

Introduction to Rational Functions

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Summary. In this article we formalize rational functions as pairs of polynomials and define some basic notions including the degree and evaluation of rational functions [8]. The main goal of the article is to provide properties of rational functions necessary to prove a theorem on the stability of networks.

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The notation and terminology used in this paper are introduced in the following articles: [14], [3], [4], [5], [18], [20], [16], [17], [1], [15], [2], [6], [12], [10], [11], [22], [19], [21], [9], [13], [23], and [7].

1. PRELIMINARIES

One can prove the following three propositions:

- (1) Let L be an add-associative right zeroed right complementable right distributive non empty double loop structure, a be an element of L , and p, q be finite sequences of elements of L . Suppose $\text{len } p = \text{len } q$ and for every element i of \mathbb{N} such that $i \in \text{dom } p$ holds $q_i = a \cdot p_i$. Then $\sum q = a \cdot \sum p$.
- (2) Let L be an add-associative right zeroed right complementable right distributive non empty double loop structure, f be a finite sequence of elements of L , and i, j be elements of \mathbb{N} . If $i \in \text{dom } f$ and $j = i - 1$, then $\text{Ins}(f_{\uparrow i}, j, f_i) = f$.
- (3) Let L be an add-associative right zeroed right complementable associative unital right distributive commutative non empty double loop structure, f be a finite sequence of elements of L , and i be an element of \mathbb{N} . If $i \in \text{dom } f$, then $\prod f = f_i \cdot \prod(f_{\uparrow i})$.

Let L be an add-associative right zeroed right complementable well unital associative left distributive commutative almost left invertible integral domain-like non trivial double loop structure and let x, y be non zero elements of L . Note that $\frac{x}{y}$ is non zero.

Let us note that every add-associative right zeroed right complementable right distributive non empty double loop structure which is integral domain-like is also almost left cancelable and every add-associative right zeroed right complementable left distributive non empty double loop structure which is integral domain-like is also almost right cancelable.

Let x, y be integers. Note that $\max(x, y)$ is integer and $\min(x, y)$ is integer.

One can prove the following proposition

$$(4) \quad \text{For all integers } x, y, z \text{ holds } \max(x + y, x + z) = x + \max(y, z).$$

2. MORE ON POLYNOMIALS

Let L be a non empty zero structure and let p be a polynomial of L . We say that p is zero if and only if:

$$(\text{Def. 1}) \quad p = \mathbf{0}.L.$$

We say that p is constant if and only if:

$$(\text{Def. 2}) \quad \deg p \leq 0.$$

Let L be a non trivial zero structure. One can verify that there exists a polynomial of L which is non zero.

Let L be a non empty zero structure. One can verify that $\mathbf{0}.L$ is zero and constant.

Let L be a non degenerated multiplicative loop with zero structure. Note that $\mathbf{1}.L$ is non zero.

Let L be a non empty multiplicative loop with zero structure. Note that $\mathbf{1}.L$ is constant.

Let L be a non empty zero structure. One can verify that every polynomial of L which is zero is also constant. Note that every polynomial of L which is non constant is also non zero.

Let L be a non trivial zero structure. One can verify that there exists a polynomial of L which is non constant.

Let L be a well unital non degenerated non empty double loop structure, let z be an element of L , and let k be an element of \mathbb{N} . Observe that $\text{rpoly}(k, z)$ is non zero.

Let L be an add-associative right zeroed right complementable distributive non degenerated double loop structure. One can check that Polynom-Ring L is non degenerated.

Let L be an integral domain-like add-associative right zeroed right complementable distributive non trivial double loop structure. Observe that Polynom-Ring L is integral domain-like.

Next we state two propositions:

- (5) Let L be an add-associative right zeroed right complementable right distributive associative non empty double loop structure, p, q be polynomials of L , and a be an element of L . Then $(a \cdot p) * q = a \cdot (p * q)$.
- (6) Let L be an add-associative right zeroed right complementable right distributive commutative associative non empty double loop structure, p, q be polynomials of L , and a be an element of L . Then $p*(a \cdot q) = a \cdot (p*q)$.

Let L be an add-associative right zeroed right complementable well unital commutative associative distributive almost left invertible non trivial double loop structure, let p be a non zero polynomial of L , and let a be a non zero element of L . Note that $a \cdot p$ is non zero.

