

On L^1 Space Formed by Complex-Valued Partial Functions

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Summary. In this article, we formalized L^1 space formed by complex-valued partial functions [11], [15]. The real-valued case was formalized in [22] and this article is its generalization.

MML identifier: LPSPACC1, version: 8.0.01 5.4.1165

The notation and terminology used here have been introduced in the following papers: [4], [10], [5], [19], [17], [6], [7], [1], [22], [3], [18], [13], [16], [8], [14], [23], [24], [12], [20], [21], [2], and [9].

1. PRELIMINARIES OF COMPLEX LINEAR SPACE

Let D be a non empty set and let E be a complex-membered set. One can verify that every element of $D \dot{\rightarrow} E$ is complex-valued.

Let D be a non empty set, let E be a complex-membered set, and let F_1, F_2 be elements of $D \dot{\rightarrow} E$. Then $F_1 + F_2$ is an element of $D \dot{\rightarrow} \mathbb{C}$. Then $F_1 - F_2$ is an element of $D \dot{\rightarrow} \mathbb{C}$. Then $F_1 \cdot F_2$ is an element of $D \dot{\rightarrow} \mathbb{C}$. Then F_1/F_2 is an element of $D \dot{\rightarrow} \mathbb{C}$.

Let D be a non empty set, let E be a complex-membered set, let F be an element of $D \dot{\rightarrow} E$, and let a be a complex number. Then $a \cdot F$ is an element of $D \dot{\rightarrow} \mathbb{C}$.

Let V be a non empty CLS structure and let V_1 be a subset of V . We say that V_1 is multiplicatively closed if and only if:

- (Def. 1) For every complex number a and for every vector v of V such that $v \in V_1$ holds $a \cdot v \in V_1$.

Next we state the proposition

- (1) Let V be a complex linear space and V_1 be a subset of V . Then V_1 is linearly closed if and only if V_1 is add closed and multiplicatively closed.

Let V be a non empty CLS structure. One can verify that there exists a non empty subset of V which is add closed and multiplicatively closed.

Let X be a non empty CLS structure and let X_1 be a multiplicatively closed non empty subset of X . The functor $\cdot_{(X_1)}$ yields a function from $\mathbb{C} \times X_1$ into X_1 and is defined by:

- (Def. 2) $\cdot_{(X_1)} = (\text{the external multiplication of } X) \upharpoonright (\mathbb{C} \times X_1)$.

In the sequel a, b, r denote complex numbers and V denotes a complex linear space.

We now state two propositions:

- (2) Let V be an Abelian add-associative right zeroed vector distributive scalar distributive scalar associative scalar unital non empty CLS structure, V_1 be a non empty subset of V , d_1 be an element of V_1 , A be a binary operation on V_1 , and M be a function from $\mathbb{C} \times V_1$ into V_1 . Suppose $d_1 = 0_V$ and $A = (\text{the addition of } V) \upharpoonright (V_1)$ and $M = (\text{the external multiplication of } V) \upharpoonright (\mathbb{C} \times V_1)$. Then $\langle V_1, d_1, A, M \rangle$ is Abelian, add-associative, right zeroed, vector distributive, scalar distributive, scalar associative, and scalar unital.
- (3) Let V be an Abelian add-associative right zeroed vector distributive scalar distributive scalar associative scalar unital non empty CLS structure and V_1 be an add closed multiplicatively closed non empty subset of V . Suppose $0_V \in V_1$. Then $\langle V_1, 0_V (\in V_1), \text{add} \upharpoonright (V_1, V), \cdot_{(V_1)} \rangle$ is Abelian, add-associative, right zeroed, vector distributive, scalar distributive, scalar associative, and scalar unital.

2. QUASI-COMPLEX LINEAR SPACE OF PARTIAL FUNCTIONS

We follow the rules: A, B are non empty sets and f, g, h are elements of $A \dot{\rightarrow} \mathbb{C}$.

Let us consider A . The functor $\text{multcpfunc } A$ yielding a binary operation on $A \dot{\rightarrow} \mathbb{C}$ is defined as follows:

- (Def. 3) For all elements f, g of $A \dot{\rightarrow} \mathbb{C}$ holds $(\text{multcpfunc } A)(f, g) = f \cdot g$.

Let us consider A . The functor $\text{multcomplexcpfunc } A$ yielding a function from $\mathbb{C} \times (A \dot{\rightarrow} \mathbb{C})$ into $A \dot{\rightarrow} \mathbb{C}$ is defined by:

- (Def. 4) For every complex number a and for every element f of $A \dot{\rightarrow} \mathbb{C}$ holds $(\text{multcomplexcpfunc } A)(a, f) = a \cdot f$.

