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Cayley-Dickson Construction[1]

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Summary. Cayley-Dickson construction produces a sequence of normed algebras over real numbers. Its consequent applications result in complex numbers, quaternions, octonions, etc. In this paper we formalize the construction and prove its basic properties.

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The notation and terminology used here have been introduced in the following papers: [22], [12], [3], [1], [9], [8], [16], [13], [4], [5], [19], [15], [17], [14], [2], [6], [23], [20], [18], [21], [10], [11], and [7].

1. Preliminaries

We use the following convention: u, v, x, y, z, X, Y are sets and r, s are real numbers.

One can prove the following proposition

(1) For all real numbers a, b, c, d holds $(a+b)^{\mathbf{2}} + (c+d)^{\mathbf{2}} \leq (\sqrt{a^{\mathbf{2}} + c^{\mathbf{2}}} + \sqrt{b^{\mathbf{2}} + d^{\mathbf{2}}})^{\mathbf{2}}$.

Let X be a non trivial real normed space and let x be a non zero element of X. One can verify that $\|x\|$ is positive.

Let c be a non zero complex number. Note that $c^{\mathbf{2}}$ is non zero.

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Let x be a non empty set. Observe that $\langle x \rangle$ is non-empty.

Let us note that there exists a finite 0-sequence which is non-empty.

Let f, g be non-empty finite 0-sequences. Observe that $f \frown g$ is non-empty.

Let x, y be non empty sets. One can verify that $\langle x, y \rangle$ is non-empty.

The following propositions are true:

(2) If $\langle u \rangle = \langle x \rangle$, then $u = x$.

(3) If $\langle u, v \rangle = \langle x, y \rangle$, then $u = x$ and $v = y$.

(4) If $x \in X$, then $\langle x \rangle \in \prod \langle X \rangle$.

(5) If $z \in \prod \langle X \rangle$, then there exists x such that $x \in X$ and $z = \langle x \rangle$.

(6) If $x \in X$ and $y \in Y$, then $\langle x, y \rangle \in \prod \langle X, Y \rangle$.

(7) If $z \in \prod \langle X, Y \rangle$, then there exist x, y such that $x \in X$ and $y \in Y$ and $z = \langle x, y \rangle$.

Let D be a set. The functor $\operatorname{binop} D$ yielding a binary operation on D is defined by:

(Def. 1) $\operatorname{binop} D = D \times D \longmapsto$ the element of D.

Let D be a set. Observe that $\operatorname{binop} D$ is associative and commutative.

Let D be a set. One can verify that there exists a binary operation on D which is associative and commutative.

2. Conjunctive Normed Spaces

We introduce conjunctive normed algebra structures which are extensions of normed algebra structures and are systems

⟨ a carrier, a multiplication, an addition, an external multiplication, a one, a zero, a norm, a conjugate ⟩,

where the carrier is a set, the multiplication and the addition are binary operations on the carrier, the external multiplication is a function from $\mathbb{R} \times$ the carrier into the carrier, the one and the zero are elements of the carrier, the norm is a function from the carrier into $\mathbb{R}$, and the conjugate is a function from the carrier into the carrier.

Let us observe that there exists a conjunctive normed algebra structure which is non trivial and strict.

We use the following convention: N is a non empty conjunctive normed algebra structure and a, a_1, a_2, b, b_1, b_2 are elements of N.

Let N be a non empty conjunctive normed algebra structure and let a be an element of N. The functor $\overline{a}$ yields an element of N and is defined as follows:

(Def. 2) $\overline{a} = (\text{the conjugate of } N)(a)$.

Let N be a non empty conjunctive normed algebra structure and let a be an element of N. We say that a is properly conjugated if and only if:

(Def. 3)(i) $\overline{a} \cdot a = \|a\|^{\mathbf{2}} \cdot 1_N$ if a is non zero,

(ii) $\overline{a}$ is zero, otherwise.

Let N be a non empty conjunctive normed algebra structure. We say that N is properly conjugated if and only if:

(Def. 4) Every element of N is properly conjugated.

We say that N is additively conjugative if and only if:

(Def. 5) For all elements a, b of N holds $\overline{a+b} = \overline{a} + \overline{b}$.

We say that N is norm-wise conjugative if and only if:

(Def. 6) For every element a of N holds $\|\overline{a}\| = \|a\|$.