Let L be an integral domain-like add-associative right zeroed right complementable distributive non trivial double loop structure and let p_1, p_2 be non zero polynomials of L . Observe that $p_1 * p_2$ is non zero.

One can prove the following proposition

- (7) Let L be an add-associative right zeroed right complementable distributive Abelian integral domain-like non trivial double loop structure, p_1, p_2 be polynomials of L , and p_3 be a non zero polynomial of L . If $p_1 * p_3 = p_2 * p_3$, then $p_1 = p_2$.

Let L be a non trivial zero structure and let p be a non zero polynomial of L . One can check that $\text{degree}(p)$ is natural.

Next we state several propositions:

- (8) Let L be an add-associative right zeroed right complementable unital right distributive non empty double loop structure and p be a polynomial of L . If $\text{deg } p = 0$, then for every element x of L holds $\text{eval}(p, x) \neq 0_L$.
- (9) Let L be an Abelian add-associative right zeroed right complementable well unital associative commutative distributive almost left invertible non degenerated double loop structure, p be a polynomial of L , and x be an element of L . Then $\text{eval}(p, x) = 0_L$ if and only if $\text{rpoly}(1, x) \mid p$.
- (10) Let L be an Abelian add-associative right zeroed right complementable well unital associative commutative distributive almost left invertible integral domain-like non degenerated double loop structure, p, q be polynomials of L , and x be an element of L . If $\text{rpoly}(1, x) \mid p * q$, then $\text{rpoly}(1, x) \mid p$ or $\text{rpoly}(1, x) \mid q$.
- (11) Let L be an Abelian add-associative right zeroed right complementable well unital associative commutative distributive almost left invertible non degenerated double loop structure and f be a finite sequence of elements

of Polynom-Ring L . Suppose that for every natural number i such that $i \in \text{dom } f$ there exists an element z of L such that $f(i) = \text{rpoly}(1, z)$. Let p be a polynomial of L . If $p = \prod f$, then $p \neq \mathbf{0}_L$.

- (12) Let L be an Abelian add-associative right zeroed right complementable well unital associative commutative distributive almost left invertible integral domain-like non degenerated double loop structure and f be a finite sequence of elements of Polynom-Ring L . Suppose that for every natural number i such that $i \in \text{dom } f$ there exists an element z of L such that $f(i) = \text{rpoly}(1, z)$. Let p be a polynomial of L . Suppose $p = \prod f$. Let x be an element of L . Then $\text{eval}(p, x) = 0_L$ if and only if there exists a natural number i such that $i \in \text{dom } f$ and $f(i) = \text{rpoly}(1, x)$.

3. COMMON ROOTS OF POLYNOMIALS

Let L be a unital non empty double loop structure, let p_1, p_2 be polynomials of L , and let x be an element of L . We say that x is a common root of p_1 and p_2 if and only if:

- (Def. 3) x is a root of p_1 and x is a root of p_2 .

Let L be a unital non empty double loop structure and let p_1, p_2 be polynomials of L . We say that p_1 and p_2 have a common root if and only if:

- (Def. 4) There exists an element of L which is a common root of p_1 and p_2 .

Let L be a unital non empty double loop structure and let p_1, p_2 be polynomials of L . We introduce p_1 and p_2 have common roots as a synonym of p_1 and p_2 have a common root. We introduce p_1 and p_2 have no common roots as an antonym of p_1 and p_2 have a common root.

Next we state several propositions:

- (13) Let L be an Abelian add-associative right zeroed right complementable unital distributive non empty double loop structure, p be a polynomial of L , and x be an element of L . If x is a root of p , then x is a root of $-p$.
- (14) Let L be an Abelian add-associative right zeroed right complementable unital left distributive non empty double loop structure, p_1, p_2 be polynomials of L , and x be an element of L . If x is a common root of p_1 and p_2 , then x is a root of $p_1 + p_2$.
- (15) Let L be an Abelian add-associative right zeroed right complementable unital distributive non empty double loop structure, p_1, p_2 be polynomials of L , and x be an element of L . If x is a common root of p_1 and p_2 , then x is a root of $-(p_1 + p_2)$.
- (16) Let L be an Abelian add-associative right zeroed right complementable unital distributive non empty double loop structure, p, q be polynomials

of L , and x be an element of L . If x is a common root of p and q , then x is a root of $p + q$.

