

Simple Graphs as Simplicial Complexes: the Mycielskian of a Graph¹

Piotr Rudnicki
University of Alberta
Edmonton, Canada

Lorna Stewart
University of Alberta
Edmonton, Canada

Summary. Harary [10, p. 7] claims that Veblen [20, p. 2] first suggested to formalize simple graphs using simplicial complexes. We have developed basic terminology for simple graphs as at most 1-dimensional complexes.

We formalize this new setting and then reprove Mycielski's [12] construction resulting in a triangle-free graph with arbitrarily large chromatic number. A different formalization of similar material is in [15].

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The papers [5], [1], [4], [16], [14], [6], [9], [18], [7], [15], [2], [11], [3], [17], [13], [19], and [8] provide the terminology and notation for this paper.

1. PRELIMINARIES

One can prove the following propositions:

- (1) For all sets x, X holds $\langle x, X \rangle \notin X$.
- (2) For all sets x, X holds $\langle x, X \rangle \neq X$.
- (3) For all sets x, X holds $\langle x, X \rangle \neq x$.
- (4) For all sets x_1, y_1, x_2, y_2, X such that $x_1, x_2 \in X$ and $\{x_1, \langle y_1, X \rangle\} = \{x_2, \langle y_2, X \rangle\}$ holds $x_1 = x_2$ and $y_1 = y_2$.
- (5) For all sets X, v such that $3 \subseteq \overline{X}$ there exist sets v_1, v_2 such that $v_1, v_2 \in X$ and $v_1 \neq v$ and $v_2 \neq v$ and $v_1 \neq v_2$.
- (6) For every set x holds $S_{\{x\}} = \{\{x\}\}$.

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Let us observe that there exists a finite sequence which is finite-yielding.

The following proposition is true

- (7) Let X be a non empty finite set and P be a partition of X . If $\overline{P} < \overline{X}$, then there exist sets p, x, y such that $p \in P$ and $x, y \in p$ and $x \neq y$.

Let us note that $\bigcup\{\emptyset\}$ is empty.

Next we state three propositions:

- (8) For every set x holds $\bigcup\{\emptyset, \{x\}\} = \{x\}$.
- (9) For every set X and for every subset s of X such that s is 1-element there exists a set x such that $x \in X$ and $s = \{x\}$.
- (10) For every set X holds $\overline{\overline{\{\{X, \langle x, X \rangle\}; x \text{ ranges over elements of } X: x \in X\}}} = \overline{X}$.

Let G be a set. The functor PairsOf G yielding a subset of G is defined as follows:

- (Def. 1) For every set e holds $e \in \text{PairsOf } G$ iff $e \in G$ and $\overline{e} = 2$.

The following propositions are true:

- (11) For every set X and for every set e such that $e \in \text{PairsOf } X$ there exist sets x, y such that $x \neq y$ and $x, y \in \bigcup X$ and $e = \{x, y\}$.
- (12) For all sets X, x, y such that $x \neq y$ and $\{x, y\} \in X$ holds $\{x, y\} \in \text{PairsOf } X$.
- (13) For all sets X, x, y such that $\{x, y\} \in \text{PairsOf } X$ holds $x \neq y$ and $x, y \in \bigcup X$.
- (14) For all sets G, H such that $G \subseteq H$ holds $\text{PairsOf } G \subseteq \text{PairsOf } H$.
- (15) For every finite set X holds $\overline{\overline{\{\{x, \langle y, \bigcup X \rangle\}; x \text{ ranges over elements of } \bigcup X, y \text{ ranges over elements of } \bigcup X: \{x, y\} \in \text{PairsOf } X\}}} = 2 \cdot \overline{\text{PairsOf } X}$.
- (16) For every finite set X holds $\overline{\overline{\{\{x, y\}; x \text{ ranges over elements of } \bigcup X, y \text{ ranges over elements of } \bigcup X: \{x, y\} \in \text{PairsOf } X\}}} = 2 \cdot \overline{\text{PairsOf } X}$.