We say that N is scalar-wise conjugative if and only if:

(Def. 7) For every real number r and for every element a of N holds $r \cdot \overline{a} = \overline{r \cdot a}$.

Let D be a real-membered set, let a, m be binary operations on D, let M be a function from $\mathbb{R} \times D$ into D, let O, Z be elements of D, let n be a function from D into $\mathbb{R}$, and let c be a function from D into D. Observe that $\langle D, m, a, M, O, Z, n, c\rangle$ is real-membered.

Let D be a set, let a be an associative binary operation on D, let m be a binary operation on D, let M be a function from $\mathbb{R} \times D$ into D, let O, Z be elements of D, let n be a function from D into $\mathbb{R}$, and let c be a function from D into D. Observe that $\langle D, m, a, M, O, Z, n, c\rangle$ is add-associative.

Let D be a set, let a be a commutative binary operation on D, let m be a binary operation on D, let M be a function from $\mathbb{R} \times D$ into D, let O, Z be elements of D, let n be a function from D into $\mathbb{R}$, and let c be a function from D into D. Observe that $\langle D, m, a, M, O, Z, n, c\rangle$ is Abelian.

Let D be a set, let a be a binary operation on D, let m be an associative binary operation on D, let M be a function from $\mathbb{R} \times D$ into D, let O, Z be elements of D, let n be a function from D into $\mathbb{R}$, and let c be a function from D into D. One can verify that $\langle D, m, a, M, O, Z, n, c\rangle$ is associative.

Let D be a set, let a be a binary operation on D, let m be a commutative binary operation on D, let M be a function from $\mathbb{R} \times D$ into D, let O, Z be elements of D, let n be a function from D into $\mathbb{R}$, and let c be a function from D into D. One can check that $\langle D, m, a, M, O, Z, n, c\rangle$ is commutative.

The strict conjunctive normed algebra structure N-Real is defined by:

(Def. 8) $\text{N-Real} = \langle \mathbb{R}, \cdot_{\mathbb{R}}, +_{\mathbb{R}}, \cdot_{\mathbb{R}}, 1(\in \mathbb{R}), 0(\in \mathbb{R}), |\square|_{\mathbb{R}}, \mathrm{id}_{\mathbb{R}}\rangle$.

Let us observe that N-Real is non degenerated, real-membered, add-associative, Abelian, associative, and commutative. Let a, b be elements of N-Real and r, s be real numbers. We identify $r+s$ with $a+b$ where $a = r$ and $b = s$. We identify $r \cdot s$ with $a \cdot b$ where $a = r$ and $b = s$.

One can check the following observations:

* every Abelian non empty additive magma which is right add-cancelable is also left add-cancelable,

* every Abelian non empty additive magma which is left add-cancelable is also right add-cancelable,
* every Abelian non empty additive loop structure which is left complementable is also right complementable,
* every Abelian commutative non empty double loop structure which is left distributive is also right distributive,
* every Abelian commutative non empty double loop structure which is right distributive is also left distributive,
* every commutative non empty multiplicative loop with zero structure which is almost left invertible is also almost right invertible,
* every commutative non empty multiplicative loop with zero structure which is almost right invertible is also almost left invertible,
* every commutative non empty multiplicative loop with zero structure which is almost right cancelable is also almost left cancelable,
* every commutative non empty multiplicative loop with zero structure which is almost left cancelable is also almost right cancelable,
* every commutative non empty multiplicative magma which is right mult-cancelable is also left mult-cancelable, and
* every commutative non empty multiplicative magma which is left mult-cancelable is also right mult-cancelable.

One can verify that N-Real is right complementable and right add-cancelable.

We identify $-r$ with $-a$ where $a = r$.

We identify $r - s$ with $a - b$ where $a = r$ and $b = s$.

We identify $r \cdot s$ with $r \cdot a$ where $a = s$.

We identify $|a|$ with $\|a\|$.

The following proposition is true

(8) For every element a of N-Real holds $a \cdot a = \|a\|^{\mathbf{2}}$.

Let us observe that $\overline{a}$ reduces to a.

One can verify that N-Real is reflexive, discernible, well unital, real normed space-like, right zeroed, right distributive, vector associative, vector distributive, scalar distributive, scalar associative, scalar unital, Banach Algebra-like1, Banach Algebra-like2, Banach Algebra-like3, almost left invertible, almost left cancelable, properly conjugated, additively conjugative, norm-wise conjugative, and scalar-wise conjugative.

