

# On $L^1$ Space Formed by Complex-Valued Partial Functions

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**Summary.** In this article, we formalized  $L^1$  space formed by complex-valued partial functions [11], [15]. The real-valued case was formalized in [22] and this article is its generalization.

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The notation and terminology used here have been introduced in the following papers: [4], [10], [5], [19], [17], [6], [7], [1], [22], [3], [18], [13], [16], [8], [14], [23], [24], [12], [20], [21], [2], and [9].

## 1. PRELIMINARIES OF COMPLEX LINEAR SPACE

Let  $D$  be a non empty set and let  $E$  be a complex-membered set. One can verify that every element of  $D \dot{\rightarrow} E$  is complex-valued.

Let  $D$  be a non empty set, let  $E$  be a complex-membered set, and let  $F_1, F_2$  be elements of  $D \dot{\rightarrow} E$ . Then  $F_1 + F_2$  is an element of  $D \dot{\rightarrow} \mathbb{C}$ . Then  $F_1 - F_2$  is an element of  $D \dot{\rightarrow} \mathbb{C}$ . Then  $F_1 \cdot F_2$  is an element of  $D \dot{\rightarrow} \mathbb{C}$ . Then  $F_1/F_2$  is an element of  $D \dot{\rightarrow} \mathbb{C}$ .

Let  $D$  be a non empty set, let  $E$  be a complex-membered set, let  $F$  be an element of  $D \dot{\rightarrow} E$ , and let  $a$  be a complex number. Then  $a \cdot F$  is an element of  $D \dot{\rightarrow} \mathbb{C}$ .

Let  $V$  be a non empty CLS structure and let  $V_1$  be a subset of  $V$ . We say that  $V_1$  is multiplicatively closed if and only if:

(Def. 1) For every complex number  $a$  and for every vector  $v$  of  $V$  such that  $v \in V_1$  holds  $a \cdot v \in V_1$ .

Next we state the proposition

(1) Let  $V$  be a complex linear space and  $V_1$  be a subset of  $V$ . Then  $V_1$  is linearly closed if and only if  $V_1$  is add closed and multiplicatively closed.

Let  $V$  be a non empty CLS structure. One can verify that there exists a non empty subset of  $V$  which is add closed and multiplicatively closed.

Let  $X$  be a non empty CLS structure and let  $X_1$  be a multiplicatively closed non empty subset of  $X$ . The functor  $\cdot_{(X_1)}$  yields a function from  $\mathbb{C} \times X_1$  into  $X_1$  and is defined by:

(Def. 2)  $\cdot_{(X_1)} = (\text{the external multiplication of } X) \upharpoonright (\mathbb{C} \times X_1)$ .

In the sequel  $a, b, r$  denote complex numbers and  $V$  denotes a complex linear space.

We now state two propositions:

(2) Let  $V$  be an Abelian add-associative right zeroed vector distributive scalar distributive scalar associative scalar unital non empty CLS structure,  $V_1$  be a non empty subset of  $V$ ,  $d_1$  be an element of  $V_1$ ,  $A$  be a binary operation on  $V_1$ , and  $M$  be a function from  $\mathbb{C} \times V_1$  into  $V_1$ . Suppose  $d_1 = 0_V$  and  $A = (\text{the addition of } V) \upharpoonright (V_1)$  and  $M = (\text{the external multiplication of } V) \upharpoonright (\mathbb{C} \times V_1)$ . Then  $\langle V_1, d_1, A, M \rangle$  is Abelian, add-associative, right zeroed, vector distributive, scalar distributive, scalar associative, and scalar unital.

(3) Let  $V$  be an Abelian add-associative right zeroed vector distributive scalar distributive scalar associative scalar unital non empty CLS structure and  $V_1$  be an add closed multiplicatively closed non empty subset of  $V$ . Suppose  $0_V \in V_1$ . Then  $\langle V_1, 0_V (\in V_1), \text{add} \upharpoonright (V_1, V), \cdot_{(V_1)} \rangle$  is Abelian, add-associative, right zeroed, vector distributive, scalar distributive, scalar associative, and scalar unital.

