

# Contracting Mapping on Normed Linear Space<sup>1</sup>

Keiichi Miyajima  
Ibaraki University  
Faculty of Engineering  
Hitachi, Japan

Artur Korniłowicz<sup>2</sup>  
Institute of Informatics  
University of Białystok  
Sosnowa 64, 15-887 Białystok  
Poland

Yasunari Shidama<sup>3</sup>  
Shinshu University  
Nagano, Japan

**Summary.** In this article, we described the contracting mapping on normed linear space. Furthermore, we applied that mapping to ordinary differential equations on real normed space. Our method is based on the one presented by Schwarz [29].

MML identifier: ORDEQ\_01, version: 8.0.01 5.3.1162

The papers [28], [3], [20], [8], [26], [32], [4], [5], [18], [16], [17], [12], [34], [30], [2], [33], [23], [15], [22], [21], [24], [19], [25], [1], [6], [10], [13], [27], [9], [38], [39], [35], [36], [11], [31], [37], [14], and [7] provide the notation and terminology for this paper.

## 1. THE PRINCIPLE OF CONTRACTING MAPPING ON NORMED LINEAR SPACE

We use the following convention:  $n$  denotes a non empty element of  $\mathbb{N}$  and  $a, b, r, t$  denote real numbers.

---

<sup>1</sup>We would like to express our gratitude to Prof. Yatsuka Nakamura.

<sup>2</sup>My work has been supported by the Polish Ministry of Science and Higher Education project “Managing a Large Repository of Computer-verified Mathematical Knowledge” (N N519 385136).

<sup>3</sup>My work has been supported by JSPS KAKENHI 22300285.

Let  $f$  be a function. We say that  $f$  has unique fixpoint if and only if:

- (Def. 1) There exists a set  $x$  such that  $x$  is a fixpoint of  $f$  and for every set  $y$  such that  $y$  is a fixpoint of  $f$  holds  $x = y$ .

Next we state two propositions:

- (1) Every set  $x$  is a fixpoint of  $\{\langle x, x \rangle\}$ .
- (2) For all sets  $x, y, z$  such that  $x$  is a fixpoint of  $\{\langle y, z \rangle\}$  holds  $x = y$ .

Let  $x$  be a set. Observe that  $\{\langle x, x \rangle\}$  has unique fixpoint.

Next we state three propositions:

- (3) Let  $X$  be a real normed space and  $x$  be a point of  $X$ . If for every real number  $e$  such that  $e > 0$  holds  $\|x\| < e$ , then  $x = 0_X$ .
- (4) Let  $X$  be a real normed space and  $x, y$  be points of  $X$ . If for every real number  $e$  such that  $e > 0$  holds  $\|x - y\| < e$ , then  $x = y$ .
- (5) For all real numbers  $K, L, e$  such that  $0 < K < 1$  and  $0 < e$  there exists a natural number  $n$  such that  $|L \cdot K^n| < e$ .

Let  $X$  be a real normed space. Note that every function from  $X$  into  $X$  which is constant is also contraction.

Let  $X$  be a real Banach space. One can verify that every function from  $X$  into  $X$  which is contraction also has unique fixpoint.

One can prove the following three propositions:

- (6) Let  $X$  be a real Banach space and  $f$  be a function from  $X$  into  $X$ . Suppose  $f$  is contraction. Then there exists a point  $x_1$  of  $X$  such that  $f(x_1) = x_1$  and for every point  $x$  of  $X$  such that  $f(x) = x$  holds  $x_1 = x$ .
- (7) Let  $X$  be a real Banach space and  $f$  be a function from  $X$  into  $X$  such that there exists a natural number  $n_0$  such that  $f^{n_0}$  is contraction. Then  $f$  has unique fixpoint.
- (8) Let  $X$  be a real Banach space and  $f$  be a function from  $X$  into  $X$ . Given an element  $n_0$  of  $\mathbb{N}$  such that  $f^{n_0}$  is contraction. Then there exists a point  $x_1$  of  $X$  such that  $f(x_1) = x_1$  and for every point  $x$  of  $X$  such that  $f(x) = x$  holds  $x_1 = x$ .

## 2. THE REAL BANACH SPACE $C([A,B],X)$

We now state the proposition

- (9) Let  $X$  be a non empty closed interval subset of  $\mathbb{R}$ ,  $Y$  be a real normed space, and  $f$  be a continuous partial function from  $\mathbb{R}$  to  $Y$ . If  $\text{dom } f = X$ , then  $f$  is a bounded function from  $X$  into  $Y$ .

Let  $X$  be a non empty closed interval subset of  $\mathbb{R}$  and let  $Y$  be a real normed space. The continuous functions of  $X$  and  $Y$  yields a subset of the set of bounded real sequences from  $X$  into  $Y$  and is defined by the condition (Def. 2).

(Def. 2) Let  $x$  be a set. Then  $x \in$  the continuous functions of  $X$  and  $Y$  if and only if there exists a continuous partial function  $f$  from  $\mathbb{R}$  to  $Y$  such that  $x = f$  and  $\text{dom } f = X$ .

Let  $X$  be a non empty closed interval subset of  $\mathbb{R}$  and let  $Y$  be a real normed space. Note that the continuous functions of  $X$  and  $Y$  is non empty.

Let  $X$  be a non empty closed interval subset of  $\mathbb{R}$  and let  $Y$  be a real normed space. Observe that the continuous functions of  $X$  and  $Y$  is linearly closed.