One can verify that there exists a non empty conjunctive normed algebra structure which is strict, non degenerated, real-membered, reflexive, discernible, zeroed, complementable, add-associative, Abelian, associative, commutative, distributive, well unital, add-cancelable, vector associative, vector distributive, scalar distributive, scalar associative, scalar unital, Banach Algebra-like1, Banach Algebra-like2, Banach Algebra-like3, properly conjugated, additively conjugative, norm-wise conjugative, scalar-wise conjugative, almost left invertible, almost left cancelable, and real normed space-like.

One can check that $0_{\text{N-Real}}$ is non left invertible and non right invertible.

We identify r^{-1} with a^{-1} where $a = r$.

Let X be a discernible non trivial conjunctive normed algebra structure and let x be a non zero element of X. One can check that $\|x\|$ is non zero.

Let us mention that every non zero element of N-Real is non empty.

Let us observe that every non zero element of N-Real is mult-cancelable.

Let N be a properly conjugated non empty conjunctive normed algebra structure. Observe that every element of N is properly conjugated.

Let N be a properly conjugated non empty conjunctive normed algebra structure and let a be a zero element of N. Observe that $\overline{a}$ is zero.

Let us observe that $\overline{0_N}$ reduces to 0_N.

Let N be a properly conjugated discernible add-associative right zeroed right complementable left distributive scalar distributive scalar associative scalar unital vector distributive non degenerated conjunctive normed algebra structure and let a be a non zero element of N. Note that $\overline{a}$ is non zero.

The following propositions are true:

(9) Suppose that N is add-associative, right zeroed, right complementable, properly conjugated, reflexive, scalar distributive, scalar unital, vector distributive, and left distributive. Let given a. Then $\overline{a} \cdot a = \|a\|^{\mathbf{2}} \cdot 1_N$.

Let N be left unital Banach Algebra-like2 almost right cancelable properly conjugated scalar unital nonempty conjunctive normed algebra structure. Let us observe that $\overline{\overline{a}}$ reduces to a.

Let N be right unital Banach Algebra-like2 almost right cancelable properly conjugated scalar unital nonempty conjunctive normed algebra structure. Let us observe that $\overline{1_N}$ reduces to 1_N.

(10) Suppose that N is properly conjugated, reflexive, discernible, real normed space-like, vector distributive, scalar distributive, scalar associative, scalar unital, Abelian, add-associative, right zeroed, right complementable, associative, distributive, well unital, non degenerated, and almost left invertible. Then $\overline{-a} = -\overline{a}$.

(11) Suppose that N is properly conjugated, reflexive, discernible, real normed space-like, vector distributive, scalar distributive, scalar associative, scalar unital, Abelian, add-associative, right zeroed, right complementable, associative, distributive, well unital, non degenerated, almost left invertible, and additively conjugative. Then $\overline{a - b} = \overline{a} - \overline{b}$.

3. Cayley-Dickson Construction

Let N be a non empty conjunctive normed algebra structure. The functor Cayley-Dickson N yielding a strict conjunctive normed algebra structure is defined by the conditions (Def. 9).

(Def. 9)(i) The carrier of Cayley-Dickson $N = \prod\langle$the carrier of N, the carrier of $N\rangle$,

(ii) the zero of Cayley-Dickson $N = \langle 0_N, 0_N\rangle$,

(iii) the one of Cayley-Dickson $N = \langle 1_N, 0_N\rangle$,

(iv) for all elements a_1, a_2, b_1, b_2 of N holds (the addition of Cayley-Dickson N)$(\langle a_1, b_1\rangle, \langle a_2, b_2\rangle) = \langle a_1 + a_2, b_1 + b_2\rangle$ and (the multiplication of Cayley-Dickson N)$(\langle a_1, b_1\rangle, \langle a_2, b_2\rangle) = \langle a_1 \cdot a_2 - \overline{b_2} \cdot b_1, b_2 \cdot a_1 + b_1 \cdot \overline{a_2}\rangle$,

(v) for every real number r and for all elements a, b of N holds (the external multiplication of Cayley-Dickson N)$(r, \langle a, b\rangle) = \langle r \cdot a, r \cdot b\rangle$, and

(vi) for all elements a, b of N holds (the norm of Cayley-Dickson N)$(\langle a, b\rangle) = \sqrt{\|a\|^{\mathbf{2}} + \|b\|^{\mathbf{2}}}$ and (the conjugate of Cayley-Dickson N)$(\langle a, b\rangle) = \langle \overline{a}, -b\rangle$.