Let D be a non empty set. The functor $\text{addecpfunc } D$ yields a binary operation on $D \rightarrow \mathbb{C}$ and is defined as follows:

(Def. 5) For all elements F_1, F_2 of $D \rightarrow \mathbb{C}$ holds $(\text{addecpfunc } D)(F_1, F_2) = F_1 + F_2$.

Let A be a set. The functor $\text{CPFuncZero } A$ yields an element of $A \rightarrow \mathbb{C}$ and is defined by:

(Def. 6) $\text{CPFuncZero } A = A \mapsto 0_{\mathbb{C}}$.

Let A be a set. The functor $\text{CPFuncUnit } A$ yielding an element of $A \rightarrow \mathbb{C}$ is defined as follows:

(Def. 7) $\text{CPFuncUnit } A = A \mapsto 1_{\mathbb{C}}$.

The following propositions are true:

(4) $h = (\text{addecpfunc } A)(f, g)$ iff $\text{dom } h = \text{dom } f \cap \text{dom } g$ and for every element x of A such that $x \in \text{dom } h$ holds $h(x) = f(x) + g(x)$.

(5) $h = (\text{multcpfunc } A)(f, g)$ iff $\text{dom } h = \text{dom } f \cap \text{dom } g$ and for every element x of A such that $x \in \text{dom } h$ holds $h(x) = f(x) \cdot g(x)$.

(6) $\text{CPFuncZero } A \neq \text{CPFuncUnit } A$.

(7) $h = (\text{multcomplexcpfunc } A)(a, f)$ iff $\text{dom } h = \text{dom } f$ and for every element x of A such that $x \in \text{dom } f$ holds $h(x) = a \cdot f(x)$.

Let us consider A . Note that $\text{addecpfunc } A$ is commutative and associative.

Observe that $\text{multcpfunc } A$ is commutative and associative.

One can prove the following propositions:

(8) $\text{CPFuncUnit } A$ is a unity w.r.t. $\text{multcpfunc } A$.

(9) $\text{CPFuncZero } A$ is a unity w.r.t. $\text{addecpfunc } A$.

(10) $(\text{addecpfunc } A)(f, (\text{multcomplexcpfunc } A)(-1_{\mathbb{C}}, f)) = \text{CPFuncZero } A \upharpoonright \text{dom } f$.

(11) $(\text{multcomplexcpfunc } A)(1_{\mathbb{C}}, f) = f$.

(12) $(\text{multcomplexcpfunc } A)(a, (\text{multcomplexcpfunc } A)(b, f)) = (\text{multcomplexcpfunc } A)(a \cdot b, f)$.

(13) $(\text{addecpfunc } A)((\text{multcomplexcpfunc } A)(a, f), (\text{multcomplexcpfunc } A)(b, f)) = (\text{multcomplexcpfunc } A)(a + b, f)$.

(14) $(\text{multcpfunc } A)(f, (\text{addecpfunc } A)(g, h)) = (\text{addecpfunc } A)((\text{multcpfunc } A)(f, g), (\text{multcpfunc } A)(f, h))$.

(15) $(\text{multcpfunc } A)((\text{multcomplexcpfunc } A)(a, f), g) = (\text{multcomplexcpfunc } A)(a, (\text{multcpfunc } A)(f, g))$.

Let us consider A . The functor $\text{CLSp PFunc } A$ yields a non empty CLS structure and is defined as follows:

(Def. 8) $\text{CLSp PFunc } A = \langle A \rightarrow \mathbb{C}, \text{CPFuncZero } A, \text{addecpfunc } A, \text{multcomplexcpfunc } A \rangle$.

In the sequel u, v, w are vectors of $\text{CLSp PFunc } A$.

Note that CLSp PFunc A is strict, Abelian, add-associative, right zeroed, vector distributive, scalar distributive, scalar associative, and scalar unital.

3. QUASI-COMPLEX LINEAR SPACE OF INTEGRABLE FUNCTIONS

For simplicity, we use the following convention: X is a non empty set, x is an element of X , S is a σ -field of subsets of X , M is a σ -measure on S , E, A are elements of S , and f, g, h, f_1, g_1 are partial functions from X to \mathbb{C} .

Let us consider X and let f be a partial function from X to \mathbb{C} . Note that $|f|$ is non-negative.

Next we state the proposition

- (16) Let f be a partial function from X to \mathbb{C} . Suppose $\text{dom } f \in S$ and for every x such that $x \in \text{dom } f$ holds $0 = f(x)$. Then f is integrable on M and $\int f \, dM = 0$.

Let X be a non empty set, let S be a σ -field of subsets of X , and let M be a σ -measure on S . The functor $L_1\text{CFunctions } M$ yielding a non empty subset of CLSp PFunc X is defined by the condition (Def. 9).

