FORMALIZED MATHEMATICS
Vol. 20, No. 4, Pages 257–263, 2012
DOI: 10.2478/v10037-012-0030-0

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Posterior Probability on Finite Set[1]

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Summary. In [14] we formalized probability and probability distribution on a finite sample space. In this article first we propose a formalization of the class of finite sample spaces whose element's probability distributions are equivalent with each other. Next, we formalize the probability measure of the class of sample spaces we have formalized above. Finally, we formalize the sampling and posterior probability.

MML identifier: DIST_2, version: 8.0.01 5.3.1162

The notation and terminology used in this paper have been introduced in the following papers: [11], [1], [14], [17], [3], [5], [20], [10], [6], [7], [4], [19], [22], [25], [18], [2], [8], [13], [15], [12], [23], [24], [16], [21], and [9].

1. Equivalent Distributed Finite and Distributed Sample Spaces

The following propositions are true:

(1) Let Y be a non empty finite set and s be a finite sequence of elements of Y. If $Y = \{1\}$ and $s = \langle 1 \rangle$, then $\mathrm{FDprobSEQ}\, s = \langle 1 \rangle$.

(2) Let S be a non empty finite set, p be a probability distribution finite sequence on S, and s be a finite sequence of elements of S. If $\mathrm{FDprobSEQ}\, s = p$, then $\mathrm{distribution}(p, S) =$ the equivalence class of s and $s \in \mathrm{distribution}(p, S)$.

(3) Let S be a non empty finite set and x be an element of S. Then $x \in \mathrm{rng}\, \mathrm{CFS}(S)$ and there exists a natural number n such that $n \in \mathrm{dom}\, \mathrm{CFS}(S)$ and $x = (\mathrm{CFS}(S))(n)$ and $n \in \mathrm{Seg}\, \overline{\overline{S}}$.

[1]This work is supported by JSPS KAKENHI 21240001.

ISSN 1426–2630(p), 1898-9934(e)

Let S be a non empty finite set. One can check that every non empty finite set is non empty.

Let S be a non empty finite set and let D be an element of the distribution family of S. We see that the element of D is a finite sequence of elements of S.

One can prove the following proposition

(4) Let S be a non empty finite set, D be an element of the distribution family of S, and s, t be elements of D. Then s and t are probability equivalent.

Let S be a non empty finite set and let D be an element of the distribution family of S. We introduce D is well distributed as a synonym of D has non empty elements.

We now state the proposition

(5) Let S be a non empty finite set and s be a finite sequence of elements of S. Then for every set x holds $\mathrm{Prob}_{\mathrm{D}}(x, s) = 0$ if and only if s is empty.

Let S be a non empty finite set. Observe that every non empty finite set which is well distributed

We now state the proposition

(6) Let S be a non empty finite set and D be an element of the distribution family of S. Then D is not well distributed if and only if $D = \{\varepsilon_S\}$.

Let S be a non empty finite set. An equivalent distributed sample spaces family of S is a well distributed element of the distribution family of S.

Let S be a non empty finite set. One can verify that the uniform distribution S is well distributed.

One can prove the following proposition

(7) Let S be a non empty finite set and D be an equivalent distributed sample spaces family of S. Then $(\mathrm{GenProbSEQ}\, S)(D)$ is a probability distribution finite sequence on S.

2. Probability Measure of Equivalent Distributed Finite and Distributed Sample Spaces

Let S be a non empty finite set and let a be an element of S. The functor $|\bullet : a|_{\mathbb{N}}$ yielding an element of $\mathbb{N}$ is defined by:

(Def. 1) $|\bullet : a|_{\mathbb{N}} = a \hookleftarrow \mathrm{CFS}(S)$.

Let S be a non empty finite set and let D be an equivalent distributed sample spaces family of S. The probability finite sequence of D yields a probability distribution finite sequence on S and is defined by:

(Def. 2) The probability finite sequence of $D = (\mathrm{GenProbSEQ}\, S)(D)$.

Let j_1 be a *Boolean*-valued function. The true event of j_1 yielding an event of $\mathrm{dom}\, j_1$ is defined as follows:

(Def. 3) The true event of $j_1 = {j_1}^{-1}(\{\mathit{true}\})$.

The following proposition is true

(8) Let S be a non empty finite set, f be an S-valued function, and j_1 be a function from S into *Boolean*. Then the true event of $j_1 \cdot f$ is an event of $\operatorname{dom} f$.