Let X be a finite set. Note that PairsOf X is finite.

Let X be a set. We say that X is void if and only if:

- (Def. 2) $X = \{\emptyset\}$.

One can verify that there exists a set which is void.

Let us observe that every set which is void is also finite.

Let G be a void set. Observe that $\bigcup G$ is empty.

Next we state two propositions:

- (17) For every set X such that X is void holds $\text{PairsOf } X = \emptyset$.
- (18) For every set X such that $\bigcup X = \emptyset$ holds $X = \emptyset$ or $X = \{\emptyset\}$.

Let X be a set. We say that X is pair free if and only if:

(Def. 3) PairsOf X is empty.

We now state the proposition

(19) For all sets X, x such that $\overline{\bigcup X} = 1$ holds X is pair free.

Let us observe that there exists a set which is finite-membered and non empty.

Let X be a finite-membered set and let Y be a set. Observe that $X \cap Y$ is finite-membered and $X \setminus Y$ is finite-membered.

2. SIMPLE GRAPHS AS SIMPLICIAL COMPLEXES

Let n be a natural number and let X be a set. We say that X is at most n -dimensional if and only if:

(Def. 4) For every set x such that $x \in X$ holds $\overline{x} \subseteq n + 1$.

Let n be a natural number. Observe that every set which is at most n -dimensional is also finite-membered.

Let n be a natural number. Observe that there exists a set which is at most n -dimensional, subset-closed, and non empty.

Next we state two propositions:

(20) For every subset-closed non empty set G holds $\emptyset \in G$.

(21) Let n be a natural number, X be an at most n -dimensional set, and x be an element of X . If $x \in X$, then $\overline{x} \leq n + 1$.

Let n be a natural number and let X, Y be at most n -dimensional sets. Note that $X \cup Y$ is at most n -dimensional.

Let n be a natural number, let X be an at most n -dimensional set, and let Y be a set. Note that $X \cap Y$ is at most n -dimensional and $X \setminus Y$ is at most n -dimensional.

Let n be a natural number and let X be an at most n -dimensional set. Observe that every at most n -dimensional set is at most n -dimensional.

Let s be a set. We say that s is simple graph-like if and only if:

(Def. 5) s is at most 1-dimensional, subset-closed, and non empty.

Let us note that every set which is simple graph-like is also at most 1-dimensional, subset-closed, and non empty and every set which is at most 1-dimensional, subset-closed, and non empty is also simple graph-like.

The following proposition is true

(22) $\{\emptyset\}$ is simple graph-like.

One can verify that $\{\emptyset\}$ is simple graph-like.

One can verify that there exists a set which is simple graph-like.

A simple graph is a simple graph-like set.

One can verify that there exists a simple graph which is void and there exists a simple graph which is finite.

Let G be a set. We introduce Vertices G as a synonym of $\bigcup G$. We introduce Edges G as a synonym of PairsOf G .

Let X be a set. We introduce X is edgesless as a synonym of X is pair free.

We now state three propositions:

- (23) For every simple graph G such that Vertices G is finite holds G is finite.
- (24) For every simple graph G and for every set x holds $x \in$ Vertices G iff $\{x\} \in G$.
- (25) For every set x holds $\{\emptyset, \{x\}\}$ is a simple graph.

Let X be a finite finite-membered set. The functor order X yielding a natural number is defined by:

(Def. 6) order $X = \overline{\bigcup X}$.

Let X be a finite set. The functor size X yielding a natural number is defined by:

(Def. 7) size $X = \overline{\overline{\text{PairsOf } X}}$.

Next we state the proposition

- (26) For every finite simple graph G holds order $G \leq \overline{G}$.

Let G be a simple graph. A vertex of G is an element of Vertices G . An edge of G is an element of Edges G .