Let  $X$  be a non empty closed interval subset of  $\mathbb{R}$  and let  $Y$  be a real normed space. The  $\mathbb{R}$ -vector space of continuous functions of  $X$  and  $Y$  yielding a strict real linear space is defined by the condition (Def. 3).

(Def. 3) The  $\mathbb{R}$ -vector space of continuous functions of  $X$  and  $Y =$  (the continuous functions of  $X$  and  $Y$ , Zero(the continuous functions of  $X$  and  $Y$ , the set of bounded real sequences from  $X$  into  $Y$ ), Add(the continuous functions of  $X$  and  $Y$ , the set of bounded real sequences from  $X$  into  $Y$ ), Mult(the continuous functions of  $X$  and  $Y$ , the set of bounded real sequences from  $X$  into  $Y$ )).

Let  $X$  be a non empty closed interval subset of  $\mathbb{R}$  and let  $Y$  be a real normed space. Observe that the  $\mathbb{R}$ -vector space of continuous functions of  $X$  and  $Y$  is Abelian, add-associative, right zeroed, right complementable, vector distributive, scalar distributive, scalar associative, and scalar unital.

One can prove the following three propositions:

(10) Let  $X$  be a non empty closed interval subset of  $\mathbb{R}$ ,  $Y$  be a real normed space,  $f, g, h$  be vectors of the  $\mathbb{R}$ -vector space of continuous functions of  $X$  and  $Y$ , and  $f_g, g_g, h_g$  be continuous partial functions from  $\mathbb{R}$  to  $Y$ . Suppose  $f_g = f$  and  $g_g = g$  and  $h_g = h$  and  $\text{dom } f_g = X$  and  $\text{dom } g_g = X$  and  $\text{dom } h_g = X$ . Then  $h = f + g$  if and only if for every element  $x$  of  $X$  holds  $(h_g)_x = (f_g)_x + (g_g)_x$ .

(11) Let  $X$  be a non empty closed interval subset of  $\mathbb{R}$ ,  $Y$  be a real normed space,  $f, h$  be vectors of the  $\mathbb{R}$ -vector space of continuous functions of  $X$  and  $Y$ , and  $f_g, h_g$  be continuous partial functions from  $\mathbb{R}$  to  $Y$ . Suppose  $f_g = f$  and  $h_g = h$  and  $\text{dom } f_g = X$  and  $\text{dom } h_g = X$ . Then  $h = a \cdot f$  if and only if for every element  $x$  of  $X$  holds  $(h_g)_x = a \cdot (f_g)_x$ .

(12) Let  $X$  be a non empty closed interval subset of  $\mathbb{R}$  and  $Y$  be a real normed space. Then  $0_{\text{the } \mathbb{R}\text{-vector space of continuous functions of } X \text{ and } Y} = X \mapsto 0_Y$ .

Let  $X$  be a non empty closed interval subset of  $\mathbb{R}$  and let  $Y$  be a real normed space. The continuous functions norm of  $X$  and  $Y$  yields a function from the continuous functions of  $X$  and  $Y$  into  $\mathbb{R}$  and is defined as follows:

(Def. 4) The continuous functions norm of  $X$  and  $Y = \text{BdFuncsNorm}(X, Y)$  | the continuous functions of  $X$  and  $Y$ .

Let  $X$  be a non empty closed interval subset of  $\mathbb{R}$ , let  $Y$  be a real normed

space, and let  $f$  be a set. Let us assume that  $f \in$  the continuous functions of  $X$  and  $Y$ . The functor  $\text{modetrans}(f, X, Y)$  yielding a continuous partial function from  $\mathbb{R}$  to  $Y$  is defined by:

(Def. 5)  $\text{modetrans}(f, X, Y) = f$  and  $\text{dom modetrans}(f, X, Y) = X$ .

Let  $X$  be a non empty closed interval subset of  $\mathbb{R}$  and let  $Y$  be a real normed space. The  $\mathbb{R}$ -norm space of continuous functions of  $X$  and  $Y$  yields a strict non empty normed structure and is defined by the condition (Def. 6).

(Def. 6) The  $\mathbb{R}$ -norm space of continuous functions of  $X$  and  $Y = \langle$ the continuous functions of  $X$  and  $Y$ , Zero(the continuous functions of  $X$  and  $Y$ , the set of bounded real sequences from  $X$  into  $Y$ ), Add(the continuous functions of  $X$  and  $Y$ , the set of bounded real sequences from  $X$  into  $Y$ ), Mult(the continuous functions of  $X$  and  $Y$ , the set of bounded real sequences from  $X$  into  $Y$ ), the continuous functions norm of  $X$  and  $Y$  $\rangle$ .