- (17) Let L be an Abelian add-associative right zeroed right complementable well unital associative commutative distributive almost left invertible non trivial double loop structure and p_1, p_2 be polynomials of L . If $p_1 \mid p_2$ and p_1 has roots, then p_1 and p_2 have common roots.

Let L be a unital non empty double loop structure and let p, q be polynomials of L . The common roots of p and q yields a subset of L and is defined by:

- (Def. 5) The common roots of p and $q = \{x \in L: x \text{ is a common root of } p \text{ and } q\}$.

4. NORMALIZED POLYNOMIALS

Let L be a non empty zero structure and let p be a polynomial of L . The leading coefficient of p yields an element of L and is defined by:

- (Def. 6) The leading coefficient of $p = p(\text{len } p - 1)$.

We introduce $LC p$ as a synonym of the leading coefficient of p .

Let L be a non trivial double loop structure and let p be a non zero polynomial of L . One can check that $LC p$ is non zero.

One can prove the following proposition

- (18) Let L be an add-associative right zeroed right complementable well unital commutative associative distributive almost left invertible non empty double loop structure, p be a polynomial of L , and a be an element of L . Then $LC(a \cdot p) = a \cdot LC p$.

Let L be a non empty double loop structure and let p be a polynomial of L . We say that p is normalized if and only if:

- (Def. 7) $LC p = 1_L$.

Let L be an add-associative right zeroed right complementable well unital commutative associative distributive almost left invertible non trivial double loop structure and let p be a non zero polynomial of L . One can check that $\frac{1_L}{LC p} \cdot p$ is normalized.

Let L be a field and let p be a non zero polynomial of L . One can verify that $\text{NormPolynomial } p$ is normalized.

5. RATIONAL FUNCTIONS

Let L be a non trivial multiplicative loop with zero structure. Rational function of L is defined by:

- (Def. 8) There exists a polynomial p_1 of L and there exists a non zero polynomial p_2 of L such that it = $\langle p_1, p_2 \rangle$.

Let L be a non trivial multiplicative loop with zero structure, let p_1 be a polynomial of L , and let p_2 be a non zero polynomial of L . Then $\langle p_1, p_2 \rangle$ is a rational function of L .

Let L be a non trivial multiplicative loop with zero structure and let z be a rational function of L . Then z_1 is a polynomial of L . Then z_2 is a non zero polynomial of L .

Let L be a non trivial multiplicative loop with zero structure and let z be a rational function of L . We say that z is zero if and only if:

(Def. 9) $z_1 = \mathbf{0}$. L .

Let L be a non trivial multiplicative loop with zero structure. One can check that there exists a rational function of L which is non zero.

Next we state the proposition

(19) Let L be a non trivial multiplicative loop with zero structure and z be a rational function of L . Then $z = \langle z_1, z_2 \rangle$.

Let L be an add-associative right zeroed right complementable distributive unital non trivial double loop structure and let z be a rational function of L . We say that z is irreducible if and only if:

(Def. 10) z_1 and z_2 have no common roots.

Let L be an add-associative right zeroed right complementable distributive unital non trivial double loop structure and let z be a rational function of L . We introduce z is reducible as an antonym of z is irreducible.

Let L be an add-associative right zeroed right complementable distributive unital non trivial double loop structure and let z be a rational function of L . We say that z is normalized if and only if:

(Def. 11) z is irreducible and z_2 is normalized.

Let L be an add-associative right zeroed right complementable distributive unital non trivial double loop structure. Observe that every rational function of L which is normalized is also irreducible.

Let L be an Abelian add-associative right zeroed right complementable well unital associative distributive commutative almost left invertible integral domain-like non trivial double loop structure and let z be a rational function of L . Note that $\text{LC}(z_2)$ is non zero.

Let L be an Abelian add-associative right zeroed right complementable well unital associative distributive commutative almost left invertible integral domain-like non trivial double loop structure and let z be a rational function of L . The norm rational function of z yields a rational function of L and is defined by:

(Def. 12) The norm rational function of $z = \langle \frac{1_L}{\text{LC}(z_2)} \cdot z_1, \frac{1_L}{\text{LC}(z_2)} \cdot z_2 \rangle$.

Let L be an Abelian add-associative right zeroed right complementable well unital associative distributive commutative almost left invertible integral

domain-like non trivial double loop structure and let z be a rational function of L . We introduce $\text{NormRatF } z$ as a synonym of the norm rational function of z .