The following propositions are true:

- (27) For every simple graph G holds $G = \{\emptyset\} \cup S_{(\text{Vertices } G)} \cup \text{Edges } G$.
- (28) For every simple graph G such that Vertices $G = \emptyset$ holds G is void.
- (29) Let G be a simple graph and x be a set. If $x \in G$ and $x \neq \emptyset$, then there exists a set y such that $x = \{y\}$ and $y \in$ Vertices G or $x \in$ Edges G .
- (30) For every simple graph G and for every set x such that Vertices $G = \{x\}$ holds $G = \{\emptyset, \{x\}\}$.
- (31) For every set X there exists a simple graph G such that G is edgesless and Vertices $G = X$.

Let G be a simple graph and let v be an element of Vertices G . The functor Adjacent(v) yielding a subset of Vertices G is defined by:

(Def. 8) For every element x of Vertices G holds $x \in$ Adjacent(v) iff $\{v, x\} \in$ Edges G .

Let X be a set. A simple graph is called a simple graph of X if:

(Def. 9) Vertices it = X .

Let X be a set. The functor CompleteSGraph X yields a simple graph of X and is defined by:

(Def. 10) CompleteSGraph $X = \{V; V \text{ ranges over finite subsets of } X: \overline{V} \leq 2\}$.

One can prove the following proposition

- (32) For every simple graph G such that for all sets x, y such that $x, y \in \text{Vertices } G$ holds $\{x, y\} \in G$ holds $G = \text{CompleteSGraph } \text{Vertices } G$.

Let X be a finite set. One can check that $\text{CompleteSGraph } X$ is finite.

The following propositions are true:

- (33) For every set X and for every set x such that $x \in X$ holds $\{x\} \in \text{CompleteSGraph } X$.
- (34) For every set X and for all sets x, y such that $x, y \in X$ holds $\{x, y\} \in \text{CompleteSGraph } X$.
- (35) $\text{CompleteSGraph } \emptyset = \{\emptyset\}$.
- (36) For every set x holds $\text{CompleteSGraph } \{x\} = \{\emptyset, \{x\}\}$.
- (37) For all sets x, y holds $\text{CompleteSGraph } \{x, y\} = \{\emptyset, \{x\}, \{y\}, \{x, y\}\}$.
- (38) For all sets X, Y such that $X \subseteq Y$ holds $\text{CompleteSGraph } X \subseteq \text{CompleteSGraph } Y$.
- (39) For every simple graph G and for every set x such that $x \in \text{Vertices } G$ holds $\text{CompleteSGraph } \{x\} \subseteq G$.

Let G be a simple graph. One can check that there exists a subset of G which is simple graph-like.

Let G be a simple graph. A subgraph of G is a simple graph-like subset of G .

Let G be a simple graph. The functor $\text{Complement } G$ yields a simple graph and is defined as follows:

- (Def. 11) $\text{Complement } G = \text{CompleteSGraph } \text{Vertices } G \setminus \text{Edges } G$.

Let us observe that the functor $\text{Complement } G$ is involutive.

Next we state two propositions:

- (40) For every simple graph G holds $\text{Vertices } G = \text{Vertices } \text{Complement } G$.
- (41) Let G be a simple graph and x, y be sets. If $x \neq y$ and $x, y \in \text{Vertices } G$, then $\{x, y\} \in \text{Edges } G$ iff $\{x, y\} \notin \text{Edges } \text{Complement } G$.

3. INDUCED SUBGRAPHS

Let G be a simple graph and let L be a set. The subgraph induced by G yielding a subset of G is defined by:

- (Def. 12) The subgraph induced by $G = G \cap 2^L$.

Let G be a simple graph and let L be a set. Observe that the subgraph induced by G is simple graph-like.

Next we state two propositions:

- (42) For every simple graph G holds $G =$ the subgraph induced by G .
- (43) For every simple graph G and for every set L holds the subgraph induced by $G =$ the subgraph induced by G .

Let G be a finite simple graph and let L be a set. Observe that the subgraph induced by G is finite.

Let G be a simple graph and let L be a finite set. One can check that the subgraph induced by G is finite.