We now state several propositions:

- (13) Let  $X$  be a non empty closed interval subset of  $\mathbb{R}$ ,  $Y$  be a real normed space, and  $f$  be a continuous partial function from  $\mathbb{R}$  to  $Y$ . If  $\text{dom } f = X$ , then  $\text{modetrans}(f, X, Y) = f$ .
- (14) Let  $X$  be a non empty closed interval subset of  $\mathbb{R}$  and  $Y$  be a real normed space. Then  $X \mapsto 0_Y = 0_{\text{the } \mathbb{R}\text{-norm space of continuous functions of } X \text{ and } Y}$ .
- (15) Let  $X$  be a non empty closed interval subset of  $\mathbb{R}$ ,  $Y$  be a real normed space,  $f, g, h$  be points of the  $\mathbb{R}$ -norm space of continuous functions of  $X$  and  $Y$ , and  $f_g, g_g, h_g$  be continuous partial functions from  $\mathbb{R}$  to  $Y$ . Suppose  $f_g = f$  and  $g_g = g$  and  $h_g = h$  and  $\text{dom } f_g = X$  and  $\text{dom } g_g = X$  and  $\text{dom } h_g = X$ . Then  $h = f + g$  if and only if for every element  $x$  of  $X$  holds  $(h_g)_x = (f_g)_x + (g_g)_x$ .
- (16) Let  $X$  be a non empty closed interval subset of  $\mathbb{R}$ ,  $Y$  be a real normed space,  $f, h$  be points of the  $\mathbb{R}$ -norm space of continuous functions of  $X$  and  $Y$ , and  $f_g, h_g$  be continuous partial functions from  $\mathbb{R}$  to  $Y$ . Suppose  $f_g = f$  and  $h_g = h$  and  $\text{dom } f_g = X$  and  $\text{dom } h_g = X$ . Then  $h = a \cdot f$  if and only if for every element  $x$  of  $X$  holds  $(h_g)_x = a \cdot (f_g)_x$ .
- (17) Let  $X$  be a non empty closed interval subset of  $\mathbb{R}$ ,  $Y$  be a real normed space,  $f$  be a point of the  $\mathbb{R}$ -norm space of continuous functions of  $X$  and  $Y$ , and  $g$  be a point of the real normed space of bounded functions from  $X$  into  $Y$ . If  $f = g$ , then  $\|f\| = \|g\|$ .
- (18) Let  $X$  be a non empty closed interval subset of  $\mathbb{R}$ ,  $Y$  be a real normed space,  $f, g$  be points of the  $\mathbb{R}$ -norm space of continuous functions of  $X$  and  $Y$ , and  $f_1, g_1$  be points of the real normed space of bounded functions from  $X$  into  $Y$ . If  $f_1 = f$  and  $g_1 = g$ , then  $f + g = f_1 + g_1$ .
- (19) Let  $X$  be a non empty closed interval subset of  $\mathbb{R}$ ,  $Y$  be a real normed space,  $f$  be a point of the  $\mathbb{R}$ -norm space of continuous functions of  $X$  and

$Y$ , and  $f_1$  be a point of the real normed space of bounded functions from  $X$  into  $Y$ . If  $f_1 = f$ , then  $a \cdot f = a \cdot f_1$ .

Let  $X$  be a non empty closed interval subset of  $\mathbb{R}$  and let  $Y$  be a real normed space. Observe that the  $\mathbb{R}$ -norm space of continuous functions of  $X$  and  $Y$  is reflexive, discernible, real normed space-like, vector distributive, scalar distributive, scalar associative, scalar unital, Abelian, add-associative, right zeroed, and right complementable.

One can prove the following propositions:

- (20) Let  $X$  be a non empty closed interval subset of  $\mathbb{R}$ ,  $Y$  be a real normed space,  $f, g, h$  be points of the  $\mathbb{R}$ -norm space of continuous functions of  $X$  and  $Y$ , and  $f_9, g_9, h_9$  be continuous partial functions from  $\mathbb{R}$  to  $Y$ . Suppose  $f_9 = f$  and  $g_9 = g$  and  $h_9 = h$  and  $\text{dom } f_9 = X$  and  $\text{dom } g_9 = X$  and  $\text{dom } h_9 = X$ . Then  $h = f - g$  if and only if for every element  $x$  of  $X$  holds  $(h_9)_x = (f_9)_x - (g_9)_x$ .
- (21) Let  $X$  be a non empty closed interval subset of  $\mathbb{R}$ ,  $Y$  be a real normed space,  $f, g$  be points of the  $\mathbb{R}$ -norm space of continuous functions of  $X$  and  $Y$ , and  $f_1, g_1$  be points of the real normed space of bounded functions from  $X$  into  $Y$ . If  $f_1 = f$  and  $g_1 = g$ , then  $f - g = f_1 - g_1$ .

Let  $X$  be a non empty closed interval subset of  $\mathbb{R}$  and let  $Y$  be a real normed space. Note that there exists a subset of the real normed space of bounded functions from  $X$  into  $Y$  which is closed.

The following two propositions are true:

- (22) Let  $X$  be a non empty closed interval subset of  $\mathbb{R}$  and  $Y$  be a real normed space. Then the continuous functions of  $X$  and  $Y$  is a closed subset of the real normed space of bounded functions from  $X$  into  $Y$ .
- (23) Let  $X$  be a non empty closed interval subset of  $\mathbb{R}$ ,  $Y$  be a real normed space, and  $s_1$  be a sequence of the  $\mathbb{R}$ -norm space of continuous functions of  $X$  and  $Y$ . Suppose  $Y$  is complete and  $s_1$  is Cauchy sequence by norm. Then  $s_1$  is convergent.

Let  $X$  be a non empty closed interval subset of  $\mathbb{R}$  and let  $Y$  be a real Banach space. One can check that the  $\mathbb{R}$ -norm space of continuous functions of  $X$  and  $Y$  is complete.