Let L be an Abelian add-associative right zeroed right complementable well unital associative distributive commutative almost left invertible integral domain-like non trivial double loop structure and let z be a non zero rational function of L . Observe that the norm rational function of z is non zero.

Let L be a non degenerated multiplicative loop with zero structure. The functor $0.L$ yields a rational function of L and is defined by:

(Def. 13) $0.L = \langle \mathbf{0}.L, \mathbf{1}.L \rangle$.

The functor $1.L$ yields a rational function of L and is defined as follows:

(Def. 14) $1.L = \langle \mathbf{1}.L, \mathbf{1}.L \rangle$.

Let L be an add-associative right zeroed right complementable distributive associative well unital non degenerated double loop structure. One can check that $0.L$ is normalized.

Let L be a non degenerated multiplicative loop with zero structure. Note that $1.L$ is non zero.

Let L be an add-associative right zeroed right complementable distributive associative well unital non degenerated double loop structure. One can verify that $1.L$ is irreducible.

Let L be an add-associative right zeroed right complementable distributive associative well unital non degenerated double loop structure. Observe that there exists a rational function of L which is irreducible and non zero.

Let L be an add-associative right zeroed right complementable distributive Abelian associative well unital non degenerated double loop structure and let x be an element of L . One can check that $\langle \text{rpoly}(1, x), \text{rpoly}(1, x) \rangle$ is reducible and non zero as a rational function of L .

Let L be an add-associative right zeroed right complementable distributive Abelian associative well unital non degenerated double loop structure. Observe that there exists a rational function of L which is reducible and non zero.

Let L be an add-associative right zeroed right complementable distributive associative well unital non degenerated double loop structure. One can verify that there exists a rational function of L which is normalized.

Let L be a non degenerated multiplicative loop with zero structure. One can verify that $0.L$ is zero.

Let L be an add-associative right zeroed right complementable distributive associative well unital non degenerated double loop structure. One can check that $1.L$ is normalized.

Let L be an integral domain-like add-associative right zeroed right complementable distributive non trivial double loop structure and let p, q be rational functions of L . The functor $p + q$ yields a rational function of L and is defined by:

(Def. 15) $p + q = \langle p_1 * q_2 + p_2 * q_1, p_2 * q_2 \rangle$.

Let L be an integral domain-like add-associative right zeroed right complementable distributive non trivial double loop structure and let p, q be rational functions of L . The functor $p * q$ yielding a rational function of L is defined by:

(Def. 16) $p * q = \langle p_1 * q_1, p_2 * q_2 \rangle$.

One can prove the following proposition

- (20) Let L be an add-associative right zeroed right complementable well unital commutative associative distributive almost left invertible non trivial double loop structure, p be a rational function of L , and a be a non zero element of L . Then $\langle a \cdot p_1, a \cdot p_2 \rangle$ is irreducible if and only if p is irreducible.

6. NORMALIZED RATIONAL FUNCTIONS

We now state the proposition

- (21) Let L be an Abelian add-associative right zeroed right complementable well unital associative distributive commutative integral domain-like non trivial double loop structure and z be a rational function of L . Then there exists a rational function z_1 of L and there exists a non zero polynomial z_2 of L such that
- (i) $z = \langle z_2 * (z_1)_1, z_2 * (z_1)_2 \rangle$,
 - (ii) z_1 is irreducible, and
 - (iii) there exists a finite sequence f of elements of Polynom-Ring L such that $z_2 = \prod f$ and for every element i of \mathbb{N} such that $i \in \text{dom } f$ there exists an element x of L such that x is a common root of z_1 and z_2 and $f(i) = \text{rpoly}(1, x)$.

Let L be an Abelian add-associative right zeroed right complementable well unital associative distributive commutative almost left invertible integral domain-like non trivial double loop structure and let z be a rational function of L . The functor $\text{NF } z$ yielding a rational function of L is defined by:

- (Def. 17)(i) There exists a rational function z_1 of L and there exists a non zero polynomial z_2 of L such that $z = \langle z_2 * (z_1)_1, z_2 * (z_1)_2 \rangle$ and z_1 is irreducible and $\text{NF } z =$ the norm rational function of z_1 and there exists a finite sequence f of elements of Polynom-Ring L such that $z_2 = \prod f$ and for every element i of \mathbb{N} such that $i \in \text{dom } f$ there exists an element x of L such that x is a common root of z_1 and z_2 and $f(i) = \text{rpoly}(1, x)$ if z is non zero,
- (ii) $\text{NF } z = 0$. L , otherwise.