- (Def. 9) $L_1\text{CFunctions } M = \{f; f \text{ ranges over partial functions from } X \text{ to } \mathbb{C}; \bigvee_{N_1: \text{element of } S} (M(N_1) = 0 \wedge \text{dom } f = N_1^c \wedge f \text{ is integrable on } M)\}$.

The following propositions are true:

- (17) If $f, g \in L_1\text{CFunctions } M$, then $f + g \in L_1\text{CFunctions } M$.
 (18) If $f \in L_1\text{CFunctions } M$, then $a \cdot f \in L_1\text{CFunctions } M$.

Note that $L_1\text{CFunctions } M$ is multiplicatively closed and add closed.

The functor CLSp $L_1\text{Func } M$ yielding a non empty CLS structure is defined by:

- (Def. 10) $\text{CLSp } L_1\text{Func } M = \langle L_1\text{CFunctions } M, 0_{\text{CLSp PFunc } X} (\in L_1\text{CFunctions } M), \text{add } |(L_1\text{CFunctions } M, \text{CLSp PFunc } X), \cdot_{L_1\text{CFunctions } M} \rangle$.

One can verify that CLSp $L_1\text{Func } M$ is strict, Abelian, add-associative, right zeroed, vector distributive, scalar distributive, scalar associative, and scalar unital.

4. QUOTIENT SPACE OF QUASI-COMPLEX LINEAR SPACE OF INTEGRABLE FUNCTIONS

In the sequel v, u are vectors of CLSp $L_1\text{Func } M$.

Next we state two propositions:

- (19) If $f = v$ and $g = u$, then $f + g = v + u$.
 (20) If $f = u$, then $a \cdot f = a \cdot u$.

Let X be a non empty set, let S be a σ -field of subsets of X , let M be a σ -measure on S , and let f, g be partial functions from X to \mathbb{C} . We say that f a.e.cpfunc = g and M if and only if:

(Def. 11) There exists an element E of S such that $M(E) = 0$ and $f|E^c = g|E^c$.

We now state several propositions:

- (21) Suppose $f = u$. Then
 - (i) $u + (-1_{\mathbb{C}}) \cdot u = (X \mapsto 0_{\mathbb{C}})|\text{dom } f$, and
 - (ii) there exist partial functions v, g from X to \mathbb{C} such that $v, g \in L_1\text{CFunctions } M$ and $v = u + (-1_{\mathbb{C}}) \cdot u$ and $g = X \mapsto 0_{\mathbb{C}}$ and v a.e.cpfunc = g and M .
- (22) f a.e.cpfunc = f and M .
- (23) If f a.e.cpfunc = g and M , then g a.e.cpfunc = f and M .
- (24) If f a.e.cpfunc = g and M and g a.e.cpfunc = h and M , then f a.e.cpfunc = h and M .
- (25) If f a.e.cpfunc = f_1 and M and g a.e.cpfunc = g_1 and M , then $f + g$ a.e.cpfunc = $f_1 + g_1$ and M .
- (26) If f a.e.cpfunc = g and M , then $a \cdot f$ a.e.cpfunc = $a \cdot g$ and M .

Let X be a non empty set, let S be a σ -field of subsets of X , and let M be a σ -measure on S . The almost zero cfunctions of M yields a non empty subset of $\text{CLSp } L_1\text{Funct } M$ and is defined by the condition (Def. 12).

(Def. 12) The almost zero cfunctions of $M = \{f; f \text{ ranges over partial functions from } X \text{ to } \mathbb{C}: f \in L_1\text{CFunctions } M \wedge f \text{ a.e.cpfunc} = X \mapsto 0_{\mathbb{C}} \text{ and } M\}$.

One can prove the following proposition

- (27) $(X \mapsto 0_{\mathbb{C}}) + (X \mapsto 0_{\mathbb{C}}) = X \mapsto 0_{\mathbb{C}}$ and $a \cdot (X \mapsto 0_{\mathbb{C}}) = X \mapsto 0_{\mathbb{C}}$.

Let X be a non empty set, let S be a σ -field of subsets of X , and let M be a σ -measure on S . One can check that the almost zero cfunctions of M is add closed and multiplicatively closed.

One can prove the following proposition

- (28) $0_{\text{CLSp } L_1\text{Funct } M} = X \mapsto 0_{\mathbb{C}}$ and $0_{\text{CLSp } L_1\text{Funct } M} \in$ the almost zero cfunctions of M .

Let X be a non empty set, let S be a σ -field of subsets of X , and let M be a σ -measure on S . The clsp almost zero functions of M yields a non empty CLS structure and is defined by the condition (Def. 13).