We now state four propositions:

- (24) Let  $X$  be a non empty closed interval subset of  $\mathbb{R}$ ,  $Y$  be a real normed space,  $v$  be a point of the  $\mathbb{R}$ -norm space of continuous functions of  $X$  and  $Y$ , and  $g$  be a partial function from  $\mathbb{R}$  to  $Y$ . If  $g = v$ , then for every real number  $t$  such that  $t \in X$  holds  $\|g_t\| \leq \|v\|$ .
- (25) Let  $X$  be a non empty closed interval subset of  $\mathbb{R}$ ,  $Y$  be a real normed space,  $K$  be a real number,  $v$  be a point of the  $\mathbb{R}$ -norm space of continuous functions of  $X$  and  $Y$ , and  $g$  be a partial function from  $\mathbb{R}$  to  $Y$ . Suppose

$g = v$  and for every real number  $t$  such that  $t \in X$  holds  $\|g_t\| \leq K$ . Then  $\|v\| \leq K$ .

- (26) Let  $X$  be a non empty closed interval subset of  $\mathbb{R}$ ,  $Y$  be a real normed space,  $v_1, v_2$  be points of the  $\mathbb{R}$ -norm space of continuous functions of  $X$  and  $Y$ , and  $g_1, g_2$  be partial functions from  $\mathbb{R}$  to  $Y$ . Suppose  $g_1 = v_1$  and  $g_2 = v_2$ . Let  $t$  be a real number. If  $t \in X$ , then  $\|(g_1)_t - (g_2)_t\| \leq \|v_1 - v_2\|$ .
- (27) Let  $X$  be a non empty closed interval subset of  $\mathbb{R}$ ,  $Y$  be a real normed space,  $K$  be a real number,  $v_1, v_2$  be points of the  $\mathbb{R}$ -norm space of continuous functions of  $X$  and  $Y$ , and  $g_1, g_2$  be partial functions from  $\mathbb{R}$  to  $Y$ . Suppose  $g_1 = v_1$  and  $g_2 = v_2$  and for every real number  $t$  such that  $t \in X$  holds  $\|(g_1)_t - (g_2)_t\| \leq K$ . Then  $\|v_1 - v_2\| \leq K$ .

### 3. DIFFERENTIAL EQUATIONS

The following propositions are true:

- (28) Let  $n, i$  be natural numbers,  $f$  be a partial function from  $\mathbb{R}$  to  $\mathcal{R}^n$ , and  $A$  be a subset of  $\mathbb{R}$ . Then  $\text{proj}(i, n) \cdot (f \upharpoonright A) = (\text{proj}(i, n) \cdot f) \upharpoonright A$ .
- (29) For every continuous partial function  $g$  from  $\mathbb{R}$  to  $\mathcal{R}^n$  such that  $\text{dom } g = [a, b]$  holds  $g \upharpoonright [a, b]$  is bounded.
- (30) For every continuous partial function  $g$  from  $\mathbb{R}$  to  $\mathcal{R}^n$  such that  $\text{dom } g = [a, b]$  holds  $g$  is integrable on  $[a, b]$ .
- (31) Let  $f, F$  be partial functions from  $\mathbb{R}$  to  $\mathcal{R}^n$ . Suppose  $a \leq b$  and  $\text{dom } f = [a, b]$  and  $\text{dom } F = [a, b]$  and  $f$  is continuous and for every real number  $t$  such that  $t \in [a, b]$  holds  $F(t) = \int_a^t f(x)dx$ . Let  $x$  be a real number. If  $x \in [a, b]$ , then  $F$  is continuous in  $x$ .
- (32) For every continuous partial function  $f$  from  $\mathbb{R}$  to  $\langle \mathcal{E}^n, \|\cdot\| \rangle$  such that  $\text{dom } f = [a, b]$  holds  $f \upharpoonright [a, b]$  is bounded.
- (33) For every continuous partial function  $f$  from  $\mathbb{R}$  to  $\langle \mathcal{E}^n, \|\cdot\| \rangle$  such that  $\text{dom } f = [a, b]$  holds  $f$  is integrable on  $[a, b]$ .
- (34) Let  $f$  be a continuous partial function from  $\mathbb{R}$  to  $\langle \mathcal{E}^n, \|\cdot\| \rangle$  and  $F$  be a partial function from  $\mathbb{R}$  to  $\langle \mathcal{E}^n, \|\cdot\| \rangle$ . Suppose  $a \leq b$  and  $\text{dom } f = [a, b]$  and  $\text{dom } F = [a, b]$  and for every real number  $t$  such that  $t \in [a, b]$  holds  $F(t) = \int_a^t f(x)dx$ . Let  $x$  be a real number. If  $x \in [a, b]$ , then  $F$  is continuous in  $x$ .
- (35) Let  $R$  be a partial function from  $\mathbb{R}$  to  $\mathbb{R}$ . Suppose  $R$  is total. Then  $R$  is rest-like if and only if for every real number  $r$  such that  $r > 0$  there exists

a real number  $d$  such that  $d > 0$  and for every real number  $z$  such that  $z \neq 0$  and  $|z| < d$  holds  $|z|^{-1} \cdot |R_z| < r$ .

In the sequel  $Z$  denotes an open subset of  $\mathbb{R}$ ,  $y_0$  denotes a vector of  $\langle \mathcal{E}^n, \|\cdot\| \rangle$ , and  $G$  denotes a function from  $\langle \mathcal{E}^n, \|\cdot\| \rangle$  into  $\langle \mathcal{E}^n, \|\cdot\| \rangle$ .