## 2. QUASI-COMPLEX LINEAR SPACE OF PARTIAL FUNCTIONS

We follow the rules:  $A, B$  are non empty sets and  $f, g, h$  are elements of  $A \dot{\rightarrow} \mathbb{C}$ .

Let us consider  $A$ . The functor  $\text{multcpfunc } A$  yielding a binary operation on  $A \dot{\rightarrow} \mathbb{C}$  is defined as follows:

(Def. 3) For all elements  $f, g$  of  $A \dot{\rightarrow} \mathbb{C}$  holds  $(\text{multcpfunc } A)(f, g) = f \cdot g$ .

Let us consider  $A$ . The functor  $\text{multcomplexcpfunc } A$  yielding a function from  $\mathbb{C} \times (A \dot{\rightarrow} \mathbb{C})$  into  $A \dot{\rightarrow} \mathbb{C}$  is defined by:

(Def. 4) For every complex number  $a$  and for every element  $f$  of  $A \dot{\rightarrow} \mathbb{C}$  holds  $(\text{multcomplexcpfunc } A)(a, f) = a \cdot f$ .

Let  $D$  be a non empty set. The functor  $\text{addecpfunc } D$  yields a binary operation on  $D \rightarrow \mathbb{C}$  and is defined as follows:

(Def. 5) For all elements  $F_1, F_2$  of  $D \rightarrow \mathbb{C}$  holds  $(\text{addecpfunc } D)(F_1, F_2) = F_1 + F_2$ .

Let  $A$  be a set. The functor  $\text{CPFuncZero } A$  yields an element of  $A \rightarrow \mathbb{C}$  and is defined by:

(Def. 6)  $\text{CPFuncZero } A = A \mapsto 0_{\mathbb{C}}$ .

Let  $A$  be a set. The functor  $\text{CPFuncUnit } A$  yielding an element of  $A \rightarrow \mathbb{C}$  is defined as follows:

(Def. 7)  $\text{CPFuncUnit } A = A \mapsto 1_{\mathbb{C}}$ .

The following propositions are true:

(4)  $h = (\text{addecpfunc } A)(f, g)$  iff  $\text{dom } h = \text{dom } f \cap \text{dom } g$  and for every element  $x$  of  $A$  such that  $x \in \text{dom } h$  holds  $h(x) = f(x) + g(x)$ .

(5)  $h = (\text{multcpfunc } A)(f, g)$  iff  $\text{dom } h = \text{dom } f \cap \text{dom } g$  and for every element  $x$  of  $A$  such that  $x \in \text{dom } h$  holds  $h(x) = f(x) \cdot g(x)$ .

(6)  $\text{CPFuncZero } A \neq \text{CPFuncUnit } A$ .

(7)  $h = (\text{multcomplexcpfunc } A)(a, f)$  iff  $\text{dom } h = \text{dom } f$  and for every element  $x$  of  $A$  such that  $x \in \text{dom } f$  holds  $h(x) = a \cdot f(x)$ .

Let us consider  $A$ . Note that  $\text{addecpfunc } A$  is commutative and associative.

Observe that  $\text{multcpfunc } A$  is commutative and associative.

One can prove the following propositions:

(8)  $\text{CPFuncUnit } A$  is a unity w.r.t.  $\text{multcpfunc } A$ .

(9)  $\text{CPFuncZero } A$  is a unity w.r.t.  $\text{addecpfunc } A$ .

(10)  $(\text{addecpfunc } A)(f, (\text{multcomplexcpfunc } A)(-1_{\mathbb{C}}, f)) = \text{CPFuncZero } A \upharpoonright \text{dom } f$ .

(11)  $(\text{multcomplexcpfunc } A)(1_{\mathbb{C}}, f) = f$ .

(12)  $(\text{multcomplexcpfunc } A)(a, (\text{multcomplexcpfunc } A)(b, f)) = (\text{multcomplexcpfunc } A)(a \cdot b, f)$ .