Let L be an Abelian add-associative right zeroed right complementable well unital associative distributive commutative almost left invertible integral

domain-like non trivial double loop structure and let z be a rational function of L . Observe that $\text{NF } z$ is normalized and irreducible.

Let L be an Abelian add-associative right zeroed right complementable well unital associative distributive commutative almost left invertible integral domain-like non trivial double loop structure and let z be a non zero rational function of L . One can verify that $\text{NF } z$ is non zero.

One can prove the following propositions:

- (22) Let L be an Abelian add-associative right zeroed right complementable well unital associative distributive commutative almost left invertible integral domain-like non trivial double loop structure, z be a non zero rational function of L , z_1 be a rational function of L , and z_2 be a non zero polynomial of L . Suppose that
 - (i) $z = \langle z_2 * (z_1)_1, z_2 * (z_1)_2 \rangle$,
 - (ii) z_1 is irreducible, and
 - (iii) there exists a finite sequence f of elements of Polynom-Ring L such that $z_2 = \prod f$ and for every element i of \mathbb{N} such that $i \in \text{dom } f$ there exists an element x of L such that x is a common root of z_1 and z_2 and $f(i) = \text{rpoly}(1, x)$.
 Then $\text{NF } z =$ the norm rational function of z_1 .
- (23) Let L be an Abelian add-associative right zeroed right complementable well unital associative distributive commutative almost left invertible integral domain-like non trivial double loop structure. Then $\text{NF } 0. L = 0. L$.
- (24) Let L be an Abelian add-associative right zeroed right complementable well unital associative distributive commutative almost left invertible integral domain-like non trivial double loop structure. Then $\text{NF } 1. L = 1. L$.
- (25) Let L be an Abelian add-associative right zeroed right complementable well unital associative distributive commutative almost left invertible integral domain-like non trivial double loop structure and z be an irreducible non zero rational function of L . Then $\text{NF } z =$ the norm rational function of z .
- (26) Let L be an Abelian add-associative right zeroed right complementable well unital associative distributive commutative almost left invertible integral domain-like non trivial double loop structure, z be a rational function of L , and x be an element of L . Then $\text{NF } \langle \text{rpoly}(1, x) * z_1, \text{rpoly}(1, x) * z_2 \rangle = \text{NF } z$.
- (27) Let L be an Abelian add-associative right zeroed right complementable well unital associative distributive commutative almost left invertible integral domain-like non trivial double loop structure and z be a rational function of L . Then $\text{NF } \text{NF } z = \text{NF } z$.
- (28) Let L be an Abelian add-associative right zeroed right complementable well unital associative distributive commutative almost left invertible in-

tegral domain-like non degenerated double loop structure and z be a non zero rational function of L . Then z is irreducible if and only if there exists an element a of L such that $a \neq 0_L$ and $\langle a \cdot z_1, a \cdot z_2 \rangle = \text{NF } z$.

7. DEGREE OF RATIONAL FUNCTIONS

Let L be an Abelian add-associative right zeroed right complementable well unital associative distributive commutative almost left invertible integral domain-like non trivial double loop structure and let z be a rational function of L . The functor $\text{degree}(z)$ yielding an integer is defined as follows:

(Def. 18) $\text{degree}(z) = \max(\text{degree}((\text{NF } z)_1), \text{degree}((\text{NF } z)_2))$.

Let L be an Abelian add-associative right zeroed right complementable well unital associative distributive commutative almost left invertible integral domain-like non trivial double loop structure and let z be a rational function of L . We introduce $\text{deg } z$ as a synonym of $\text{degree}(z)$.

Next we state two propositions:

- (29) Let L be an Abelian add-associative right zeroed right complementable well unital associative distributive commutative almost left invertible integral domain-like non trivial double loop structure and z be a rational function of L . Then $\text{degree}(z) \leq \max(\text{degree}(z_1), \text{degree}(z_2))$.
- (30) Let L be an Abelian add-associative right zeroed right complementable well unital associative distributive commutative almost left invertible integral domain-like non trivial double loop structure and z be a non zero rational function of L . Then z is irreducible if and only if $\text{degree}(z) = \max(\text{degree}(z_1), \text{degree}(z_2))$.