Let S be a non empty finite set, let D be an equivalent distributed sample spaces family of S, let s be an element of D, and let j_1 be a function from S into *Boolean*. The functor $\mathrm{Prob}(j_1, s)$ yielding a real number is defined as follows:

(Def. 4) $\mathrm{Prob}(j_1, s) = \frac{\overline{\overline{\text{the true event of } j_1 \cdot s}}}{\operatorname{len} s}$.

The following propositions are true:

(9) Let S be a non empty finite set, D be an equivalent distributed sample spaces family of S, s be an element of D, j_1 be a function from S into *Boolean*, F be a non empty finite set, and E be an event of F. If $F = \operatorname{dom} s$ and $E =$ the true event of $j_1 \cdot s$, then $\mathrm{Prob}(j_1, s) = \mathrm{P}(E)$.

(10) Let S be a non empty finite set, D be an equivalent distributed sample spaces family of S, a be an element of S, s be an element of D, and j_1 be a function from S into *Boolean*. If for every set x holds $x = a$ iff $j_1(x) = \mathit{true}$, then $\mathrm{Prob}(j_1, s) = \mathrm{Prob}_{\mathrm{D}}(a, s)$.

Let S be a set, let s be a finite sequence of elements of S, and let A be a subset of $\operatorname{dom} s$. The functor $\mathrm{extract}(s, A)$ yielding a finite sequence of elements of S is defined by:

(Def. 5) $\mathrm{extract}(s, A) = s \cdot \mathrm{CFS}(A)$.

We now state several propositions:

(11) Let S be a set, s be a finite sequence of elements of S, and A be a subset of $\operatorname{dom} s$. Then $\operatorname{len} \mathrm{extract}(s, A) = \overline{\overline{A}}$ and for every natural number i such that $i \in \operatorname{dom} \mathrm{extract}(s, A)$ holds $(\mathrm{extract}(s, A))(i) = s((\mathrm{CFS}(A))(i))$.

(12) Let S be a non empty finite set, s be a finite sequence of elements of S, A be a subset of $\operatorname{dom} s$, and f be a function. If $f = \mathrm{CFS}(A)$, then $\mathrm{extract}(s, A) \cdot f^{-1} = s \restriction A$.

(13) Let S be a non empty finite set, f be an S-valued function, j_1 be a function from S into *Boolean*, and n be a set. Suppose $n \in \operatorname{dom} f$. Then $n \in$ the true event of $j_1 \cdot f$ if and only if $f(n) \in$ the true event of j_1.

(14) Let S be a non empty finite set, f be an S-valued function, and j_1 be a function from S into *Boolean*. Then the true event of $j_1 \cdot f = f^{-1}$(the true event of j_1).

(15) Let S be a non empty finite set, D be an equivalent distributed sample spaces family of S, s be an element of D, and j_1 be a function from S into *Boolean*. Then there exists a subset A of $\operatorname{dom} \mathrm{freqSEQ}\, s$ such that $A =$ the true event of $j_1 \cdot \mathrm{CFS}(S)$ and $\overline{\overline{\text{the true event of } j_1 \cdot s}} =$

$\sum$ extract(freqSEQ s, A).

(16) Let S be a non empty finite set, D be an equivalent distributed sample spaces family of S, and s be an element of D. Then freqSEQ $s = \operatorname{len} s \cdot$ FDprobSEQ s.

(17) Let S be a non empty finite set, D be an equivalent distributed sample spaces family of S, s, t be elements of D, and j_1 be a function from S into *Boolean*. Then $\operatorname{Prob}(j_1, s) = \operatorname{Prob}(j_1, t)$.

Let S be a non empty finite set, let D be an equivalent distributed sample spaces family of S, and let j_1 be a function from S into *Boolean*. The functor $\operatorname{Prob}(j_1, D)$ yielding a real number is defined by:

(Def. 6) For every element s of D holds $\operatorname{Prob}(j_1, D) = \operatorname{Prob}(j_1, s)$.

Next we state the proposition

(18) For every non empty finite set S and for every element s of S^* and for every function j_1 from S into *Boolean* holds $\operatorname{Coim}(j_1 \cdot s, \mathit{true}) \in 2^{\operatorname{dom} s}$.

Let S be a set and let S_1 be a subset of S. The membership decision of S_1 yielding a function from S into *Boolean* is defined as follows:

(Def. 7) The membership decision of $S_1 = \chi_{(S_1),S}$.