One can prove the following three propositions:

- (44) For all simple graphs G, H such that $G \subseteq H$ holds $G \subseteq$ the subgraph induced by H .
- (45) For every simple graph G and for every set L holds $\text{Vertices (the subgraph induced by } G) = \text{Vertices } G \cap L$.
- (46) For every simple graph G and for every set x such that $x \in \text{Vertices } G$ holds the subgraph induced by $G = \{\emptyset, \{x\}\}$.

4. CLIQUE, CLIQUE NUMBER, CLIQUE COVER

Let G be a simple graph. We say that G is a clique if and only if:

(Def. 13) $G = \text{CompleteSGraph Vertices } G$.

The following propositions are true:

- (47) Let G be a simple graph. Suppose that for all sets x, y such that $x \neq y$ and $x, y \in \text{Vertices } G$ holds $\{x, y\} \in \text{Edges } G$. Then G is a clique.
- (48) $\{\emptyset\}$ is a clique.

Observe that there exists a simple graph which is a clique. Let G be a simple graph. Note that there exists a subgraph of G which is a clique.

Let G be a simple graph. A clique of G is a clique subgraph of G .

Next we state the proposition

- (49) For every set X holds $\text{CompleteSGraph } X$ is a clique.

Let X be a set. One can check that $\text{CompleteSGraph } X$ is a clique.

Next we state two propositions:

- (50) For every simple graph G and for every set x such that $x \in \text{Vertices } G$ holds $\{\emptyset, \{x\}\}$ is a clique of G .
- (51) Let G be a simple graph and x, y be sets. If $x, y \in \text{Vertices } G$ and $\{x, y\} \in G$, then $\{\emptyset, \{x\}, \{y\}, \{x, y\}\}$ is a clique of G .

Let G be a simple graph. Observe that there exists a clique of G which is finite.

We now state two propositions:

- (52) For every simple graph G and for every set x such that $x \in \bigcup G$ there exists a finite clique C of G such that $\text{Vertices } C = \{x\}$.
- (53) For every a clique simple graph C and for all sets u, v such that $u, v \in \text{Vertices } C$ holds $\{u, v\} \in C$.

Let G be a simple graph. We say that G has finite clique number if and only if:

(Def. 14) There exists a finite clique C of G such that for every finite clique D of G holds order $D \leq$ order C .

Let us note that there exists a simple graph which has finite clique number.

Let us observe that every simple graph which is finite also has finite clique number.

Let G be a simple graph with finite clique number. The functor $\omega(G)$ yielding a natural number is defined as follows:

(Def. 15) There exists a finite clique C of G such that order $C = \omega(G)$ and for every finite clique T of G holds order $T \leq \omega(G)$.

We now state several propositions:

(54) For every simple graph G with finite clique number such that $\omega(G) = 0$ holds Vertices $G = \emptyset$.

(55) For every void simple graph G holds $\omega(G) = 0$.

(56) Let G be a simple graph and x, y be sets. If $\{x, y\} \in G$, then the subgraph induced by G is a clique of G .

(57) For every simple graph G with finite clique number such that Edges $G \neq \emptyset$ holds $\omega(G) \geq 2$.

(58) For all simple graphs G, H with finite clique number such that $G \subseteq H$ holds $\omega(G) \leq \omega(H)$.

(59) For every finite set X holds $\omega(\text{CompleteSGraph } X) = \overline{\overline{X}}$.

Let G be a simple graph and let P be a partition of Vertices G . We say that P is clique-wise if and only if:

(Def. 16) For every set x such that $x \in P$ holds the subgraph induced by G is a clique of G .

Let G be a simple graph. Observe that there exists a partition of Vertices G which is clique-wise.

Let G be a simple graph. A clique-partition of G is a clique-wise partition of Vertices G .

Let G be a void simple graph. Note that every partition of Vertices G which is empty is also clique-wise.

Let G be a simple graph. We say that G has finite clique cover if and only if:

(Def. 17) There exists a clique-partition of G which is finite.

One can verify that every simple graph which is finite also has finite clique cover.

Let G be a simple graph with finite clique cover. Note that there exists a clique-partition of G which is finite.