One can prove the following propositions:

- (36) Let  $f$  be a continuous partial function from  $\mathbb{R}$  to  $\langle \mathcal{E}^n, \|\cdot\| \rangle$  and  $g$  be a partial function from  $\mathbb{R}$  to  $\langle \mathcal{E}^n, \|\cdot\| \rangle$ . Suppose  $a \leq b$  and  $\text{dom } f = [a, b]$  and  $\text{dom } g = [a, b]$  and  $Z = ]a, b[$  and for every real number  $t$  such that  $t \in [a, b]$  holds  $g(t) = y_0 + \int_a^t f(x)dx$ . Then  $g$  is continuous and  $g_a = y_0$  and  $g$  is differentiable on  $Z$  and for every real number  $t$  such that  $t \in Z$  holds  $g'(t) = f_t$ .
- (37) For every natural number  $i$  and for all points  $y_1, y_2$  of  $\langle \mathcal{E}^n, \|\cdot\| \rangle$  holds  $(\text{proj}(i, n))(y_1 + y_2) = (\text{proj}(i, n))(y_1) + (\text{proj}(i, n))(y_2)$ .
- (38) For every natural number  $i$  and for every point  $y_1$  of  $\langle \mathcal{E}^n, \|\cdot\| \rangle$  and for every real number  $r$  holds  $(\text{proj}(i, n))(r \cdot y_1) = r \cdot (\text{proj}(i, n))(y_1)$ .
- (39) Let  $g$  be a partial function from  $\mathbb{R}$  to  $\langle \mathcal{E}^n, \|\cdot\| \rangle$ ,  $x_0$  be a real number, and  $i$  be a natural number. Suppose  $1 \leq i \leq n$  and  $g$  is differentiable in  $x_0$ . Then  $\text{proj}(i, n) \cdot g$  is differentiable in  $x_0$  and  $(\text{proj}(i, n))(g'(x_0)) = (\text{proj}(i, n) \cdot g)'(x_0)$ .
- (40) Let  $f$  be a partial function from  $\mathbb{R}$  to  $\langle \mathcal{E}^n, \|\cdot\| \rangle$  and  $X$  be a set. Suppose that for every natural number  $i$  such that  $1 \leq i \leq n$  holds  $(\text{proj}(i, n) \cdot f) \upharpoonright X$  is constant. Then  $f \upharpoonright X$  is constant.
- (41) Let  $f$  be a partial function from  $\mathbb{R}$  to  $\langle \mathcal{E}^n, \|\cdot\| \rangle$ . Suppose  $]a, b[ \subseteq \text{dom } f$  and  $f$  is differentiable on  $]a, b[$  and for every real number  $x$  such that  $x \in ]a, b[$  holds  $f'(x) = 0_{\langle \mathcal{E}^n, \|\cdot\| \rangle}$ . Then  $f \upharpoonright ]a, b[$  is constant.
- (42) Let  $f$  be a continuous partial function from  $\mathbb{R}$  to  $\langle \mathcal{E}^n, \|\cdot\| \rangle$ . Suppose  $a < b$  and  $[a, b] = \text{dom } f$  and  $f \upharpoonright ]a, b[$  is constant. Let  $x$  be a real number. If  $x \in [a, b]$ , then  $f(x) = f(a)$ .
- (43) Let  $y, G_1$  be continuous partial functions from  $\mathbb{R}$  to  $\langle \mathcal{E}^n, \|\cdot\| \rangle$  and  $g$  be a partial function from  $\mathbb{R}$  to  $\langle \mathcal{E}^n, \|\cdot\| \rangle$ . Suppose that  $a < b$  and  $Z = ]a, b[$  and  $\text{dom } y = [a, b]$  and  $\text{dom } g = [a, b]$  and  $\text{dom } G_1 = [a, b]$  and  $y$  is differentiable on  $Z$  and  $y_a = y_0$  and for every real number  $t$  such that  $t \in Z$  holds  $y'(t) = (G_1)_t$  and for every real number  $t$  such that  $t \in [a, b]$  holds  $g(t) = y_0 + \int_a^t G_1(x)dx$ . Then  $y = g$ .
- (44) Let  $a, b, c, d$  be real numbers and  $f$  be a partial function from  $\mathbb{R}$  to  $\langle \mathcal{E}^n, \|\cdot\| \rangle$ . Suppose that  $a \leq b$  and  $f$  is integrable on  $[a, b]$  and  $\|f\|$  is integrable on  $[a, b]$  and  $f \upharpoonright [a, b]$  is bounded and  $[a, b] \subseteq \text{dom } f$  and  $c, d \in [a, b]$ . Then

$\|f\|$  is integrable on  $[\min(c, d), \max(c, d)]$  and  $\|f\| \upharpoonright [\min(c, d), \max(c, d)]$  is bounded and  $\|\int_c^d f(x)dx\| \leq \int_{\min(c,d)}^{\max(c,d)} \|f\|(x)dx$ .

- (45) Let  $a, b, c, d, e$  be real numbers and  $f$  be a partial function from  $\mathbb{R}$  to  $\langle \mathcal{E}^n, \|\cdot\| \rangle$ . Suppose that  $a \leq b$  and  $c \leq d$  and  $f$  is integrable on  $[a, b]$  and  $\|f\|$  is integrable on  $[a, b]$  and  $f \upharpoonright [a, b]$  is bounded and  $[a, b] \subseteq \text{dom } f$  and  $c, d \in [a, b]$  and for every real number  $x$  such that  $x \in [c, d]$  holds  $\|f_x\| \leq e$ .

