

Cayley-Dickson Construction¹

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Summary. Cayley-Dickson construction produces a sequence of normed algebras over real numbers. Its consequent applications result in complex numbers, quaternions, octonions, etc. In this paper we formalize the construction and prove its basic properties.

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The notation and terminology used here have been introduced in the following papers: [22], [12], [3], [1], [9], [8], [16], [13], [4], [5], [19], [15], [17], [14], [2], [6], [23], [20], [18], [21], [10], [11], and [7].

1. PRELIMINARIES

We use the following convention: u, v, x, y, z, X, Y are sets and r, s are real numbers.

One can prove the following proposition

- (1) For all real numbers a, b, c, d holds $(a + b)^2 + (c + d)^2 \leq (\sqrt{a^2 + c^2} + \sqrt{b^2 + d^2})^2$.

Let X be a non trivial real normed space and let x be a non zero element of X . One can verify that $\|x\|$ is positive.

Let c be a non zero complex number. Note that c^2 is non zero.

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Let x be a non empty set. Observe that $\langle x \rangle$ is non-empty.

Let us note that there exists a finite 0-sequence which is non-empty.

Let f, g be non-empty finite 0-sequences. Observe that $f \wedge g$ is non-empty.

Let x, y be non empty sets. One can verify that $\langle x, y \rangle$ is non-empty.

The following propositions are true:

- (2) If $\langle u \rangle = \langle x \rangle$, then $u = x$.
- (3) If $\langle u, v \rangle = \langle x, y \rangle$, then $u = x$ and $v = y$.
- (4) If $x \in X$, then $\langle x \rangle \in \prod \langle X \rangle$.
- (5) If $z \in \prod \langle X \rangle$, then there exists x such that $x \in X$ and $z = \langle x \rangle$.
- (6) If $x \in X$ and $y \in Y$, then $\langle x, y \rangle \in \prod \langle X, Y \rangle$.
- (7) If $z \in \prod \langle X, Y \rangle$, then there exist x, y such that $x \in X$ and $y \in Y$ and $z = \langle x, y \rangle$.

Let D be a set. The functor binop D yielding a binary operation on D is defined by:

(Def. 1) $\text{binop } D = D \times D \mapsto \text{the element of } D$.

Let D be a set. Observe that binop D is associative and commutative.

Let D be a set. One can verify that there exists a binary operation on D which is associative and commutative.

2. CONJUNCTIVE NORMED SPACES

We introduce conjunctive normed algebra structures which are extensions of normed algebra structures and are systems

$\langle \text{a carrier, a multiplication, an addition, an external multiplication, a one, a zero, a norm, a conjugate} \rangle$,

where the carrier is a set, the multiplication and the addition are binary operations on the carrier, the external multiplication is a function from $\mathbb{R} \times$ the carrier into the carrier, the one and the zero are elements of the carrier, the norm is a function from the carrier into \mathbb{R} , and the conjugate is a function from the carrier into the carrier.

Let us observe that there exists a conjunctive normed algebra structure which is non trivial and strict.

We use the following convention: N is a non empty conjunctive normed algebra structure and a, a_1, a_2, b, b_1, b_2 are elements of N .

Let N be a non empty conjunctive normed algebra structure and let a be an element of N . The functor \bar{a} yields an element of N and is defined as follows:

(Def. 2) $\bar{a} = (\text{the conjugate of } N)(a)$.

Let N be a non empty conjunctive normed algebra structure and let a be an element of N . We say that a is properly conjugated if and only if:

- (Def. 3)(i) $\bar{a} \cdot a = \|a\|^2 \cdot 1_N$ if a is non zero,
 (ii) \bar{a} is zero, otherwise.

Let N be a non empty conjunctive normed algebra structure. We say that N is properly conjugated if and only if:

- (Def. 4) Every element of N is properly conjugated.

We say that N is additively conjugative if and only if:

- (Def. 5) For all elements a, b of N holds $\overline{a + b} = \bar{a} + \bar{b}$.

We say that N is norm-wise conjugative if and only if:

- (Def. 6) For every element a of N holds $\|\bar{a}\| = \|a\|$.

We say that N is scalar-wise conjugative if and only if:

- (Def. 7) For every real number r and for every element a of N holds $r \cdot \bar{a} = \bar{r \cdot a}$.