The following propositions are true:

(19) For every non empty finite set S and for every subset B_1 of S there exists a function j_1 from S into *Boolean* such that $\operatorname{Coim}(j_1, \mathit{true}) = B_1$.

(20) Let S be a non empty finite set, s be an element of S^*, f be a function from S into *Boolean*, and F be a σ-field of subsets of $\operatorname{dom} s$. If $F = 2^{\operatorname{dom} s}$, then $\operatorname{Coim}(f \cdot s, \mathit{true})$ is an event of F.

(21) Let S be a non empty finite set, s be an element of S^*, and f, g be functions from S into *Boolean*. Then $\operatorname{Coim}((f \vee g) \cdot s, \mathit{true}) = \operatorname{Coim}(f \cdot s, \mathit{true}) \cup \operatorname{Coim}(g \cdot s, \mathit{true})$.

(22) Let S be a non empty finite set, s be an element of S^*, and f, g be functions from S into *Boolean*. Then $\operatorname{Coim}((f \wedge g) \cdot s, \mathit{true}) = \operatorname{Coim}(f \cdot s, \mathit{true}) \cap \operatorname{Coim}(g \cdot s, \mathit{true})$.

(23) Let S be a non empty finite set, s be an element of S^*, and f be a function from S into *Boolean*. Then $\operatorname{Coim}(\neg f \cdot s, \mathit{true}) = \operatorname{dom} s \setminus \operatorname{Coim}(f \cdot s, \mathit{true})$.

(24) Let S be a non empty finite set, D be an equivalent distributed sample spaces family of S, s be an element of D, and f, g be functions from S into *Boolean*. Then $\operatorname{Prob}(f \vee g, s) = \frac{\overline{\overline{(\text{the true event of } f \cdot s) \cup (\text{the true event of } g \cdot s)}}}{\operatorname{len} s}$.

(25) Let S be a non empty finite set, D be an equivalent distributed sample spaces family of S, s be an element of D, and f, g be functions from S into *Boolean*. Then $\operatorname{Prob}(f \wedge g, s) = \frac{\overline{\overline{(\text{the true event of } f \cdot s) \cap (\text{the true event of } g \cdot s)}}}{\operatorname{len} s}$.

(26) Let S be a non empty finite set, D be an equivalent distributed sample spaces family of S, s be an element of D, and f be a function from S into

Boolean. Then $\mathrm{Prob}(\neg f, s) = 1 - \mathrm{Prob}(f, s)$.

(27) Let S be a non empty finite set, D be an equivalent distributed sample spaces family of S, and f, g be functions from S into *Boolean*. Then $\mathrm{Prob}(f \vee g, D) = (\mathrm{Prob}(f, D) + \mathrm{Prob}(g, D)) - \mathrm{Prob}(f \wedge g, D)$.

(28) Let S be a non empty finite set, D be an equivalent distributed sample spaces family of S, and f be a function from S into *Boolean*. Then $\mathrm{Prob}(\neg f, D) = 1 - \mathrm{Prob}(f, D)$.

(29) Let S be a non empty finite set, D be an equivalent distributed sample spaces family of S, and f be a function from S into *Boolean*. If $f = \chi_{S,S}$, then $\mathrm{Prob}(f, D) = 1$.

(30) Let S be a non empty finite set, D be an equivalent distributed sample spaces family of S, and f be a function from S into *Boolean*. Then $0 \leq \mathrm{Prob}(f, D)$.

(31) Let S be a non empty finite set, A, B be sets, and f, g be functions from S into *Boolean*. If $A \subseteq S$ and $B \subseteq S$ and $f = \chi_{A,S}$ and $g = \chi_{B,S}$, then $\chi_{A \cup B,S} = f \vee g$.

(32) Let S be a non empty finite set, D be an equivalent distributed sample spaces family of S, A, B be sets, and f, g be functions from S into *Boolean*. If $A \subseteq S$ and $B \subseteq S$ and A misses B and $f = \chi_{A,S}$ and $g = \chi_{B,S}$, then $\mathrm{Prob}(f \wedge g, D) = 0$.

Let S be a non empty finite set and let D be an equivalent distributed sample spaces family of S. A function from *Boolean*^{S} into $\mathbb{R}$ is said to be a probability on D if:

(Def. 8) For every element j_1 of *Boolean*^{S} holds $\mathrm{it}(j_1) = \mathrm{Prob}(j_1, D)$.

Let S be a non empty finite set and let D be an equivalent distributed sample spaces family of S. The trivial probability of D yields a probability on the trivial σ-field of S and is defined by the condition (Def. 9).