Then  $\|\int_c^d f(x)dx\| \leq e \cdot (d - c)$  and  $\|\int_d^c f(x)dx\| \leq e \cdot (d - c)$ .

- (46) Let  $n$  be a natural number and  $g$  be a function from  $\mathbb{R}$  into  $\mathbb{R}$ . Suppose that for every real number  $x$  holds  $g(x) = (x - a)^{n+1}$ . Let  $x$  be a real number. Then  $g$  is differentiable in  $x$  and  $g'(x) = (n + 1) \cdot (x - a)^n$ .

- (47) Let  $n$  be a natural number and  $g$  be a function from  $\mathbb{R}$  into  $\mathbb{R}$ . Suppose that for every real number  $x$  holds  $g(x) = \frac{(x-a)^{n+1}}{(n+1)!}$ . Let  $x$  be a real number. Then  $g$  is differentiable in  $x$  and  $g'(x) = \frac{(x-a)^n}{n!}$ .

- (48) Let  $f, g$  be partial functions from  $\mathbb{R}$  to  $\mathbb{R}$ . Suppose that  $a \leq t$  and  $[a, t] \subseteq \text{dom } f$  and  $f$  is integrable on  $[a, t]$  and  $f \upharpoonright [a, t]$  is bounded and  $[a, t] \subseteq \text{dom } g$  and  $g$  is integrable on  $[a, t]$  and  $g \upharpoonright [a, t]$  is bounded and for every real number  $x$  such that  $x \in [a, t]$  holds  $f(x) \leq g(x)$ . Then

$$\int_a^t f(x)dx \leq \int_a^t g(x)dx.$$

Let  $n$  be a non empty element of  $\mathbb{N}$ , let  $y_0$  be a vector of  $\langle \mathcal{E}^n, \|\cdot\| \rangle$ , let  $G$  be a function from  $\langle \mathcal{E}^n, \|\cdot\| \rangle$  into  $\langle \mathcal{E}^n, \|\cdot\| \rangle$ , and let  $a, b$  be real numbers. Let us assume that  $a \leq b$  and  $G$  is continuous on  $\text{dom } G$ . The functor  $\text{Fredholm}(G, a, b, y_0)$  yielding a function from the  $\mathbb{R}$ -norm space of continuous functions of  $[a, b]$  and  $\langle \mathcal{E}^n, \|\cdot\| \rangle$  into the  $\mathbb{R}$ -norm space of continuous functions of  $[a, b]$  and  $\langle \mathcal{E}^n, \|\cdot\| \rangle$  is defined by the condition (Def. 7).

- (Def. 7) Let  $x$  be a vector of the  $\mathbb{R}$ -norm space of continuous functions of  $[a, b]$  and  $\langle \mathcal{E}^n, \|\cdot\| \rangle$ . Then there exist continuous partial functions  $f, g, G_1$  from  $\mathbb{R}$  to  $\langle \mathcal{E}^n, \|\cdot\| \rangle$  such that  $x = f$  and  $(\text{Fredholm}(G, a, b, y_0))(x) = g$  and  $\text{dom } f = [a, b]$  and  $\text{dom } g = [a, b]$  and  $G_1 = G \cdot f$  and for every real number

$$t \text{ such that } t \in [a, b] \text{ holds } g(t) = y_0 + \int_a^t G_1(x)dx.$$

We now state several propositions:

- (49) Suppose  $a \leq b$  and  $0 < r$  and for all vectors  $y_1, y_2$  of  $\langle \mathcal{E}^n, \|\cdot\| \rangle$  holds  $\|G_{y_1} - G_{y_2}\| \leq r \cdot \|y_1 - y_2\|$ . Let  $u, v$  be vectors of the  $\mathbb{R}$ -norm space of continuous functions of  $[a, b]$  and  $\langle \mathcal{E}^n, \|\cdot\| \rangle$  and  $g, h$  be continuous partial



functions from  $\mathbb{R}$  to  $\langle \mathcal{E}^n, \|\cdot\| \rangle$ . Suppose  $g = (\text{Fredholm}(G, a, b, y_0))(u)$  and  $h = (\text{Fredholm}(G, a, b, y_0))(v)$ . Let  $t$  be a real number. If  $t \in [a, b]$ , then  $\|g_t - h_t\| \leq r \cdot (t - a) \cdot \|u - v\|$ .