Let D be a real-membered set, let a, m be binary operations on D , let M be a function from $\mathbb{R} \times D$ into D , let O, Z be elements of D , let n be a function from D into \mathbb{R} , and let c be a function from D into D . Observe that $\langle D, m, a, M, O, Z, n, c \rangle$ is real-membered.

Let D be a set, let a be an associative binary operation on D , let m be a binary operation on D , let M be a function from $\mathbb{R} \times D$ into D , let O, Z be elements of D , let n be a function from D into \mathbb{R} , and let c be a function from D into D . Observe that $\langle D, m, a, M, O, Z, n, c \rangle$ is add-associative.

Let D be a set, let a be a commutative binary operation on D , let m be a binary operation on D , let M be a function from $\mathbb{R} \times D$ into D , let O, Z be elements of D , let n be a function from D into \mathbb{R} , and let c be a function from D into D . Observe that $\langle D, m, a, M, O, Z, n, c \rangle$ is Abelian.

Let D be a set, let a be a binary operation on D , let m be an associative binary operation on D , let M be a function from $\mathbb{R} \times D$ into D , let O, Z be elements of D , let n be a function from D into \mathbb{R} , and let c be a function from D into D . One can verify that $\langle D, m, a, M, O, Z, n, c \rangle$ is associative.

Let D be a set, let a be a binary operation on D , let m be a commutative binary operation on D , let M be a function from $\mathbb{R} \times D$ into D , let O, Z be elements of D , let n be a function from D into \mathbb{R} , and let c be a function from D into D . One can check that $\langle D, m, a, M, O, Z, n, c \rangle$ is commutative.

The strict conjunctive normed algebra structure N-Real is defined by:

- (Def. 8) N-Real = $\langle \mathbb{R}, \cdot_{\mathbb{R}}, +_{\mathbb{R}}, \cdot_{\mathbb{R}}, 1(\in \mathbb{R}), 0(\in \mathbb{R}), |\square|_{\mathbb{R}}, \text{id}_{\mathbb{R}} \rangle$.

Let us observe that N-Real is non degenerated, real-membered, add-associative, Abelian, associative, and commutative. Let a, b be elements of N-Real and r, s be real numbers. We identify $r + s$ with $a + b$ where $a = r$ and $b = s$. We identify $r \cdot s$ with $a \cdot b$ where $a = r$ and $b = s$.

One can check the following observations:

- * every Abelian non empty additive magma which is right add-cancelable is also left add-cancelable,

- * every Abelian non empty additive magma which is left add-cancelable is also right add-cancelable,
- * every Abelian non empty additive loop structure which is left complementable is also right complementable,
- * every Abelian commutative non empty double loop structure which is left distributive is also right distributive,
- * every Abelian commutative non empty double loop structure which is right distributive is also left distributive,
- * every commutative non empty multiplicative loop with zero structure which is almost left invertible is also almost right invertible,
- * every commutative non empty multiplicative loop with zero structure which is almost right invertible is also almost left invertible,
- * every commutative non empty multiplicative loop with zero structure which is almost right cancelable is also almost left cancelable,
- * every commutative non empty multiplicative loop with zero structure which is almost left cancelable is also almost right cancelable,
- * every commutative non empty multiplicative magma which is right mult-cancelable is also left mult-cancelable, and
- * every commutative non empty multiplicative magma which is left mult-cancelable is also right mult-cancelable.

One can verify that N-Real is right complementable and right add-cancelable.

We identify $-r$ with $-a$ where $a = r$.

We identify $r - s$ with $a - b$ where $a = r$ and $b = s$.

We identify $r \cdot s$ with $r \cdot a$ where $a = s$.

We identify $|a|$ with $\|a\|$.

The following proposition is true

- (8) For every element a of N-Real holds $a \cdot a = \|a\|^2$.

Let us observe that \bar{a} reduces to a .

One can verify that N-Real is reflexive, discernible, well unital, real normed space-like, right zeroed, right distributive, vector associative, vector distributive, scalar distributive, scalar associative, scalar unital, Banach Algebra-like1, Banach Algebra-like2, Banach Algebra-like3, almost left invertible, almost left cancelable, properly conjugated, additively conjugative, norm-wise conjugative, and scalar-wise conjugative.

One can verify that there exists a non empty conjunctive normed algebra structure which is strict, non degenerated, real-membered, reflexive, discernible, zeroed, complementable, add-associative, Abelian, associative, commutative, distributive, well unital, add-cancelable, vector associative, vector distributive, scalar distributive, scalar associative, scalar unital, Banach Algebra-like1, Banach Algebra-like2, Banach Algebra-like3, properly conjugated, additively con-

jugative, norm-wise conjugative, scalar-wise conjugative, almost left invertible, almost left cancelable, and real normed space-like.