8. EVALUATION OF RATIONAL FUNCTIONS

Let L be a field, let z be a rational function of L , and let x be an element of L . The functor $\text{eval}(z, x)$ yielding an element of L is defined by:

(Def. 19) $\text{eval}(z, x) = \frac{\text{eval}(z_1, x)}{\text{eval}(z_2, x)}$.

The following propositions are true:

- (31) For every field L and for every element x of L holds $\text{eval}(0, L, x) = 0_L$.
- (32) For every field L and for every element x of L holds $\text{eval}(1, L, x) = 1_L$.
- (33) Let L be a field, p, q be rational functions of L , and x be an element of L . If $\text{eval}(p_2, x) \neq 0_L$ and $\text{eval}(q_2, x) \neq 0_L$, then $\text{eval}(p + q, x) = \text{eval}(p, x) + \text{eval}(q, x)$.
- (34) Let L be a field, p, q be rational functions of L , and x be an element of L . If $\text{eval}(p_2, x) \neq 0_L$ and $\text{eval}(q_2, x) \neq 0_L$, then $\text{eval}(p * q, x) = \text{eval}(p, x) \cdot \text{eval}(q, x)$.

- (35) Let L be a field, p be a rational function of L , and x be an element of L . If $\text{eval}(p_2, x) \neq 0_L$, then $\text{eval}(\text{the norm rational function of } p, x) = \text{eval}(p, x)$.
- (36) Let L be a field, p be a rational function of L , and x be an element of L . If $\text{eval}(p_2, x) \neq 0_L$, then x is a common root of p_1 and p_2 or $\text{eval}(\text{NF } p, x) = \text{eval}(p, x)$.

REFERENCES

- [1] Grzegorz Bancerek. The fundamental properties of natural numbers. *Formalized Mathematics*, 1(1):41–46, 1990.
- [2] Grzegorz Bancerek. The ordinal numbers. *Formalized Mathematics*, 1(1):91–96, 1990.
- [3] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Formalized Mathematics*, 1(1):107–114, 1990.
- [4] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(1):55–65, 1990.
- [5] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(1):153–164, 1990.
- [6] Czesław Byliński. Partial functions. *Formalized Mathematics*, 1(2):357–367, 1990.
- [7] Czesław Byliński. Some basic properties of sets. *Formalized Mathematics*, 1(1):47–53, 1990.
- [8] H. Heuser. *Lehrbuch der Analysis*. B.G. Teubner Stuttgart, 1990.
- [9] Eugeniusz Kusak, Wojciech Leończuk, and Michał Muzalewski. Abelian groups, fields and vector spaces. *Formalized Mathematics*, 1(2):335–342, 1990.
- [10] Robert Milewski. The evaluation of polynomials. *Formalized Mathematics*, 9(2):391–395, 2001.
- [11] Robert Milewski. Fundamental theorem of algebra. *Formalized Mathematics*, 9(3):461–470, 2001.
- [12] Robert Milewski. The ring of polynomials. *Formalized Mathematics*, 9(2):339–346, 2001.
- [13] Michał Muzalewski. Construction of rings and left-, right-, and bi-modules over a ring. *Formalized Mathematics*, 2(1):3–11, 1991.
- [14] Michał Muzalewski and Lesław W. Szczerba. Construction of finite sequences over ring and left-, right-, and bi-modules over a ring. *Formalized Mathematics*, 2(1):97–104, 1991.
- [15] Jan Popiołek. Real normed space. *Formalized Mathematics*, 2(1):111–115, 1991.
- [16] Christoph Schwarzweller and Agnieszka Rowińska-Schwarzweller. Schur's theorem on the stability of networks. *Formalized Mathematics*, 14(4):135–142, 2006, doi:10.2478/v10037-006-0017-9.
- [17] Michał J. Trybulec. Integers. *Formalized Mathematics*, 1(3):501–505, 1990.
- [18] Wojciech A. Trybulec. Groups. *Formalized Mathematics*, 1(5):821–827, 1990.
- [19] Wojciech A. Trybulec. Vectors in real linear space. *Formalized Mathematics*, 1(2):291–296, 1990.
- [20] Wojciech A. Trybulec. Lattice of subgroups of a group. Frattini subgroup. *Formalized Mathematics*, 2(1):41–47, 1991.
- [21] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(1):67–71, 1990.
- [22] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1(1):73–83, 1990.
- [23] Hiroshi Yamazaki and Yasunari Shidama. Algebra of vector functions. *Formalized Mathematics*, 3(2):171–175, 1992.

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