Let G be a simple graph with finite clique cover and let S be a subset of Vertices G . One can verify that the subgraph induced by G has finite clique cover.

Let G be a simple graph with finite clique cover. The functor $\kappa(G)$ yielding a natural number is defined by:

(Def. 18) There exists a finite clique-partition C of G such that $\overline{C} = \kappa(G)$ and for every finite clique-partition C of G holds $\kappa(G) \leq \overline{C}$.

5. STABLE SET, COLORING

Let G be a simple graph and let S be a subset of Vertices G . We say that S is stable if and only if:

(Def. 19) For all sets x, y such that $x \neq y$ and $x, y \in S$ holds $\{x, y\} \notin G$.

We now state two propositions:

(60) For every simple graph G holds $\emptyset_{\text{Vertices } G}$ is stable.

(61) For every simple graph G and for every subset S of Vertices G and for every set v such that $S = \{v\}$ holds S is stable.

Let G be a simple graph. Observe that every subset of Vertices G which is trivial is also stable.

Let G be a simple graph. Note that there exists a subset of Vertices G which is stable.

Let G be a simple graph. A stable set of G is a stable subset of Vertices G .

The following two propositions are true:

(62) For every simple graph G and for all sets x, y such that $x, y \in \text{Vertices } G$ and $\{x, y\} \notin G$ holds $\{x, y\}$ is a stable set of G .

(63) For every simple graph G with finite clique number such that $\omega(G) = 1$ holds Vertices G is a stable set of G .

Let G be a simple graph. Note that there exists a stable set of G which is finite.

One can prove the following proposition

(64) For every simple graph G and for every stable set A of G holds every subset of A is a stable set of G .

Let G be a simple graph and let P be a partition of Vertices G . We say that P is stable-wise if and only if:

(Def. 20) For every set x such that $x \in P$ holds x is a stable set of G .

The following proposition is true

(65) For every simple graph G holds $\text{SmallestPartition}(\text{Vertices } G)$ is stable-wise.

Let G be a simple graph. Note that there exists a partition of Vertices G which is stable-wise. A coloring of G is a stable-wise partition of Vertices G . We say that G is finitely colorable if and only if:

(Def. 21) There exists a coloring of G which is finite.

One can verify that there exists a simple graph which is finitely colorable.

Let us note that every simple graph which is finite is also finitely colorable.

Let G be a finitely colorable simple graph. Note that there exists a coloring of G which is finite.

We now state two propositions:

(66) Let G be a simple graph, S be a clique of G , and L be a set. If $L \subseteq$ Vertices S , then the subgraph induced by G is a clique of G .

(67) Let G be a simple graph, C be a coloring of G , and S be a subset of Vertices G . Then $C|_S$ is a coloring of the subgraph induced by G .

Let G be a finitely colorable simple graph and let S be a set. One can check that the subgraph induced by G is finitely colorable. The functor $\chi(G)$ yielding a natural number is defined as follows:

(Def. 22) There exists a finite coloring C of G such that $\overline{C} = \chi(G)$ and for every finite coloring C of G holds $\chi(G) \leq \overline{C}$.

One can prove the following three propositions:

(68) For all finitely colorable simple graphs G, H such that $G \subseteq H$ holds $\chi(G) \leq \chi(H)$.

(69) For every finite set X holds $\chi(\text{CompleteSGraph } X) = \overline{X}$.

(70) Let G be a finitely colorable simple graph, C be a finite coloring of G , and c be a set. Suppose $c \in C$ and $\overline{C} = \chi(G)$. Then there exists an element v of Vertices G such that $v \in c$ and for every element d of C such that $d \neq c$ there exists an element w of Vertices G such that $w \in \text{Adjacent}(v)$ and $w \in d$.

Let G be a simple graph. We say that G has finite stability number if and only if:

(Def. 23) There exists a finite stable set A of G such that for every finite stable set B of G holds $\overline{B} \leq \overline{A}$.

One can check that every simple graph which is finite also has finite stability number.

Let G be a simple graph with finite stability number. Observe that every stable set of G is finite.