(Def. 9) Let x be a set. Suppose $x \in$ the trivial σ-field of S. Then there exists a function c_1 from S into *Boolean* such that $c_1 = \chi_{x,S}$ and (the trivial probability of D)$(x) = \mathrm{Prob}(c_1, D)$.

3. Sampling and Posterior Probability

Let S be a non empty finite set and let D be an equivalent distributed sample spaces family of S. An element of S is called a sample of D if:

(Def. 10) There exists an element s of D such that $\mathrm{it} \in \mathrm{rng}\, s$.

Let S be a non empty finite set, let D be an equivalent distributed sample spaces family of S, and let x be a sample of D. The functor $\mathrm{Prob}\, x$ yielding a real number is defined as follows:

(Def. 11) $\mathrm{Prob}\, x = \mathrm{Prob}(\text{the membership decision of } \{x\}, D)$.

One can prove the following proposition

(33) Let S be a non empty finite set, D be an equivalent distributed sample spaces family of S, and x be a sample of D. Then $\mathrm{Prob}\, x =$ (the probability finite sequence of D)$(|\bullet : x|_{\mathbb{N}})$.

A non empty subset of S is said to be a sampling RNG of D if:

(Def. 12) There exists a sample x of D such that $x \in$ it.

Let S be a non empty finite set, let D be an equivalent distributed sample spaces family of S, and let X be a sampling RNG of D. The functor $\mathrm{Prob}\, X$ yielding a real number is defined as follows:

(Def. 13) $\mathrm{Prob}\, X = \mathrm{Prob}($the membership decision of X, $D)$.

We now state several propositions:

(34) Let S be a non empty finite set, X be a subset of S, s, t be finite sequences of elements of S, S_2 be a subset of $\mathrm{dom}\, s$, and x be a subset of X. If $S_2 = s^{-1}(X)$ and $t = \mathrm{extract}(s, S_2)$, then $\overline{\overline{s^{-1}(x)}} = \overline{\overline{t^{-1}(x)}}$.

(35) Let S be a non empty finite set, X be a subset of S, s, t be finite sequences of elements of S, S_2 be a subset of $\mathrm{dom}\, s$, and x be a set. If $S_2 = s^{-1}(X)$ and $t = \mathrm{extract}(s, S_2)$ and $x \in X$, then $\mathrm{frequency}(x, s) = \mathrm{frequency}(x, t)$.

(36) Let S be a non empty finite set, D be an element of the distribution family of S, and s be a finite sequence of elements of S. If $s \in D$, then $D =$ the equivalence class of s.

(37) Let S be a non empty finite set, X be a subset of S, and s be a finite sequence of elements of S. Then $s^{-1}(X) =$ the true event of (the membership decision of $X) \cdot s$.

(38) Let S be a non empty finite set, X be a non empty subset of S, D be an equivalent distributed sample spaces family of S, s_1, s_2 be elements of D, t_1, t_2 be finite sequences of elements of S, S_3 be a subset of $\mathrm{dom}\, s_1$, and S_4 be a subset of $\mathrm{dom}\, s_2$. Suppose $S_3 = {s_1}^{-1}(X)$ and $t_1 = \mathrm{extract}(s_1, S_3)$ and $S_4 = {s_2}^{-1}(X)$ and $t_2 = \mathrm{extract}(s_2, S_4)$. Then t_1 and t_2 are probability equivalent.

The conditional subset of X yields an equivalent distributed sample spaces family of S and is defined by the condition (Def. 14).

(Def. 14) There exists an element s of D and there exists a finite sequence t of elements of S and there exists a subset S_2 of $\mathrm{dom}\, s$ such that $S_2 = s^{-1}(X)$ and $t = \mathrm{extract}(s, S_2)$ and $t \in$ the conditional subset of X.

Let f be a function from S into *Boolean*. The functor $\mathrm{Prob}(f, X)$ yielding a real number is defined by:

(Def. 15) $\mathrm{Prob}(f, X) = \mathrm{Prob}(f,$ the conditional subset of $X)$.

One can prove the following proposition

(39) Let S be a non empty finite set, D be an equivalent distributed sample spaces family of S, X be a sampling RNG of D, and f be a function from S into *Boolean*. Then $\mathrm{Prob}(f, X) \cdot \mathrm{Prob}\, X = \mathrm{Prob}(f \wedge$ the membership decision of X, $D)$.

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Received July 4, 2012