One can check that $0_{N\text{-Real}}$ is non left invertible and non right invertible.

We identify r^{-1} with a^{-1} where $a = r$.

Let X be a discernible non trivial conjunctive normed algebra structure and let x be a non zero element of X . One can check that $\|x\|$ is non zero.

Let us mention that every non zero element of $N\text{-Real}$ is non empty.

Let us observe that every non zero element of $N\text{-Real}$ is mult-cancelable.

Let N be a properly conjugated non empty conjunctive normed algebra structure. Observe that every element of N is properly conjugated.

Let N be a properly conjugated non empty conjunctive normed algebra structure and let a be a zero element of N . Observe that \bar{a} is zero.

Let us observe that $\overline{0_N}$ reduces to 0_N .

Let N be a properly conjugated discernible add-associative right zeroed right complementable left distributive scalar distributive scalar associative scalar unital vector distributive non degenerated conjunctive normed algebra structure and let a be a non zero element of N . Note that \bar{a} is non zero.

The following propositions are true:

- (9) Suppose that N is add-associative, right zeroed, right complementable, properly conjugated, reflexive, scalar distributive, scalar unital, vector distributive, and left distributive. Let given a . Then $\bar{a} \cdot a = \|a\|^2 \cdot 1_N$.

Let N be left unital Banach Algebra-like2 almost right cancelable properly conjugated scalar unital nonempty conjunctive normed algebra structure.

Let us observe that $\bar{\bar{a}}$ reduces to a .

Let N be right unital Banach Algebra-like2 almost right cancelable properly conjugated scalar unital nonempty conjunctive normed algebra structure. Let us observe that $\overline{1_N}$ reduces to 1_N .

- (10) Suppose that N is properly conjugated, reflexive, discernible, real normed space-like, vector distributive, scalar distributive, scalar associative, scalar unital, Abelian, add-associative, right zeroed, right complementable, associative, distributive, well unital, non degenerated, and almost left invertible. Then $\overline{-a} = -\bar{a}$.

- (11) Suppose that N is properly conjugated, reflexive, discernible, real normed space-like, vector distributive, scalar distributive, scalar associative, scalar unital, Abelian, add-associative, right zeroed, right complementable, associative, distributive, well unital, non degenerated, almost left invertible, and additively conjugative. Then $\overline{a - b} = \bar{a} - \bar{b}$.

3. CAYLEY-DICKSON CONSTRUCTION

Let N be a non empty conjunctive normed algebra structure. The functor Cayley-Dickson N yielding a strict conjunctive normed algebra structure is defined by the conditions (Def. 9).

- (Def. 9)(i) The carrier of Cayley-Dickson $N = \coprod$ (the carrier of N , the carrier of N),
- (ii) the zero of Cayley-Dickson $N = \langle 0_N, 0_N \rangle$,
- (iii) the one of Cayley-Dickson $N = \langle 1_N, 0_N \rangle$,
- (iv) for all elements a_1, a_2, b_1, b_2 of N holds (the addition of Cayley-Dickson N)($\langle a_1, b_1 \rangle, \langle a_2, b_2 \rangle$) = $\langle a_1 + a_2, b_1 + b_2 \rangle$ and (the multiplication of Cayley-Dickson N)($\langle a_1, b_1 \rangle, \langle a_2, b_2 \rangle$) = $\langle a_1 \cdot a_2 - \overline{b_2} \cdot b_1, b_2 \cdot a_1 + b_1 \cdot \overline{a_2} \rangle$,
- (v) for every real number r and for all elements a, b of N holds (the external multiplication of Cayley-Dickson N)($r, \langle a, b \rangle$) = $\langle r \cdot a, r \cdot b \rangle$, and
- (vi) for all elements a, b of N holds (the norm of Cayley-Dickson N)($\langle a, b \rangle$) = $\sqrt{\|a\|^2 + \|b\|^2}$ and (the conjugate of Cayley-Dickson N)($\langle a, b \rangle$) = $\langle \overline{a}, -b \rangle$.

In the sequel c, c_1, c_2 are elements of Cayley-Dickson N .

Let N be a non empty conjunctive normed algebra structure. Note that Cayley-Dickson N is non empty.