In the sequel c, c_1, c_2 are elements of Cayley-Dickson N.

Let N be a non empty conjunctive normed algebra structure. Note that Cayley-Dickson N is non empty.

We now state two propositions:

(12) There exist elements a, b of N such that $c = \langle a, b\rangle$.

(13) For every element c of Cayley-Dickson Cayley-Dickson N there exist a_1, b_1, a_2, b_2 such that $c = \langle\langle a_1, b_1\rangle, \langle a_2, b_2\rangle\rangle$.

Let us consider N, a, b. Then $\langle a, b\rangle$ is an element of Cayley-Dickson N.

Let us consider N and let a, b be zero elements of N. Observe that $\langle a, b\rangle$ is zero.

Let N be a non degenerated non empty conjunctive normed algebra structure, let a be a non zero element of N, and let b be an element of N. One can check that $\langle a, b\rangle$ is non zero.

Let N be a reflexive non empty conjunctive normed algebra structure. Note that Cayley-Dickson N is reflexive.

Let N be a discernible non empty conjunctive normed algebra structure. Observe that Cayley-Dickson N is discernible.

We now state a number of propositions:

(14) If a is left complementable and b is left complementable, then $\langle a, b\rangle$ is left complementable.

(15) If $\langle a, b\rangle$ is left complementable, then a is left complementable and b is left complementable.

(16) If a is right complementable and b is right complementable, then $\langle a, b\rangle$ is right complementable.

(17) If $\langle a, b\rangle$ is right complementable, then a is right complementable and b is right complementable.

(18) If a is complementable and b is complementable, then $\langle a, b\rangle$ is complementable.

(19) If $\langle a, b\rangle$ is complementable, then a is complementable and b is complementable.

(20) If a is left add-cancelable and b is left add-cancelable, then $\langle a, b\rangle$ is left add-cancelable.

(21) If $\langle a, b\rangle$ is left add-cancelable, then a is left add-cancelable and b is left add-cancelable.

(22) If a is right add-cancelable and b is right add-cancelable, then $\langle a, b\rangle$ is right add-cancelable.

(23) If $\langle a, b\rangle$ is right add-cancelable, then a is right add-cancelable and b is right add-cancelable.

(24) If a is add-cancelable and b is add-cancelable, then $\langle a, b\rangle$ is add-cancelable.

(25) If $\langle a, b\rangle$ is add-cancelable, then a is add-cancelable and b is add-cancelable.

(26) If $\langle a, b\rangle$ is left complementable and right add-cancelable, then $-\langle a, b\rangle = \langle -a, -b\rangle$.

Let N be an add-associative non empty conjunctive normed algebra structure. Observe that Cayley-Dickson N is add-associative.

Let N be a right zeroed non empty conjunctive normed algebra structure. Observe that Cayley-Dickson N is right zeroed.

Let N be a left zeroed non empty conjunctive normed algebra structure. One can verify that Cayley-Dickson N is left zeroed.

Let N be a right complementable non empty conjunctive normed algebra structure. One can check that Cayley-Dickson N is right complementable.

Let N be a left complementable non empty conjunctive normed algebra structure. One can check that Cayley-Dickson N is left complementable.

Let N be an Abelian non empty conjunctive normed algebra structure. Observe that Cayley-Dickson N is Abelian.

One can prove the following propositions:

(27) If N is add-associative, right zeroed, and right complementable, then $-\langle a, b\rangle = \langle -a, -b\rangle$.

(28) If N is add-associative, right zeroed, and right complementable, then $\langle a_1, b_1\rangle - \langle a_2, b_2\rangle = \langle a_1 - a_2, b_1 - b_2\rangle$.

Let N be a well unital add-associative right zeroed right complementable distributive Banach Algebra-like2 properly conjugated scalar unital almost right cancelable non empty conjunctive normed algebra structure. Observe that

Cayley-Dickson N is well unital.

Let N be a non degenerated non empty conjunctive normed algebra structure. One can check that Cayley-Dickson N is non degenerated.

Let N be an additively conjugative add-associative right zeroed right complementable Abelian non empty conjunctive normed algebra structure. One can verify that Cayley-Dickson N is additively conjugative.

Let N be a norm-wise conjugative reflexive discernible real normed space-like vector distributive scalar distributive scalar associative scalar unital Abelian add-associative right zeroed right complementable non empty conjunctive normed algebra structure. Observe that Cayley-Dickson N is norm-wise conjugative.