(13)  $(\text{addecpfunc } A)((\text{multcomplexcpfunc } A)(a, f), (\text{multcomplexcpfunc } A)(b, f)) = (\text{multcomplexcpfunc } A)(a + b, f)$ .

(14)  $(\text{multcpfunc } A)(f, (\text{addecpfunc } A)(g, h)) = (\text{addecpfunc } A)((\text{multcpfunc } A)(f, g), (\text{multcpfunc } A)(f, h))$ .

(15)  $(\text{multcpfunc } A)((\text{multcomplexcpfunc } A)(a, f), g) = (\text{multcomplexcpfunc } A)(a, (\text{multcpfunc } A)(f, g))$ .

Let us consider  $A$ . The functor  $\text{CLSp PFunc } A$  yields a non empty CLS structure and is defined as follows:

(Def. 8)  $\text{CLSp PFunc } A = \langle A \rightarrow \mathbb{C}, \text{CPFuncZero } A, \text{addecpfunc } A, \text{multcomplexcpfunc } A \rangle$ .

In the sequel  $u, v, w$  are vectors of  $\text{CLSp PFunc } A$ .

Note that CLSp PFunc  $A$  is strict, Abelian, add-associative, right zeroed, vector distributive, scalar distributive, scalar associative, and scalar unital.

### 3. QUASI-COMPLEX LINEAR SPACE OF INTEGRABLE FUNCTIONS

For simplicity, we use the following convention:  $X$  is a non empty set,  $x$  is an element of  $X$ ,  $S$  is a  $\sigma$ -field of subsets of  $X$ ,  $M$  is a  $\sigma$ -measure on  $S$ ,  $E, A$  are elements of  $S$ , and  $f, g, h, f_1, g_1$  are partial functions from  $X$  to  $\mathbb{C}$ .

Let us consider  $X$  and let  $f$  be a partial function from  $X$  to  $\mathbb{C}$ . Note that  $|f|$  is non-negative.

Next we state the proposition

- (16) Let  $f$  be a partial function from  $X$  to  $\mathbb{C}$ . Suppose  $\text{dom } f \in S$  and for every  $x$  such that  $x \in \text{dom } f$  holds  $0 = f(x)$ . Then  $f$  is integrable on  $M$  and  $\int f \, dM = 0$ .

Let  $X$  be a non empty set, let  $S$  be a  $\sigma$ -field of subsets of  $X$ , and let  $M$  be a  $\sigma$ -measure on  $S$ . The functor  $L_1\text{CFunctions } M$  yielding a non empty subset of CLSp PFunc  $X$  is defined by the condition (Def. 9).

- (Def. 9)  $L_1\text{CFunctions } M = \{f; f \text{ ranges over partial functions from } X \text{ to } \mathbb{C}; \bigvee_{N_1: \text{element of } S} (M(N_1) = 0 \wedge \text{dom } f = N_1^c \wedge f \text{ is integrable on } M)\}$ .

The following propositions are true:

- (17) If  $f, g \in L_1\text{CFunctions } M$ , then  $f + g \in L_1\text{CFunctions } M$ .  
 (18) If  $f \in L_1\text{CFunctions } M$ , then  $a \cdot f \in L_1\text{CFunctions } M$ .

Note that  $L_1\text{CFunctions } M$  is multiplicatively closed and add closed.

The functor CLSp  $L_1\text{Func } M$  yielding a non empty CLS structure is defined by:

- (Def. 10)  $\text{CLSp } L_1\text{Func } M = \langle L_1\text{CFunctions } M, 0_{\text{CLSp PFunc } X} (\in L_1\text{CFunctions } M), \text{add } |(L_1\text{CFunctions } M, \text{CLSp PFunc } X), \cdot_{L_1\text{CFunctions } M} \rangle$ .

One can verify that CLSp  $L_1\text{Func } M$  is strict, Abelian, add-associative, right zeroed, vector distributive, scalar distributive, scalar associative, and scalar unital.