We now state two propositions:

- (12) There exist elements a, b of N such that $c = \langle a, b \rangle$.
- (13) For every element c of Cayley-Dickson Cayley-Dickson N there exist a_1, b_1, a_2, b_2 such that $c = \langle \langle a_1, b_1 \rangle, \langle a_2, b_2 \rangle \rangle$.

Let us consider N, a, b . Then $\langle a, b \rangle$ is an element of Cayley-Dickson N .

Let us consider N and let a, b be zero elements of N . Observe that $\langle a, b \rangle$ is zero.

Let N be a non degenerated non empty conjunctive normed algebra structure, let a be a non zero element of N , and let b be an element of N . One can check that $\langle a, b \rangle$ is non zero.

Let N be a reflexive non empty conjunctive normed algebra structure. Note that Cayley-Dickson N is reflexive.

Let N be a discernible non empty conjunctive normed algebra structure. Observe that Cayley-Dickson N is discernible.

We now state a number of propositions:

- (14) If a is left complementable and b is left complementable, then $\langle a, b \rangle$ is left complementable.
- (15) If $\langle a, b \rangle$ is left complementable, then a is left complementable and b is left complementable.
- (16) If a is right complementable and b is right complementable, then $\langle a, b \rangle$ is right complementable.

- (17) If $\langle a, b \rangle$ is right complementable, then a is right complementable and b is right complementable.
- (18) If a is complementable and b is complementable, then $\langle a, b \rangle$ is complementable.
- (19) If $\langle a, b \rangle$ is complementable, then a is complementable and b is complementable.
- (20) If a is left add-cancelable and b is left add-cancelable, then $\langle a, b \rangle$ is left add-cancelable.
- (21) If $\langle a, b \rangle$ is left add-cancelable, then a is left add-cancelable and b is left add-cancelable.
- (22) If a is right add-cancelable and b is right add-cancelable, then $\langle a, b \rangle$ is right add-cancelable.
- (23) If $\langle a, b \rangle$ is right add-cancelable, then a is right add-cancelable and b is right add-cancelable.
- (24) If a is add-cancelable and b is add-cancelable, then $\langle a, b \rangle$ is add-cancelable.
- (25) If $\langle a, b \rangle$ is add-cancelable, then a is add-cancelable and b is add-cancelable.
- (26) If $\langle a, b \rangle$ is left complementable and right add-cancelable, then $-\langle a, b \rangle = \langle -a, -b \rangle$.

Let N be an add-associative non empty conjunctive normed algebra structure. Observe that Cayley-Dickson N is add-associative.

Let N be a right zeroed non empty conjunctive normed algebra structure. Observe that Cayley-Dickson N is right zeroed.

Let N be a left zeroed non empty conjunctive normed algebra structure. One can verify that Cayley-Dickson N is left zeroed.

Let N be a right complementable non empty conjunctive normed algebra structure. One can check that Cayley-Dickson N is right complementable.

Let N be a left complementable non empty conjunctive normed algebra structure. One can check that Cayley-Dickson N is left complementable.

Let N be an Abelian non empty conjunctive normed algebra structure. Observe that Cayley-Dickson N is Abelian.

One can prove the following propositions:

- (27) If N is add-associative, right zeroed, and right complementable, then $-\langle a, b \rangle = \langle -a, -b \rangle$.
- (28) If N is add-associative, right zeroed, and right complementable, then $\langle a_1, b_1 \rangle - \langle a_2, b_2 \rangle = \langle a_1 - a_2, b_1 - b_2 \rangle$.

Let N be a well unital add-associative right zeroed right complementable distributive Banach Algebra-like2 properly conjugated scalar unital almost right cancelable non empty conjunctive normed algebra structure. Observe that

Cayley-Dickson N is well unital.

Let N be a non degenerated non empty conjunctive normed algebra structure. One can check that Cayley-Dickson N is non degenerated.

Let N be an additively conjugative add-associative right zeroed right complementable Abelian non empty conjunctive normed algebra structure. One can verify that Cayley-Dickson N is additively conjugative.

Let N be a norm-wise conjugative reflexive discernible real normed space-like vector distributive scalar distributive scalar associative scalar unital Abelian add-associative right zeroed right complementable non empty conjunctive normed algebra structure. Observe that Cayley-Dickson N is norm-wise conjugative.