Let us note that there exists a simple graph which is non void and has finite stability number.

Let G be a simple graph with finite stability number. The functor $\alpha(G)$ yielding a natural number is defined as follows:

(Def. 24) There exists a finite stable set A of G such that $\overline{\overline{A}} = \alpha(G)$ and for every finite stable set T of G holds $\overline{\overline{T}} \leq \alpha(G)$.

Let G be a non void simple graph with finite stability number. One can check that $\alpha(G)$ is positive.

Next we state the proposition

(71) For every simple graph G with finite stability number such that $\alpha(G) = 1$ holds G is a clique.

Let us observe that every simple graph which has finite clique number and finite stability number is also finite.

We now state four propositions:

(72) For every simple graph G and for every clique C of G holds Vertices C is a stable set of Complement G .

(73) For every simple graph G and for every clique C of Complement G holds Vertices C is a stable set of G .

(74) For every simple graph G and for every stable set C of G holds the subgraph induced by Complement G is a clique of Complement G .

(75) For every simple graph G and for every stable set C of Complement G holds the subgraph induced by G is a clique of G .

Let G be a simple graph with finite clique number. One can check that Complement G has finite stability number.

Let G be a simple graph with finite stability number. Note that Complement G has finite clique number.

We now state several propositions:

(76) For every simple graph G with finite clique number holds $\omega(G) = \alpha(\text{Complement } G)$.

(77) For every simple graph G with finite stability number holds $\alpha(G) = \omega(\text{Complement } G)$.

(78) For every simple graph G holds every clique-partition of Complement G is a coloring of G .

(79) For every simple graph G holds every clique-partition of G is a coloring of Complement G .

(80) For every simple graph G holds every coloring of G is a clique-partition of Complement G .

(81) For every simple graph G holds every coloring of Complement G is a clique-partition of G .

Let G be a finitely colorable simple graph. One can check that Complement G has finite clique cover.

Let G be a simple graph with finite clique cover.

One can check that Complement G is finitely colorable.

One can prove the following propositions:

- (82) For every finitely colorable simple graph G holds $\chi(G) = \kappa(\text{Complement } G)$.
- (83) For every simple graph G with finite clique cover holds $\kappa(G) = \chi(\text{Complement } G)$.

6. MYCIELSKIAN OF A GRAPH

Let G be a simple graph. The functor Mycielskian G yielding a simple graph is defined by the condition (Def. 25).

(Def. 25) Mycielskian $G = \{\emptyset\} \cup \{\{x\} : x \text{ ranges over elements of } \bigcup G \cup \bigcup G \times \{\bigcup G\} \cup \{\bigcup G\}\} \cup \text{Edges } G \cup \{\{x, \langle y, \bigcup G \rangle\} : x \text{ ranges over elements of } \bigcup G, y \text{ ranges over elements of } \bigcup G : \{x, y\} \in \text{Edges } G\} \cup \{\{\bigcup G, \langle x, \bigcup G \rangle\} : x \text{ ranges over elements of } \bigcup G : x \in \text{Vertices } G\}$.

We now state several propositions:

- (84) For every simple graph G holds $G \subseteq \text{Mycielskian } G$.
- (85) Let G be a simple graph and v be a set. Then $v \in \text{Vertices Mycielskian } G$ if and only if one of the following conditions is satisfied:
- (i) $v \in \bigcup G$, or
 - (ii) there exists a set x such that $x \in \bigcup G$ and $v = \langle x, \bigcup G \rangle$, or
 - (iii) $v = \bigcup G$.
- (86) For every simple graph G holds $\text{Vertices Mycielskian } G = \bigcup G \cup \bigcup G \times \{\bigcup G\} \cup \{\bigcup G\}$.
- (87) For every simple graph G holds $\bigcup G \in \bigcup \text{Mycielskian } G$.
- (88) For every void simple graph G holds $\text{Mycielskian } G = \{\emptyset, \{\bigcup G\}\}$.

Let G be a finite simple graph. Note that Mycielskian G is finite.

