

Isomorphisms of Direct Products of Finite Cyclic Groups

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Summary. In this article, we formalize that every finite cyclic group is isomorphic to a direct product of finite cyclic groups which orders are relative prime. This theorem is closely related to the Chinese Remainder theorem ([18]) and is a useful lemma to prove the basis theorem for finite abelian groups and the fundamental theorem of finite abelian groups. Moreover, we formalize some facts about the product of a finite sequence of abelian groups.

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The notation and terminology used in this paper are introduced in the following articles: [5], [1], [2], [4], [11], [6], [7], [20], [17], [18], [19], [3], [8], [13], [15], [16], [12], [23], [21], [10], [22], [14], and [9].

Let G be an Abelian add-associative right zeroed right complementable non empty additive loop structure. Note that $\langle G \rangle$ is non empty and Abelian group yielding as a finite sequence.

Let G, F be Abelian add-associative right zeroed right complementable non empty additive loop structures. Note that $\langle G, F \rangle$ is non empty and Abelian group yielding as a finite sequence.

We now state the proposition

- (1) Let X be an Abelian group. Then there exists a homomorphism I from X to $\prod \langle X \rangle$ such that I is bijective and for every element x of X holds $I(x) = \langle x \rangle$.

Let G, F be non empty Abelian group yielding finite sequences. Note that $G \wedge F$ is Abelian group yielding.

One can prove the following propositions:

- (2) Let X, Y be Abelian groups. Then there exists a homomorphism I from $X \times Y$ to $\prod \langle X, Y \rangle$ such that I is bijective and for every element x of X and for every element y of Y holds $I(x, y) = \langle x, y \rangle$.
- (3) Let X, Y be sequences of groups. Then there exists a homomorphism I from $\prod X \times \prod Y$ to $\prod (X \wedge Y)$ such that
 - (i) I is bijective, and
 - (ii) for every element x of $\prod X$ and for every element y of $\prod Y$ there exist finite sequences x_1, y_1 such that $x = x_1$ and $y = y_1$ and $I(x, y) = x_1 \wedge y_1$.
- (4) Let G, F be Abelian groups. Then
 - (i) for every set x holds x is an element of $\prod \langle G, F \rangle$ iff there exists an element x_1 of G and there exists an element x_2 of F such that $x = \langle x_1, x_2 \rangle$,
 - (ii) for all elements x, y of $\prod \langle G, F \rangle$ and for all elements x_1, y_1 of G and for all elements x_2, y_2 of F such that $x = \langle x_1, x_2 \rangle$ and $y = \langle y_1, y_2 \rangle$ holds $x + y = \langle x_1 + y_1, x_2 + y_2 \rangle$,
 - (iii) $0_{\prod \langle G, F \rangle} = \langle 0_G, 0_F \rangle$, and
 - (iv) for every element x of $\prod \langle G, F \rangle$ and for every element x_1 of G and for every element x_2 of F such that $x = \langle x_1, x_2 \rangle$ holds $-x = \langle -x_1, -x_2 \rangle$.
- (5) Let G, F be Abelian groups. Then
 - (i) for every set x holds x is an element of $G \times F$ iff there exists an element x_1 of G and there exists an element x_2 of F such that $x = \langle x_1, x_2 \rangle$,
 - (ii) for all elements x, y of $G \times F$ and for all elements x_1, y_1 of G and for all elements x_2, y_2 of F such that $x = \langle x_1, x_2 \rangle$ and $y = \langle y_1, y_2 \rangle$ holds $x + y = \langle x_1 + y_1, x_2 + y_2 \rangle$,
 - (iii) $0_{G \times F} = \langle 0_G, 0_F \rangle$, and
 - (iv) for every element x of $G \times F$ and for every element x_1 of G and for every element x_2 of F such that $x = \langle x_1, x_2 \rangle$ holds $-x = \langle -x_1, -x_2 \rangle$.
- (6) Let G, H, I be groups, h be a homomorphism from G to H , and h_1 be a homomorphism from H to I . Then $h_1 \cdot h$ is a homomorphism from G to I .

Let G, H, I be groups, let h be a homomorphism from G to H , and let h_1 be a homomorphism from H to I . Then $h_1 \cdot h$ is a homomorphism from G to I .

One can prove the following propositions:

- (7) Let G, H be groups and h be a homomorphism from G to H . If h is bijective, then h^{-1} is a homomorphism from H to G .
- (8) Let X, Y be sequences of groups. Then there exists a homomorphism I from $\prod \langle \prod X, \prod Y \rangle$ to $\prod (X \wedge Y)$ such that
 - (i) I is bijective, and

- (ii) for every element x of $\prod X$ and for every element y of $\prod Y$ there exist finite sequences x_1, y_1 such that $x = x_1$ and $y = y_1$ and $I(\langle x, y \rangle) = x_1 \hat{\ } y_1$.
- (9) Let X, Y be Abelian groups. Then there exists a homomorphism I from $X \times Y$ to $X \times \prod \langle Y \rangle$ such that I is bijective and for every element x of X and for every element y of Y holds $I(x, y) = \langle x, \langle y \rangle \rangle$.
- (10) Let X be a sequence of groups and Y be an Abelian group. Then there exists a homomorphism I from $\prod X \times Y$ to $\prod (X \hat{\ } \langle Y \rangle)$ such that
 - (i) I is bijective, and
 - (ii) for every element x of $\prod X$ and for every element y of Y there exist finite sequences x_1, y_1 such that $x = x_1$ and $\langle y \rangle = y_1$ and $I(x, y) = x_1 \hat{\ } y_1$.
- (11) Let n be a non zero natural number. Then the additive loop structure of (\mathbb{Z}_n^R) is non empty, Abelian, right complementable, add-associative, and right zeroed.