### 4. QUOTIENT SPACE OF QUASI-COMPLEX LINEAR SPACE OF INTEGRABLE FUNCTIONS

In the sequel  $v, u$  are vectors of CLSp  $L_1\text{Func } M$ .

Next we state two propositions:

- (19) If  $f = v$  and  $g = u$ , then  $f + g = v + u$ .  
 (20) If  $f = u$ , then  $a \cdot f = a \cdot u$ .

Let  $X$  be a non empty set, let  $S$  be a  $\sigma$ -field of subsets of  $X$ , let  $M$  be a  $\sigma$ -measure on  $S$ , and let  $f, g$  be partial functions from  $X$  to  $\mathbb{C}$ . We say that  $f$  a.e.cpfunc =  $g$  and  $M$  if and only if:

(Def. 11) There exists an element  $E$  of  $S$  such that  $M(E) = 0$  and  $f|E^c = g|E^c$ .

We now state several propositions:

- (21) Suppose  $f = u$ . Then
  - (i)  $u + (-1_{\mathbb{C}}) \cdot u = (X \mapsto 0_{\mathbb{C}})|\text{dom } f$ , and
  - (ii) there exist partial functions  $v, g$  from  $X$  to  $\mathbb{C}$  such that  $v, g \in L_1\text{CFunctions } M$  and  $v = u + (-1_{\mathbb{C}}) \cdot u$  and  $g = X \mapsto 0_{\mathbb{C}}$  and  $v$  a.e.cpfunc =  $g$  and  $M$ .
- (22)  $f$  a.e.cpfunc =  $f$  and  $M$ .
- (23) If  $f$  a.e.cpfunc =  $g$  and  $M$ , then  $g$  a.e.cpfunc =  $f$  and  $M$ .
- (24) If  $f$  a.e.cpfunc =  $g$  and  $M$  and  $g$  a.e.cpfunc =  $h$  and  $M$ , then  $f$  a.e.cpfunc =  $h$  and  $M$ .
- (25) If  $f$  a.e.cpfunc =  $f_1$  and  $M$  and  $g$  a.e.cpfunc =  $g_1$  and  $M$ , then  $f + g$  a.e.cpfunc =  $f_1 + g_1$  and  $M$ .
- (26) If  $f$  a.e.cpfunc =  $g$  and  $M$ , then  $a \cdot f$  a.e.cpfunc =  $a \cdot g$  and  $M$ .

Let  $X$  be a non empty set, let  $S$  be a  $\sigma$ -field of subsets of  $X$ , and let  $M$  be a  $\sigma$ -measure on  $S$ . The almost zero cfunctions of  $M$  yields a non empty subset of  $\text{CLSp } L_1\text{Funct } M$  and is defined by the condition (Def. 12).

(Def. 12) The almost zero cfunctions of  $M = \{f; f \text{ ranges over partial functions from } X \text{ to } \mathbb{C}: f \in L_1\text{CFunctions } M \wedge f \text{ a.e.cpfunc} = X \mapsto 0_{\mathbb{C}} \text{ and } M\}$ .

One can prove the following proposition

- (27)  $(X \mapsto 0_{\mathbb{C}}) + (X \mapsto 0_{\mathbb{C}}) = X \mapsto 0_{\mathbb{C}}$  and  $a \cdot (X \mapsto 0_{\mathbb{C}}) = X \mapsto 0_{\mathbb{C}}$ .

Let  $X$  be a non empty set, let  $S$  be a  $\sigma$ -field of subsets of  $X$ , and let  $M$  be a  $\sigma$ -measure on  $S$ . One can check that the almost zero cfunctions of  $M$  is add closed and multiplicatively closed.

One can prove the following proposition

- (28)  $0_{\text{CLSp } L_1\text{Funct } M} = X \mapsto 0_{\mathbb{C}}$  and  $0_{\text{CLSp } L_1\text{Funct } M} \in$  the almost zero cfunctions of  $M$ .