The following propositions are true:

- (89) For every finite simple graph G holds $\text{order Mycielskian } G = 2 \cdot \text{order } G + 1$.
- (90) Let G be a simple graph and e be a set. Then $e \in \text{Edges Mycielskian } G$ if and only if one of the following conditions is satisfied:
- (i) $e \in \text{Edges } G$, or
 - (ii) there exist elements x, y of $\bigcup G$ such that $e = \{x, \langle y, \bigcup G \rangle\}$ and $\{x, y\} \in \text{Edges } G$, or
 - (iii) there exists an element y of $\bigcup G$ such that $e = \{\bigcup G, \langle y, \bigcup G \rangle\}$ and $y \in \bigcup G$.
- (91) Let G be a simple graph. Then $\text{Edges Mycielskian } G = \text{Edges } G \cup \{\{x, \langle y, \bigcup G \rangle\} : x \text{ ranges over elements of } \bigcup G, y \text{ ranges over elements of } \bigcup G : \{x, y\} \in \text{Edges } G\} \cup \{\{\bigcup G, \langle y, \bigcup G \rangle\} : y \text{ ranges over elements of } \bigcup G : y \in \bigcup G\}$.

- (92) For every finite simple graph G holds $\text{size Mycielskian } G = 3 \cdot \text{size } G + \text{order } G$.
- (93) Let G be a simple graph and s, t be sets. Suppose $\{s, t\} \in \text{Edges Mycielskian } G$. Then
- (i) $\{s, t\} \in \text{Edges } G$, or
 - (ii) $s \in \bigcup G$ or $s = \bigcup G$ but there exists a set y such that $y \in \bigcup G$ and $t = \langle y, \bigcup G \rangle$, or
 - (iii) $t \in \bigcup G$ or $t = \bigcup G$ but there exists a set y such that $y \in \bigcup G$ and $s = \langle y, \bigcup G \rangle$.
- (94) For every simple graph G and for every set u such that $\{\bigcup G, u\} \in \text{Edges Mycielskian } G$ there exists a set x such that $x \in \bigcup G$ and $u = \langle x, \bigcup G \rangle$.
- (95) For every simple graph G and for every set u such that $u \in \text{Vertices } G$ holds $\{\langle u, \bigcup G \rangle\} \in \text{Mycielskian } G$.
- (96) For every simple graph G and for every set u such that $u \in \text{Vertices } G$ holds $\{\langle u, \bigcup G \rangle, \bigcup G\} \in \text{Mycielskian } G$.
- (97) For every simple graph G and for all sets x, y holds $\{\langle x, \bigcup G \rangle, \langle y, \bigcup G \rangle\} \notin \text{Edges Mycielskian } G$.
- (98) For every simple graph G and for all sets x, y such that $x \neq y$ holds $\{\langle x, \bigcup G \rangle, \langle y, \bigcup G \rangle\} \notin \text{Mycielskian } G$.
- (99) For every simple graph G and for all sets x, y such that $\{\langle x, \bigcup G \rangle, y\} \in \text{Edges Mycielskian } G$ holds $x \neq y$ but $x \in \bigcup G$ but $y \in \bigcup G$ or $y = \bigcup G$.
- (100) For every simple graph G and for all sets x, y such that $\{\langle x, \bigcup G \rangle, y\} \in \text{Mycielskian } G$ holds $x \neq y$.
- (101) For every simple graph G and for all sets x, y such that $y \in \bigcup G$ and $\{\langle x, \bigcup G \rangle, y\} \in \text{Mycielskian } G$ holds $\{x, y\} \in G$.
- (102) For every simple graph G and for all sets x, y such that $\{x, y\} \in \text{Edges } G$ holds $\{\langle x, \bigcup G \rangle, y\} \in \text{Mycielskian } G$.
- (103) For every simple graph G and for all sets x, y such that $x, y \in \text{Vertices } G$ and $\{x, y\} \in \text{Mycielskian } G$ holds $\{x, y\} \in G$.
- (104) For every simple graph G holds $G =$ the subgraph induced by Mycielskian G .
- (105) Let G be a simple graph and C be a finite clique of Mycielskian G . If $3 \leq \text{order } C$, then for every vertex v of C holds $v \neq \bigcup G$.
- (106) For every simple graph G with finite clique number such that $\omega(G) = 0$ and for every finite clique D of Mycielskian G holds $\text{order } D \leq 1$.
- (107) For every simple graph G and for every set x such that $\text{Vertices } G = \{x\}$ holds $\text{Mycielskian } G = \{\emptyset, \{x\}, \{\langle x, \bigcup G \rangle\}, \{\bigcup G\}, \{\langle x, \bigcup G \rangle, \bigcup G\}\}$.
- (108) For every simple graph G with finite clique number such that $\omega(G) = 1$