Let N be a scalar-wise conjugative add-associative right zeroed right complementable Abelian scalar distributive scalar associative scalar unital vector distributive non empty conjunctive normed algebra structure. One can check that Cayley-Dickson N is scalar-wise conjugative.

Let N be a distributive add-associative right zeroed right complementable Abelian non empty conjunctive normed algebra structure.

Note that Cayley-Dickson N is left distributive.

Let N be a distributive add-associative right zeroed right complementable additively conjugative Abelian non empty conjunctive normed algebra structure. Note that Cayley-Dickson N is right distributive.

Let N be a reflexive discernible real normed space-like vector distributive scalar distributive scalar associative scalar unital Abelian add-associative right zeroed right complementable non empty conjunctive normed algebra structure. One can check that Cayley-Dickson N is real normed space-like.

Let N be a vector distributive non empty conjunctive normed algebra structure. Observe that Cayley-Dickson N is vector distributive.

Let N be a vector associative Banach Algebra-like3 add-associative right zeroed right complementable Abelian scalar distributive scalar associative scalar unital vector distributive non empty conjunctive normed algebra structure. Observe that Cayley-Dickson N is vector associative.

Let N be a scalar distributive non empty conjunctive normed algebra structure. One can verify that Cayley-Dickson N is scalar distributive.

Let N be a scalar associative non empty conjunctive normed algebra structure. Note that Cayley-Dickson N is scalar associative.

Let N be a scalar unital non empty conjunctive normed algebra structure. One can check that Cayley-Dickson N is scalar unital.

Let N be a reflexive Banach Algebra-like2 non empty conjunctive normed algebra structure. Observe that Cayley-Dickson N is Banach Algebra-like2.

Let N be a Banach Algebra-like3 add-associative right zeroed right complementable Abelian scalar distributive scalar associative scalar unital vector distributive vector associative scalar-wise conjugative non empty conjunctive normed algebra structure. Observe that Cayley-Dickson N is Banach Algebra-like3.

Next we state the proposition

(29) Let N be an almost left invertible associative add-associative right zeroed right complementable well unital distributive Abelian scalar distributive scalar associative scalar unital vector distributive vector associative reflexive discernible real normed space-like almost right cancelable properly conjugated additively conjugative Banach Algebra-like2 Banach Algebra-like3 non degenerated conjunctive normed algebra structure and a, b be elements of N. Suppose a is non zero or b is non zero but $\langle a, b\rangle$ is right mult-cancelable and left invertible. Then $\langle a, b\rangle^{-1} = \langle \frac{1}{\|a\|^2+\|b\|^2} \cdot \overline{a}, \frac{1}{\|a\|^2+\|b\|^2} \cdot -b\rangle$.

Let N be an add-associative right zeroed right complementable distributive scalar distributive scalar unital vector distributive discernible reflexive properly conjugated non empty conjunctive normed algebra structure. Note that Cayley-Dickson N is properly conjugated.

Let us mention that Cayley-Dickson N-Real is associative and commutative.

The following propositions are true:

(30) $\langle\langle 0_{\text{N-Real}}, 1_{\text{N-Real}}\rangle, \langle 0_{\text{N-Real}}, 0_{\text{N-Real}}\rangle\rangle \cdot \langle\langle 0_{\text{N-Real}}, 0_{\text{N-Real}}\rangle, \langle 1_{\text{N-Real}}, 0_{\text{N-Real}}\rangle\rangle = \langle\langle 0_{\text{N-Real}}, 0_{\text{N-Real}}\rangle, \langle 0_{\text{N-Real}}, 1_{\text{N-Real}}\rangle\rangle$.

(31) $\langle\langle 0_{\text{N-Real}}, 0_{\text{N-Real}}\rangle, \langle 1_{\text{N-Real}}, 0_{\text{N-Real}}\rangle\rangle \cdot \langle\langle 0_{\text{N-Real}}, 1_{\text{N-Real}}\rangle, \langle 0_{\text{N-Real}}, 0_{\text{N-Real}}\rangle\rangle = \langle\langle 0_{\text{N-Real}}, 0_{\text{N-Real}}\rangle, \langle 0_{\text{N-Real}}, -1_{\text{N-Real}}\rangle\rangle$.

One can verify that Cayley-Dickson Cayley-Dickson N-Real is associative and non commutative.