Let  $X$  be a non empty set, let  $S$  be a  $\sigma$ -field of subsets of  $X$ , and let  $M$  be a  $\sigma$ -measure on  $S$ . The clsp almost zero functions of  $M$  yields a non empty CLS structure and is defined by the condition (Def. 13).

(Def. 13) The clsp almost zero functions of  $M = \langle$ the almost zero cfunctions of  $M, 0_{\text{CLSp } L_1\text{Funct } M}(\in$  the almost zero cfunctions of  $M), \text{add} |$ (the almost zero cfunctions of  $M, \text{CLSp } L_1\text{Funct } M), \cdot$ the almost zero cfunctions of  $M\rangle$ .

Let  $X$  be a non empty set, let  $S$  be a  $\sigma$ -field of subsets of  $X$ , and let  $M$  be a  $\sigma$ -measure on  $S$ . One can check that  $\text{CLSp } L_1\text{Funct } M$  is strict, Abelian,

add-associative, right zeroed, vector distributive, scalar distributive, scalar associative, and scalar unital.

In the sequel  $v, u$  are vectors of the clsp almost zero functions of  $M$ .

One can prove the following proposition

(29) If  $f = v$  and  $g = u$ , then  $f + g = v + u$ .

Let  $X$  be a non empty set, let  $S$  be a  $\sigma$ -field of subsets of  $X$ , let  $M$  be a  $\sigma$ -measure on  $S$ , and let  $f$  be a partial function from  $X$  to  $\mathbb{C}$ . The functor  $\text{a.e-Ceq-class}(f, M)$  yields a subset of  $L_1\text{CFunctions } M$  and is defined as follows:

(Def. 14)  $\text{a.e-Ceq-class}(f, M) = \{g; g \text{ ranges over partial functions from } X \text{ to } \mathbb{C}; g \in L_1\text{CFunctions } M \wedge f \in L_1\text{CFunctions } M \wedge f \text{ a.e.cpfunc} = g \text{ and } M\}$ .

Next we state several propositions:

(30) If  $f, g \in L_1\text{CFunctions } M$ , then  $g \text{ a.e.cpfunc} = f$  and  $M$  iff  $g \in \text{a.e-Ceq-class}(f, M)$ .

(31) If  $f \in L_1\text{CFunctions } M$ , then  $f \in \text{a.e-Ceq-class}(f, M)$ .

(32) If  $f, g \in L_1\text{CFunctions } M$ , then  $\text{a.e-Ceq-class}(f, M) = \text{a.e-Ceq-class}(g, M)$  iff  $f \text{ a.e.cpfunc} = g$  and  $M$ .

(33) If  $f, g \in L_1\text{CFunctions } M$ , then  $\text{a.e-Ceq-class}(f, M) = \text{a.e-Ceq-class}(g, M)$  iff  $g \in \text{a.e-Ceq-class}(f, M)$ .

(34) If  $f, f_1, g, g_1 \in L_1\text{CFunctions } M$  and  $\text{a.e-Ceq-class}(f, M) = \text{a.e-Ceq-class}(f_1, M)$  and  $\text{a.e-Ceq-class}(g, M) = \text{a.e-Ceq-class}(g_1, M)$ , then  $\text{a.e-Ceq-class}(f + g, M) = \text{a.e-Ceq-class}(f_1 + g_1, M)$ .

(35) If  $f, g \in L_1\text{CFunctions } M$  and  $\text{a.e-Ceq-class}(f, M) = \text{a.e-Ceq-class}(g, M)$ , then  $\text{a.e-Ceq-class}(a \cdot f, M) = \text{a.e-Ceq-class}(a \cdot g, M)$ .

Let  $X$  be a non empty set, let  $S$  be a  $\sigma$ -field of subsets of  $X$ , and let  $M$  be a  $\sigma$ -measure on  $S$ . The functor  $\text{CCosetSet } M$  yields a non empty family of subsets of  $L_1\text{CFunctions } M$  and is defined by:

(Def. 15)  $\text{CCosetSet } M = \{\text{a.e-Ceq-class}(f, M); f \text{ ranges over partial functions from } X \text{ to } \mathbb{C}; f \in L_1\text{CFunctions } M\}$ .

