{1}{3} \cdot B$ and A, A_1 and C_2 are collinear and B, B_1 and C_2 are collinear and B, B_1 and A_2 are collinear and C, C_1 and A_2 are collinear and A, A_1 and B_2 are collinear and C, C_1 and B_2 are collinear. Then the area of $\Delta(A_2, B_2, C_2) = \frac{\text{the area of } \Delta(A, B, C)}{7}$.
- (21) Suppose that A, B, C form a triangle and $A_1 = (1 - l_1) \cdot B + l_1 \cdot C$ and $B_1 = (1 - m_1) \cdot C + m_1 \cdot A$ and $C_1 = (1 - n_1) \cdot A + n_1 \cdot B$ and $l_1 \neq 1$ and $m_1 \neq 1$ and $n_1 \neq 1$ and $(1 - m_1) + l_1 \cdot m_1 \neq 0$ and $(1 - l_1) + n_1 \cdot l_1 \neq 0$ and $(1 - n_1) + m_1 \cdot n_1 \neq 0$. Then $\frac{l_1}{1 - l_1} \cdot \frac{m_1}{1 - m_1} \cdot \frac{n_1}{1 - n_1} = 1$ if and only if there exists A_2 such that A, A_1 and A_2 are collinear and B, B_1 and A_2 are collinear and C, C_1 and A_2 are collinear.
- (22) Suppose A, B, C form a triangle and $A_1 = (1 - l_1) \cdot B + l_1 \cdot C$ and $B_1 = (1 - m_1) \cdot C + m_1 \cdot A$ and $C_1 = (1 - n_1) \cdot A + n_1 \cdot B$ and $l_1 \neq 1$ and $m_1 \neq 1$ and $n_1 \neq 1$. Then $\frac{l_1}{1 - l_1} \cdot \frac{m_1}{1 - m_1} \cdot \frac{n_1}{1 - n_1} = 1$ if and only if $\text{Line}(A, A_1), \text{Line}(B, B_1), \text{Line}(C, C_1)$ are concurrent.

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