Let N be a scalar-wise conjugative add-associative right zeroed right complementable Abelian scalar distributive scalar associative scalar unital vector distributive non empty conjunctive normed algebra structure. One can check that Cayley-Dickson N is scalar-wise conjugative.

Let N be a distributive add-associative right zeroed right complementable Abelian non empty conjunctive normed algebra structure.

Note that Cayley-Dickson N is left distributive.

Let N be a distributive add-associative right zeroed right complementable additively conjugative Abelian non empty conjunctive normed algebra structure. Note that Cayley-Dickson N is right distributive.

Let N be a reflexive discernible real normed space-like vector distributive scalar distributive scalar associative scalar unital Abelian add-associative right zeroed right complementable non empty conjunctive normed algebra structure. One can check that Cayley-Dickson N is real normed space-like.

Let N be a vector distributive non empty conjunctive normed algebra structure. Observe that Cayley-Dickson N is vector distributive.

Let N be a vector associative Banach Algebra-like³ add-associative right zeroed right complementable Abelian scalar distributive scalar associative scalar unital vector distributive non empty conjunctive normed algebra structure. Observe that Cayley-Dickson N is vector associative.

Let N be a scalar distributive non empty conjunctive normed algebra structure. One can verify that Cayley-Dickson N is scalar distributive.

Let N be a scalar associative non empty conjunctive normed algebra structure. Note that Cayley-Dickson N is scalar associative.

Let N be a scalar unital non empty conjunctive normed algebra structure. One can check that Cayley-Dickson N is scalar unital.

Let N be a reflexive Banach Algebra-like² non empty conjunctive normed algebra structure. Observe that Cayley-Dickson N is Banach Algebra-like².

Let N be a Banach Algebra-like³ add-associative right zeroed right complementable Abelian scalar distributive scalar associative scalar unital vector distributive vector associative scalar-wise conjugative non empty conjunctive

normed algebra structure. Observe that Cayley-Dickson N is Banach Algebra-like3.

Next we state the proposition

- (29) Let N be an almost left invertible associative add-associative right zeroed right complementable well unital distributive Abelian scalar distributive scalar associative scalar unital vector distributive vector associative reflexive discernible real normed space-like almost right cancelable properly conjugated additively conjugative Banach Algebra-like2 Banach Algebra-like3 non degenerated conjunctive normed algebra structure and a, b be elements of N . Suppose a is non zero or b is non zero but $\langle a, b \rangle$ is right mult-cancelable and left invertible. Then $\langle a, b \rangle^{-1} = \langle \frac{1}{\|a\|^2 + \|b\|^2} \cdot \bar{a}, \frac{1}{\|a\|^2 + \|b\|^2} \cdot -b \rangle$.

Let N be an add-associative right zeroed right complementable distributive scalar distributive scalar unital vector distributive discernible reflexive properly conjugated non empty conjunctive normed algebra structure. Note that Cayley-Dickson N is properly conjugated.

Let us mention that Cayley-Dickson N-Real is associative and commutative.

The following propositions are true:

- (30) $\langle \langle 0_{N-Real}, 1_{N-Real} \rangle, \langle 0_{N-Real}, 0_{N-Real} \rangle \rangle \cdot \langle \langle 0_{N-Real}, 0_{N-Real} \rangle, \langle 1_{N-Real}, 0_{N-Real} \rangle \rangle = \langle \langle 0_{N-Real}, 0_{N-Real} \rangle, \langle 0_{N-Real}, 1_{N-Real} \rangle \rangle$.
- (31) $\langle \langle 0_{N-Real}, 0_{N-Real} \rangle, \langle 1_{N-Real}, 0_{N-Real} \rangle \rangle \cdot \langle \langle 0_{N-Real}, 1_{N-Real} \rangle, \langle 0_{N-Real}, 0_{N-Real} \rangle \rangle = \langle \langle 0_{N-Real}, 0_{N-Real} \rangle, \langle 0_{N-Real}, -1_{N-Real} \rangle \rangle$.

One can verify that Cayley-Dickson Cayley-Dickson N-Real is associative and non commutative.