- (50) Suppose  $a \leq b$  and  $0 < r$  and for all vectors  $y_1, y_2$  of  $\langle \mathcal{E}^n, \|\cdot\| \rangle$  holds  $\|G_{y_1} - G_{y_2}\| \leq r \cdot \|y_1 - y_2\|$ . Let  $u, v$  be vectors of the  $\mathbb{R}$ -norm space of continuous functions of  $[a, b]$  and  $\langle \mathcal{E}^n, \|\cdot\| \rangle$ ,  $m$  be an element of  $\mathbb{N}$ , and  $g, h$  be continuous partial functions from  $\mathbb{R}$  to  $\langle \mathcal{E}^n, \|\cdot\| \rangle$ . Suppose  $g = (\text{Fredholm}(G, a, b, y_0))^{m+1}(u)$  and  $h = (\text{Fredholm}(G, a, b, y_0))^{m+1}(v)$ . Let  $t$  be a real number. If  $t \in [a, b]$ , then  $\|g_t - h_t\| \leq \frac{(r \cdot (t-a))^{m+1}}{(m+1)!} \cdot \|u - v\|$ .
- (51) Let  $m$  be a natural number. Suppose  $a \leq b$  and  $0 < r$  and for all vectors  $y_1, y_2$  of  $\langle \mathcal{E}^n, \|\cdot\| \rangle$  holds  $\|G_{y_1} - G_{y_2}\| \leq r \cdot \|y_1 - y_2\|$ . Let  $u, v$  be vectors of the  $\mathbb{R}$ -norm space of continuous functions of  $[a, b]$  and  $\langle \mathcal{E}^n, \|\cdot\| \rangle$ . Then  $\|(\text{Fredholm}(G, a, b, y_0))^{m+1}(u) - (\text{Fredholm}(G, a, b, y_0))^{m+1}(v)\| \leq \frac{(r \cdot (b-a))^{m+1}}{(m+1)!} \cdot \|u - v\|$ .
- (52) Suppose  $a < b$  and  $G$  is Lipschitzian on the carrier of  $\langle \mathcal{E}^n, \|\cdot\| \rangle$ . Then there exists a natural number  $m$  such that  $(\text{Fredholm}(G, a, b, y_0))^{m+1}$  is contraction.
- (53) If  $a < b$  and  $G$  is Lipschitzian on the carrier of  $\langle \mathcal{E}^n, \|\cdot\| \rangle$ , then  $\text{Fredholm}(G, a, b, y_0)$  has unique fixpoint.
- (54) Let  $f, g$  be continuous partial functions from  $\mathbb{R}$  to  $\langle \mathcal{E}^n, \|\cdot\| \rangle$ . Suppose  $\text{dom } f = [a, b]$  and  $\text{dom } g = [a, b]$  and  $Z = ]a, b[$  and  $a < b$  and  $G$  is Lipschitzian on the carrier of  $\langle \mathcal{E}^n, \|\cdot\| \rangle$  and  $g = (\text{Fredholm}(G, a, b, y_0))(f)$ . Then  $g_a = y_0$  and  $g$  is differentiable on  $Z$  and for every real number  $t$  such that  $t \in Z$  holds  $g'(t) = (G \cdot f)_t$ .
- (55) Let  $y$  be a continuous partial function from  $\mathbb{R}$  to  $\langle \mathcal{E}^n, \|\cdot\| \rangle$ . Suppose that  $a < b$  and  $Z = ]a, b[$  and  $G$  is Lipschitzian on the carrier of  $\langle \mathcal{E}^n, \|\cdot\| \rangle$  and  $\text{dom } y = [a, b]$  and  $y$  is differentiable on  $Z$  and  $y_a = y_0$  and for every real number  $t$  such that  $t \in Z$  holds  $y'(t) = G(y_t)$ . Then  $y$  is a fixpoint of  $\text{Fredholm}(G, a, b, y_0)$ .
- (56) Let  $y_1, y_2$  be continuous partial functions from  $\mathbb{R}$  to  $\langle \mathcal{E}^n, \|\cdot\| \rangle$ . Suppose that  $a < b$  and  $Z = ]a, b[$  and  $G$  is Lipschitzian on the carrier of  $\langle \mathcal{E}^n, \|\cdot\| \rangle$  and  $\text{dom } y_1 = [a, b]$  and  $y_1$  is differentiable on  $Z$  and  $(y_1)_a = y_0$  and for every real number  $t$  such that  $t \in Z$  holds  $y_1'(t) = G((y_1)_t)$  and  $\text{dom } y_2 = [a, b]$  and  $y_2$  is differentiable on  $Z$  and  $(y_2)_a = y_0$  and for every real number  $t$  such that  $t \in Z$  holds  $y_2'(t) = G((y_2)_t)$ . Then  $y_1 = y_2$ .
- (57) Suppose  $a < b$  and  $Z = ]a, b[$  and  $G$  is Lipschitzian on the carrier of  $\langle \mathcal{E}^n, \|\cdot\| \rangle$ . Then there exists a continuous partial function  $y$  from  $\mathbb{R}$  to  $\langle \mathcal{E}^n, \|\cdot\| \rangle$  such that  $\text{dom } y = [a, b]$  and  $y$  is differentiable on  $Z$  and  $y_a = y_0$  and for every real number  $t$  such that  $t \in Z$  holds  $y'(t) = G(y_t)$ .