We now state four propositions:

(32) $\langle\langle\langle 0_{\text{N-Real}}, 1_{\text{N-Real}}\rangle, \langle 0_{\text{N-Real}}, 0_{\text{N-Real}}\rangle\rangle, \langle\langle 0_{\text{N-Real}}, 0_{\text{N-Real}}\rangle, \langle 0_{\text{N-Real}}, 0_{\text{N-Real}}\rangle\rangle\rangle \cdot \langle\langle\langle 0_{\text{N-Real}}, 0_{\text{N-Real}}\rangle, \langle 1_{\text{N-Real}}, 0_{\text{N-Real}}\rangle\rangle, \langle\langle 0_{\text{N-Real}}, 0_{\text{N-Real}}\rangle, \langle 0_{\text{N-Real}}, 0_{\text{N-Real}}\rangle\rangle\rangle = \langle\langle\langle 0_{\text{N-Real}}, 0_{\text{N-Real}}\rangle, \langle 0_{\text{N-Real}}, 1_{\text{N-Real}}\rangle\rangle, \langle\langle 0_{\text{N-Real}}, 0_{\text{N-Real}}\rangle, \langle 0_{\text{N-Real}}, 0_{\text{N-Real}}\rangle\rangle\rangle$.

(33) $\langle\langle\langle 0_{\text{N-Real}}, 0_{\text{N-Real}}\rangle, \langle 1_{\text{N-Real}}, 0_{\text{N-Real}}\rangle\rangle, \langle\langle 0_{\text{N-Real}}, 0_{\text{N-Real}}\rangle, \langle 0_{\text{N-Real}}, 0_{\text{N-Real}}\rangle\rangle\rangle \cdot \langle\langle\langle 0_{\text{N-Real}}, 1_{\text{N-Real}}\rangle, \langle 0_{\text{N-Real}}, 0_{\text{N-Real}}\rangle\rangle, \langle\langle 0_{\text{N-Real}}, 0_{\text{N-Real}}\rangle, \langle 0_{\text{N-Real}}, 0_{\text{N-Real}}\rangle\rangle\rangle = \langle\langle\langle 0_{\text{N-Real}}, 0_{\text{N-Real}}\rangle, \langle 0_{\text{N-Real}}, -1_{\text{N-Real}}\rangle\rangle, \langle\langle 0_{\text{N-Real}}, 0_{\text{N-Real}}\rangle, \langle 0_{\text{N-Real}}, 0_{\text{N-Real}}\rangle\rangle\rangle$.

(34) $\langle\langle\langle 0_{\text{N-Real}}, 1_{\text{N-Real}}\rangle, \langle 0_{\text{N-Real}}, 0_{\text{N-Real}}\rangle\rangle, \langle\langle 0_{\text{N-Real}}, 0_{\text{N-Real}}\rangle, \langle 0_{\text{N-Real}}, 0_{\text{N-Real}}\rangle\rangle\rangle \cdot \langle\langle\langle 0_{\text{N-Real}}, 0_{\text{N-Real}}\rangle, \langle 1_{\text{N-Real}}, 0_{\text{N-Real}}\rangle\rangle, \langle\langle 0_{\text{N-Real}}, 0_{\text{N-Real}}\rangle, \langle 0_{\text{N-Real}}, 0_{\text{N-Real}}\rangle\rangle\rangle \cdot \langle\langle\langle 0_{\text{N-Real}}, 0_{\text{N-Real}}\rangle, \langle 0_{\text{N-Real}}, 0_{\text{N-Real}}\rangle\rangle, \langle\langle 0_{\text{N-Real}}, 1_{\text{N-Real}}\rangle, \langle 0_{\text{N-Real}}, 0_{\text{N-Real}}\rangle\rangle\rangle = \langle\langle\langle 0_{\text{N-Real}}, 0_{\text{N-Real}}\rangle, \langle 0_{\text{N-Real}}, 0_{\text{N-Real}}\rangle\rangle, \langle\langle 0_{\text{N-Real}}, 0_{\text{N-Real}}\rangle, \langle -1_{\text{N-Real}}, 0_{\text{N-Real}}\rangle\rangle\rangle$.