and for every finite clique D of Mycielskian G holds order $D \leq 2$.

- (109) For every simple graph G with finite clique number such that $2 \leq \omega(G)$ and for every finite clique D of Mycielskian G holds order $D \leq \omega(G)$.

Let G be a simple graph with finite clique number. Note that Mycielskian G has finite clique number.

We now state two propositions:

- (110) For every simple graph G with finite clique number such that $2 \leq \omega(G)$ holds $\omega(\text{Mycielskian } G) = \omega(G)$.

- (111) For every finitely colorable simple graph G there exists a coloring E of Mycielskian G such that $\overline{E} = 1 + \chi(G)$.

Let G be a finitely colorable simple graph. Observe that Mycielskian G is finitely colorable.

We now state the proposition

- (112) For every finitely colorable simple graph G holds $\chi(\text{Mycielskian } G) = 1 + \chi(G)$.

Let G be a simple graph. The Mycielskian sequence of G yields a many sorted set indexed by \mathbb{N} and is defined by the condition (Def. 26).

(Def. 26) There exists a function m_1 such that

- (i) the Mycielskian sequence of $G = m_1$,
- (ii) $m_1(0) = G$, and
- (iii) for every natural number k and for every simple graph G such that $G = m_1(k)$ holds $m_1(k+1) = \text{Mycielskian } G$.

We now state two propositions:

- (113) For every simple graph G holds (the Mycielskian sequence of G)(0) = G .

- (114) Let G be a simple graph and n be a natural number. Then (the Mycielskian sequence of G)(n) is a simple graph.

Let G be a simple graph and let n be a natural number. Observe that (the Mycielskian sequence of G)(n) is simple graph-like.

The following proposition is true

- (115) Let G, H be simple graphs and n be a natural number. Then (the Mycielskian sequence of G)($n+1$) = Mycielskian (the Mycielskian sequence of G)(n).

Let G be a simple graph with finite clique number and let n be a natural number. One can check that (the Mycielskian sequence of G)(n) has finite clique number.

Let G be a finitely colorable simple graph and let n be a natural number. One can check that (the Mycielskian sequence of G)(n) is finitely colorable.

Let G be a finite simple graph and let n be a natural number. Observe that (the Mycielskian sequence of G)(n) is finite.

One can prove the following propositions:

- (116) Let G be a finite simple graph and n be a natural number. Then order (the Mycielskian sequence of G)(n) = $(2^n \cdot \text{order } G + 2^n) - 1$.
- (117) Let G be a finite simple graph and n be a natural number. Then size (the Mycielskian sequence of G)(n) = $3^n \cdot \text{size } G + (3^n - 2^n) \cdot \text{order } G + ((n + 1) \text{ block } 3)$.
- (118) Let n be a natural number. Then ω ((the Mycielskian sequence of CompleteSGraph 2)(n)) = 2 and χ ((the Mycielskian sequence of CompleteSGraph 2)(n)) = $n + 2$.
- (119) For every natural number n there exists a finite simple graph G such that $\omega(G) = 2$ and $\chi(G) > n$.
- (120) For every natural number n there exists a finite simple graph G such that $\alpha(G) = 2$ and $\kappa(G) > n$.

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