## REFERENCES

- [1] Grzegorz Bancerek. The ordinal numbers. *Formalized Mathematics*, 1(1):91–96, 1990.
- [2] Józef Białas. Properties of the intervals of real numbers. *Formalized Mathematics*, 3(2):263–269, 1992.
- [3] Czesław Byliński. The complex numbers. *Formalized Mathematics*, 1(3):507–513, 1990.
- [4] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(1):55–65, 1990.
- [5] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(1):153–164, 1990.
- [6] Czesław Byliński. Partial functions. *Formalized Mathematics*, 1(2):357–367, 1990.
- [7] Czesław Byliński. Some basic properties of sets. *Formalized Mathematics*, 1(1):47–53, 1990.
- [8] Agata Darmochwał. The Euclidean space. *Formalized Mathematics*, 2(4):599–603, 1991.
- [9] Noboru Endou and Yasunari Shidama. Completeness of the real Euclidean space. *Formalized Mathematics*, 13(4):577–580, 2005.
- [10] Noboru Endou, Yasunari Shidama, and Keiichi Miyajima. Partial differentiation on normed linear spaces  $\mathcal{R}^n$ . *Formalized Mathematics*, 15(2):65–72, 2007, doi:10.2478/v10037-007-0008-5.
- [11] Noboru Endou, Yasumasa Suzuki, and Yasunari Shidama. Real linear space of real sequences. *Formalized Mathematics*, 11(3):249–253, 2003.
- [12] Noboru Endou, Katsumi Wasaki, and Yasunari Shidama. Definition of integrability for partial functions from  $\mathbb{R}$  to  $\mathbb{R}$  and integrability for continuous functions. *Formalized Mathematics*, 9(2):281–284, 2001.
- [13] Andrzej Kondracki. Basic properties of rational numbers. *Formalized Mathematics*, 1(5):841–845, 1990.
- [14] Artur Korniłowicz. Arithmetic operations on functions from sets into functional sets. *Formalized Mathematics*, 17(1):43–60, 2009, doi:10.2478/v10037-009-0005-y.
- [15] Rafał Kwiatek. Factorial and Newton coefficients. *Formalized Mathematics*, 1(5):887–890, 1990.
- [16] Keiichi Miyajima, Takahiro Kato, and Yasunari Shidama. Riemann integral of functions from  $\mathbb{R}$  into real normed space. *Formalized Mathematics*, 19(1):17–22, 2011, doi:10.2478/v10037-011-0003-8.
- [17] Keiichi Miyajima, Artur Korniłowicz, and Yasunari Shidama. Riemann integral of functions from  $\mathbb{R}$  into  $n$ -dimensional real normed space. *Formalized Mathematics*, 20(1):79–86, 2012, doi:10.2478/v10037-012-0011-3.
- [18] Keiichi Miyajima and Yasunari Shidama. Riemann integral of functions from  $\mathbb{R}$  into  $\mathcal{R}^n$ . *Formalized Mathematics*, 17(2):179–185, 2009, doi:10.2478/v10037-009-0021-y.
- [19] Keiko Narita, Artur Korniłowicz, and Yasunari Shidama. More on the continuity of real functions. *Formalized Mathematics*, 19(4):233–239, 2011, doi:10.2478/v10037-011-0032-3.
- [20] Adam Naumowicz. Conjugate sequences, bounded complex sequences and convergent complex sequences. *Formalized Mathematics*, 6(2):265–268, 1997.
- [21] Takaya Nishiyama, Artur Korniłowicz, and Yasunari Shidama. The uniform continuity of functions on normed linear spaces. *Formalized Mathematics*, 12(3):277–279, 2004.
- [22] Takaya Nishiyama, Keiji Ohkubo, and Yasunari Shidama. The continuous functions on normed linear spaces. *Formalized Mathematics*, 12(3):269–275, 2004.
- [23] Hiroyuki Okazaki, Noboru Endou, Keiko Narita, and Yasunari Shidama. Differentiable functions into real normed spaces. *Formalized Mathematics*, 19(2):69–72, 2011, doi:10.2478/v10037-011-0012-7.
- [24] Hiroyuki Okazaki, Noboru Endou, and Yasunari Shidama. More on continuous functions on normed linear spaces. *Formalized Mathematics*, 19(1):45–49, 2011, doi:10.2478/v10037-011-0008-3.
- [25] Jan Popiołek. Real normed space. *Formalized Mathematics*, 2(1):111–115, 1991.
- [26] Konrad Raczkowski and Paweł Sadowski. Real function differentiability. *Formalized Mathematics*, 1(4):797–801, 1990.
- [27] Konrad Raczkowski and Paweł Sadowski. Topological properties of subsets in real numbers. *Formalized Mathematics*, 1(4):777–780, 1990.
- [28] Piotr Rudnicki and Andrzej Trybulec. Abian’s fixed point theorem. *Formalized Mathematics*, 6(3):335–338, 1997.
- [29] Laurent Schwartz. *Cours d’analyse II, Ch. 5*. HERMANN, Paris, 1967.

- [30] Yasunari Shidama. Banach space of bounded linear operators. *Formalized Mathematics*, 12(1):39–48, 2004.
- [31] Yasumasa Suzuki. Banach space of bounded real sequences. *Formalized Mathematics*, 12(2):77–83, 2004.
- [32] Andrzej Trybulec. Binary operations applied to functions. *Formalized Mathematics*, 1(2):329–334, 1990.
- [33] Andrzej Trybulec. On the sets inhabited by numbers. *Formalized Mathematics*, 11(4):341–347, 2003.
- [34] Michał J. Trybulec. Integers. *Formalized Mathematics*, 1(3):501–505, 1990.
- [35] Wojciech A. Trybulec. Subspaces and cosets of subspaces in real linear space. *Formalized Mathematics*, 1(2):297–301, 1990.
- [36] Wojciech A. Trybulec. Vectors in real linear space. *Formalized Mathematics*, 1(2):291–296, 1990.
- [37] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(1):67–71, 1990.
- [38] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1(1):73–83, 1990.
- [39] Edmund Woronowicz. Relations defined on sets. *Formalized Mathematics*, 1(1):181–186, 1990.

*Received August 19, 2012*

---