Let  $X$  be a non empty set, let  $S$  be a  $\sigma$ -field of subsets of  $X$ , and let  $M$  be a  $\sigma$ -measure on  $S$ . The functor  $\text{addCCoset } M$  yields a binary operation on  $\text{CCosetSet } M$  and is defined by the condition (Def. 16).

(Def. 16) Let  $A, B$  be elements of  $\text{CCosetSet } M$  and  $a, b$  be partial functions from  $X$  to  $\mathbb{C}$ . If  $a \in A$  and  $b \in B$ , then  $(\text{addCCoset } M)(A, B) = \text{a.e-Ceq-class}(a + b, M)$ .

Let  $X$  be a non empty set, let  $S$  be a  $\sigma$ -field of subsets of  $X$ , and let  $M$  be a  $\sigma$ -measure on  $S$ . The functor  $\text{zeroCCoset } M$  yielding an element of  $\text{CCosetSet } M$  is defined by:

(Def. 17)  $\text{zeroCCoset } M = \text{a.e-Ceq-class}(X \mapsto 0_{\mathbb{C}}, M)$ .

Let  $X$  be a non empty set, let  $S$  be a  $\sigma$ -field of subsets of  $X$ , and let  $M$  be a  $\sigma$ -measure on  $S$ . The functor  $\text{ImultCCoset } M$  yields a function from  $\mathbb{C} \times \text{CCosetSet } M$  into  $\text{CCosetSet } M$  and is defined by the condition (Def. 18).

(Def. 18) Let  $z$  be a complex number,  $A$  be an element of  $\text{CCosetSet } M$ , and  $f$  be a partial function from  $X$  to  $\mathbb{C}$ . If  $f \in A$ , then  $(\text{ImultCCoset } M)(z, A) = \text{a.e-Ceq-class}(z \cdot f, M)$ .

Let  $X$  be a non empty set, let  $S$  be a  $\sigma$ -field of subsets of  $X$ , and let  $M$  be a  $\sigma$ -measure on  $S$ . The functor  $\text{Pre-L-CTSpace } M$  yields a strict Abelian add-associative right zeroed right complementable vector distributive scalar distributive scalar associative scalar unital non empty CLS structure and is defined by the conditions (Def. 19).

- (Def. 19)(i) The carrier of  $\text{Pre-L-CTSpace } M = \text{CCosetSet } M$ ,
- (ii) the addition of  $\text{Pre-L-CTSpace } M = \text{addCCoset } M$ ,
- (iii)  $0_{\text{Pre-L-CTSpace } M} = \text{zeroCCoset } M$ , and
- (iv) the external multiplication of  $\text{Pre-L-CTSpace } M = \text{ImultCCoset } M$ .

### 5. COMPLEX NORMED SPACE OF INTEGRABLE FUNCTIONS

Next we state several propositions:

- (36) If  $f, g \in L_1\text{CFunctions } M$  and  $f \text{ a.e.cpfunc} = g$  and  $M$ , then  $\int f \, dM = \int g \, dM$ .
- (37) If  $f$  is integrable on  $M$ , then  $\int f \, dM \in \mathbb{C}$  and  $\int |f| \, dM \in \mathbb{R}$  and  $|f|$  is integrable on  $M$ .
- (38) If  $f, g \in L_1\text{CFunctions } M$  and  $f \text{ a.e.cpfunc} = g$  and  $M$ , then  $|f| \stackrel{M}{\text{a.e.}} |g|$  and  $\int |f| \, dM = \int |g| \, dM$ .
- (39) If there exists a vector  $x$  of  $\text{Pre-L-CTSpace } M$  such that  $f, g \in x$ , then  $f \text{ a.e.cpfunc} = g$  and  $M$  and  $f, g \in L_1\text{CFunctions } M$ .
- (40) There exists a function  $N_2$  from the carrier of  $\text{Pre-L-CTSpace } M$  into  $\mathbb{R}$  such that for every point  $x$  of  $\text{Pre-L-CTSpace } M$  holds there exists a partial function  $f$  from  $X$  to  $\mathbb{C}$  such that  $f \in x$  and  $N_2(x) = \int |f| \, dM$ .