(35) $\langle\langle\langle 0_{\text{N-Real}}, 1_{\text{N-Real}}\rangle, \langle 0_{\text{N-Real}}, 0_{\text{N-Real}}\rangle\rangle, \langle\langle 0_{\text{N-Real}}, 0_{\text{N-Real}}\rangle, \langle 0_{\text{N-Real}}, 0_{\text{N-Real}}\rangle\rangle\rangle \cdot (\langle\langle\langle 0_{\text{N-Real}}, 0_{\text{N-Real}}\rangle, \langle 1_{\text{N-Real}}, 0_{\text{N-Real}}\rangle\rangle, \langle\langle 0_{\text{N-Real}}, 0_{\text{N-Real}}\rangle, \langle 0_{\text{N-Real}}, 0_{\text{N-Real}}\rangle\rangle\rangle \cdot \langle\langle\langle 0_{\text{N-Real}}, 0_{\text{N-Real}}\rangle, \langle 0_{\text{N-Real}}, 0_{\text{N-Real}}\rangle\rangle, \langle\langle 0_{\text{N-Real}}, 1_{\text{N-Real}}\rangle, \langle 0_{\text{N-Real}}, 0_{\text{N-Real}}\rangle\rangle\rangle) = \langle\langle\langle 0_{\text{N-Real}}, 0_{\text{N-Real}}\rangle, \langle 0_{\text{N-Real}}, 0_{\text{N-Real}}\rangle\rangle, \langle\langle 0_{\text{N-Real}}, 0_{\text{N-Real}}\rangle, \langle 1_{\text{N-Real}}, 0_{\text{N-Real}}\rangle\rangle\rangle$.

One can check that Cayley-Dickson Cayley-Dickson Cayley-Dickson N-Real is non associative and non commutative.

References

[1] Grzegorz Bancerek. König's theorem. *Formalized Mathematics*, 1(**3**):589–593, 1990.
[2] Grzegorz Bancerek. The ordinal numbers. *Formalized Mathematics*, 1(**1**):91–96, 1990.
[3] Czesław Byliński. Binary operations. *Formalized Mathematics*, 1(**1**):175–180, 1990.
[4] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.
[5] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.
[6] Czesław Byliński. Partial functions. *Formalized Mathematics*, 1(**2**):357–367, 1990.
[7] Czesław Byliński. Some basic properties of sets. *Formalized Mathematics*, 1(**1**):47–53, 1990.
[8] Agata Darmochwał. Finite sets. *Formalized Mathematics*, 1(**1**):165–167, 1990.
[9] Agata Darmochwał. The Euclidean space. *Formalized Mathematics*, 2(**4**):599–603, 1991.
[10] Agata Darmochwał and Yatsuka Nakamura. Metric spaces as topological spaces – fundamental concepts. *Formalized Mathematics*, 2(**4**):605–608, 1991.
[11] Eugeniusz Kusak, Wojciech Leończuk, and Michał Muzalewski. Abelian groups, fields and vector spaces. *Formalized Mathematics*, 1(**2**):335–342, 1990.
[12] Michał Muzalewski and Wojciech Skaba. From loops to abelian multiplicative groups with zero. *Formalized Mathematics*, 1(**5**):833–840, 1990.
[13] Henryk Oryszczyszyn and Krzysztof Prażmowski. Real functions spaces. *Formalized Mathematics*, 1(**3**):555–561, 1990.
[14] Jan Popiołek. Real normed space. *Formalized Mathematics*, 2(**1**):111–115, 1991.
[15] Yasunari Shidama. The Banach algebra of bounded linear operators. *Formalized Mathematics*, 12(**2**):103–108, 2004.
[16] Andrzej Trybulec. Binary operations applied to functions. *Formalized Mathematics*, 1(**2**):329–334, 1990.
[17] Andrzej Trybulec. On the sets inhabited by numbers. *Formalized Mathematics*, 11(**4**):341–347, 2003.
[18] Andrzej Trybulec and Czesław Byliński. Some properties of real numbers. *Formalized Mathematics*, 1(**3**):445–449, 1990.
[19] Wojciech A. Trybulec. Groups. *Formalized Mathematics*, 1(**5**):821–827, 1990.
[20] Wojciech A. Trybulec. Vectors in real linear space. *Formalized Mathematics*, 1(**2**):291–296, 1990.
[21] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(**1**):67–71, 1990.
[22] Tetsuya Tsunetou, Grzegorz Bancerek, and Yatsuka Nakamura. Zero-based finite sequences. *Formalized Mathematics*, 9(**4**):825–829, 2001.
[23] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1(**1**):73–83, 1990.

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