Let n be a natural number. The functor $\mathbb{Z}/n\mathbb{Z}$ yields an additive loop structure and is defined by:

(Def. 1) $\mathbb{Z}/n\mathbb{Z} =$ the additive loop structure of (\mathbb{Z}_n^R) .

Let n be a non zero natural number. Observe that $\mathbb{Z}/n\mathbb{Z}$ is non empty and strict.

Let n be a non zero natural number. Note that $\mathbb{Z}/n\mathbb{Z}$ is Abelian, right complementable, add-associative, and right zeroed.

Next we state a number of propositions:

- (12) Let X be a sequence of groups, x, y, z be elements of $\prod X$, and x_1, y_1, z_1 be finite sequences. Suppose $x = x_1$ and $y = y_1$ and $z = z_1$. Then $z = x + y$ if and only if for every element j of $\text{dom } \overline{X}$ holds $z_1(j) =$ (the addition of $X(j))(x_1(j), y_1(j))$.
- (13) For every CR-sequence m and for every natural number j and for every integer x such that $j \in \text{dom } m$ holds $x \text{ mod } \prod m \text{ mod } m(j) = x \text{ mod } m(j)$.
- (14) Let m be a CR-sequence and X be a sequence of groups. Suppose $\text{len } m = \text{len } X$ and for every element i of \mathbb{N} such that $i \in \text{dom } X$ there exists a non zero natural number m_1 such that $m_1 = m(i)$ and $X(i) = \mathbb{Z}/m_1\mathbb{Z}$. Then there exists a homomorphism I from $\mathbb{Z}/(\prod m)\mathbb{Z}$ to $\prod X$ such that for every integer x if $x \in$ the carrier of $\mathbb{Z}/(\prod m)\mathbb{Z}$, then $I(x) = \text{mod}(x, m)$.
- (15) Let X, Y be non empty sets. Then there exists a function I from $X \times Y$ into $X \times \prod \langle Y \rangle$ such that I is one-to-one and onto and for all sets x, y such that $x \in X$ and $y \in Y$ holds $I(x, y) = \langle x, \langle y \rangle \rangle$.
- (16) For every non empty set X holds $\overline{\prod \langle X \rangle} = \overline{X}$.
- (17) Let X be a non-empty finite sequence and Y be a non empty set. Then there exists a function I from $\prod X \times Y$ into $\prod (X \hat{\ } \langle Y \rangle)$ such that
 - (i) I is one-to-one and onto, and

- (ii) for all sets x, y such that $x \in \prod X$ and $y \in Y$ there exist finite sequences x_1, y_1 such that $x = x_1$ and $\langle y \rangle = y_1$ and $I(x, y) = x_1 \hat{\ } y_1$.
- (18) Let m be a finite sequence of elements of \mathbb{N} and X be a non-empty non empty finite sequence. Suppose $\text{len } m = \text{len } X$ and for every element i of \mathbb{N} such that $i \in \text{dom } X$ holds $\overline{X(i)} = m(i)$. Then $\overline{\prod X} = \prod m$.
- (19) Let m be a CR-sequence and X be a sequence of groups. Suppose $\text{len } m = \text{len } X$ and for every element i of \mathbb{N} such that $i \in \text{dom } X$ there exists a non zero natural number m_1 such that $m_1 = m(i)$ and $X(i) = \mathbb{Z}/m_1\mathbb{Z}$. Then the carrier of $\overline{\prod X} = \prod m$.
- (20) Let m be a CR-sequence, X be a sequence of groups, and I be a function from $\mathbb{Z}/(\prod m)\mathbb{Z}$ into $\prod X$. Suppose that
- (i) $\text{len } m = \text{len } X$,
 - (ii) for every element i of \mathbb{N} such that $i \in \text{dom } X$ there exists a non zero natural number m_1 such that $m_1 = m(i)$ and $X(i) = \mathbb{Z}/m_1\mathbb{Z}$, and
 - (iii) for every integer x such that $x \in$ the carrier of $\mathbb{Z}/(\prod m)\mathbb{Z}$ holds $I(x) = \text{mod}(x, m)$.
- Then I is one-to-one.
- (21) Let m be a CR-sequence and X be a sequence of groups. Suppose $\text{len } m = \text{len } X$ and for every element i of \mathbb{N} such that $i \in \text{dom } X$ there exists a non zero natural number m_1 such that $m_1 = m(i)$ and $X(i) = \mathbb{Z}/m_1\mathbb{Z}$. Then there exists a homomorphism I from $\mathbb{Z}/(\prod m)\mathbb{Z}$ to $\prod X$ such that I is bijective and for every integer x such that $x \in$ the carrier of $\mathbb{Z}/(\prod m)\mathbb{Z}$ holds $I(x) = \text{mod}(x, m)$.

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