We now state four propositions:

- (32) $\langle \langle \langle 0_{N-Real}, 1_{N-Real} \rangle, \langle 0_{N-Real}, 0_{N-Real} \rangle \rangle, \langle \langle 0_{N-Real}, 0_{N-Real} \rangle, \langle 0_{N-Real}, 0_{N-Real} \rangle \rangle \rangle \cdot \langle \langle \langle 0_{N-Real}, 0_{N-Real} \rangle, \langle 1_{N-Real}, 0_{N-Real} \rangle \rangle, \langle \langle 0_{N-Real}, 0_{N-Real} \rangle, \langle 0_{N-Real}, 0_{N-Real} \rangle \rangle \rangle = \langle \langle \langle 0_{N-Real}, 0_{N-Real} \rangle, \langle 0_{N-Real}, 1_{N-Real} \rangle \rangle, \langle \langle 0_{N-Real}, 0_{N-Real} \rangle, \langle 0_{N-Real}, 0_{N-Real} \rangle \rangle \rangle$.
- (33) $\langle \langle \langle 0_{N-Real}, 0_{N-Real} \rangle, \langle 1_{N-Real}, 0_{N-Real} \rangle \rangle, \langle \langle 0_{N-Real}, 0_{N-Real} \rangle, \langle 0_{N-Real}, 0_{N-Real} \rangle \rangle \rangle \cdot \langle \langle \langle 0_{N-Real}, 1_{N-Real} \rangle, \langle 0_{N-Real}, 0_{N-Real} \rangle \rangle, \langle \langle 0_{N-Real}, 0_{N-Real} \rangle, \langle 0_{N-Real}, 0_{N-Real} \rangle \rangle \rangle = \langle \langle \langle 0_{N-Real}, 0_{N-Real} \rangle, \langle 0_{N-Real}, -1_{N-Real} \rangle \rangle, \langle \langle 0_{N-Real}, 0_{N-Real} \rangle, \langle 0_{N-Real}, 0_{N-Real} \rangle \rangle \rangle$.
- (34) $\langle \langle \langle 0_{N-Real}, 1_{N-Real} \rangle, \langle 0_{N-Real}, 0_{N-Real} \rangle \rangle, \langle \langle 0_{N-Real}, 0_{N-Real} \rangle, \langle 0_{N-Real}, 0_{N-Real} \rangle \rangle \rangle \cdot \langle \langle \langle 0_{N-Real}, 0_{N-Real} \rangle, \langle 1_{N-Real}, 0_{N-Real} \rangle \rangle, \langle \langle 0_{N-Real}, 0_{N-Real} \rangle, \langle 0_{N-Real}, 0_{N-Real} \rangle \rangle \rangle \cdot \langle \langle \langle 0_{N-Real}, 0_{N-Real} \rangle, \langle 0_{N-Real}, 0_{N-Real} \rangle \rangle, \langle \langle 0_{N-Real}, 1_{N-Real} \rangle, \langle 0_{N-Real}, 0_{N-Real} \rangle \rangle \rangle = \langle \langle \langle 0_{N-Real}, 0_{N-Real} \rangle, \langle 0_{N-Real}, 0_{N-Real} \rangle \rangle, \langle \langle 0_{N-Real}, 0_{N-Real} \rangle, \langle -1_{N-Real}, 0_{N-Real} \rangle \rangle \rangle$.
- (35) $\langle \langle \langle 0_{N-Real}, 1_{N-Real} \rangle, \langle 0_{N-Real}, 0_{N-Real} \rangle \rangle, \langle \langle 0_{N-Real}, 0_{N-Real} \rangle, \langle 0_{N-Real}, 0_{N-Real} \rangle \rangle \rangle \cdot \langle \langle \langle \langle 0_{N-Real}, 0_{N-Real} \rangle, \langle 1_{N-Real}, 0_{N-Real} \rangle \rangle, \langle \langle 0_{N-Real}, 0_{N-Real} \rangle, \langle 0_{N-Real}, 0_{N-Real} \rangle \rangle \rangle \cdot \langle \langle \langle 0_{N-Real}, 0_{N-Real} \rangle, \langle 0_{N-Real}, 0_{N-Real} \rangle \rangle, \langle \langle 0_{N-Real}, 1_{N-Real} \rangle, \langle 0_{N-Real}, 0_{N-Real} \rangle \rangle \rangle \rangle = \langle \langle \langle 0_{N-Real}, 0_{N-Real} \rangle, \langle 0_{N-Real}, 0_{N-Real} \rangle \rangle, \langle \langle 0_{N-Real}, 0_{N-Real} \rangle, \langle 1_{N-Real}, 0_{N-Real} \rangle \rangle \rangle$.

One can check that Cayley-Dickson Cayley-Dickson Cayley-Dickson N-Real is non associative and non commutative.

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