In the sequel  $x$  is a point of  $\text{Pre-L-CTSpace } M$ .

The following two propositions are true:

- (41) If  $f \in x$ , then  $f$  is integrable on  $M$  and  $f \in L_1\text{CFunctions } M$  and  $|f|$  is integrable on  $M$ .
- (42) If  $f, g \in x$ , then  $f \text{ a.e.cpfunc} = g$  and  $M$  and  $\int f \, dM = \int g \, dM$  and  $\int |f| \, dM = \int |g| \, dM$ .

Let  $X$  be a non empty set, let  $S$  be a  $\sigma$ -field of subsets of  $X$ , and let  $M$  be a  $\sigma$ -measure on  $S$ . The functor  $L-1\text{-CNorm } M$  yields a function from the carrier of  $\text{Pre-L-CTSpace } M$  into  $\mathbb{R}$  and is defined by:

(Def. 20) For every point  $x$  of Pre-L- $\mathbb{C}$ Space  $M$  there exists a partial function  $f$  from  $X$  to  $\mathbb{C}$  such that  $f \in x$  and  $(L-1-CNorm M)(x) = \int |f| dM$ .

Let  $X$  be a non empty set, let  $S$  be a  $\sigma$ -field of subsets of  $X$ , and let  $M$  be a  $\sigma$ -measure on  $S$ . The functor L-1- $\mathbb{C}$ Space  $M$  yields a non empty complex normed space structure and is defined as follows:

(Def. 21) L-1- $\mathbb{C}$ Space  $M = \langle$ the carrier of Pre-L- $\mathbb{C}$ Space  $M$ , the zero of Pre-L- $\mathbb{C}$ Space  $M$ , the addition of Pre-L- $\mathbb{C}$ Space  $M$ , the external multiplication of Pre-L- $\mathbb{C}$ Space  $M$ , L-1-CNorm  $M$  $\rangle$ .

In the sequel  $x$  denotes a point of L-1- $\mathbb{C}$ Space  $M$ .

Next we state several propositions:

(43)(i) There exists a partial function  $f$  from  $X$  to  $\mathbb{C}$  such that  $f \in L_1\mathbb{C}Functions M$  and  $x = \text{a.e-Ceq-class}(f, M)$  and  $\|x\| = \int |f| dM$ , and

(ii) for every partial function  $f$  from  $X$  to  $\mathbb{C}$  such that  $f \in x$  holds  $\int |f| dM = \|x\|$ .

(44) If  $f \in x$ , then  $x = \text{a.e-Ceq-class}(f, M)$  and  $\|x\| = \int |f| dM$ .

(45) If  $f \in x$  and  $g \in y$ , then  $f + g \in x + y$  and if  $f \in x$ , then  $a \cdot f \in a \cdot x$ .

(46) If  $f \in L_1\mathbb{C}Functions M$  and  $\int |f| dM = 0$ , then  $f \text{ a.e.cpfunc} = X \mapsto 0_{\mathbb{C}}$  and  $M$ .

(47) If  $f, g \in L_1\mathbb{C}Functions M$ , then  $\int |f + g| dM \leq \int |f| dM + \int |g| dM$ .

Let  $X$  be a non empty set, let  $S$  be a  $\sigma$ -field of subsets of  $X$ , and let  $M$  be a  $\sigma$ -measure on  $S$ . One can check that L-1- $\mathbb{C}$ Space  $M$  is complex normed space-like, vector distributive, scalar distributive, scalar associative, scalar unital, Abelian, add-associative, right zeroed, and right complementable.

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