(Def. 13) The clsp almost zero functions of $M = \langle$ the almost zero cfunctions of $M, 0_{\text{CLSp } L_1\text{Funct } M}(\in$ the almost zero cfunctions of $M), \text{add} |$ (the almost zero cfunctions of $M, \text{CLSp } L_1\text{Funct } M), \cdot$ the almost zero cfunctions of $M\rangle$.

Let X be a non empty set, let S be a σ -field of subsets of X , and let M be a σ -measure on S . One can check that $\text{CLSp } L_1\text{Funct } M$ is strict, Abelian,

add-associative, right zeroed, vector distributive, scalar distributive, scalar associative, and scalar unital.

In the sequel v, u are vectors of the clsp almost zero functions of M .

One can prove the following proposition

(29) If $f = v$ and $g = u$, then $f + g = v + u$.

Let X be a non empty set, let S be a σ -field of subsets of X , let M be a σ -measure on S , and let f be a partial function from X to \mathbb{C} . The functor $\text{a.e-Ceq-class}(f, M)$ yields a subset of $L_1\text{CFunctions } M$ and is defined as follows:

(Def. 14) $\text{a.e-Ceq-class}(f, M) = \{g; g \text{ ranges over partial functions from } X \text{ to } \mathbb{C}; g \in L_1\text{CFunctions } M \wedge f \in L_1\text{CFunctions } M \wedge f \text{ a.e.cpfunc} = g \text{ and } M\}$.

Next we state several propositions:

(30) If $f, g \in L_1\text{CFunctions } M$, then $g \text{ a.e.cpfunc} = f$ and M iff $g \in \text{a.e-Ceq-class}(f, M)$.

(31) If $f \in L_1\text{CFunctions } M$, then $f \in \text{a.e-Ceq-class}(f, M)$.

(32) If $f, g \in L_1\text{CFunctions } M$, then $\text{a.e-Ceq-class}(f, M) = \text{a.e-Ceq-class}(g, M)$ iff $f \text{ a.e.cpfunc} = g$ and M .

(33) If $f, g \in L_1\text{CFunctions } M$, then $\text{a.e-Ceq-class}(f, M) = \text{a.e-Ceq-class}(g, M)$ iff $g \in \text{a.e-Ceq-class}(f, M)$.

(34) If $f, f_1, g, g_1 \in L_1\text{CFunctions } M$ and $\text{a.e-Ceq-class}(f, M) = \text{a.e-Ceq-class}(f_1, M)$ and $\text{a.e-Ceq-class}(g, M) = \text{a.e-Ceq-class}(g_1, M)$, then $\text{a.e-Ceq-class}(f + g, M) = \text{a.e-Ceq-class}(f_1 + g_1, M)$.

(35) If $f, g \in L_1\text{CFunctions } M$ and $\text{a.e-Ceq-class}(f, M) = \text{a.e-Ceq-class}(g, M)$, then $\text{a.e-Ceq-class}(a \cdot f, M) = \text{a.e-Ceq-class}(a \cdot g, M)$.

Let X be a non empty set, let S be a σ -field of subsets of X , and let M be a σ -measure on S . The functor $\text{CCosetSet } M$ yields a non empty family of subsets of $L_1\text{CFunctions } M$ and is defined by:

(Def. 15) $\text{CCosetSet } M = \{\text{a.e-Ceq-class}(f, M); f \text{ ranges over partial functions from } X \text{ to } \mathbb{C}; f \in L_1\text{CFunctions } M\}$.

Let X be a non empty set, let S be a σ -field of subsets of X , and let M be a σ -measure on S . The functor $\text{addCCoset } M$ yields a binary operation on $\text{CCosetSet } M$ and is defined by the condition (Def. 16).

(Def. 16) Let A, B be elements of $\text{CCosetSet } M$ and a, b be partial functions from X to \mathbb{C} . If $a \in A$ and $b \in B$, then $(\text{addCCoset } M)(A, B) = \text{a.e-Ceq-class}(a + b, M)$.

Let X be a non empty set, let S be a σ -field of subsets of X , and let M be a σ -measure on S . The functor $\text{zeroCCoset } M$ yielding an element of $\text{CCosetSet } M$ is defined by:

(Def. 17) $\text{zeroCCoset } M = \text{a.e-Ceq-class}(X \mapsto 0_{\mathbb{C}}, M)$.

Let X be a non empty set, let S be a σ -field of subsets of X , and let M be a σ -measure on S . The functor $\text{ImultCCoset } M$ yields a function from $\mathbb{C} \times \text{CCosetSet } M$ into $\text{CCosetSet } M$ and is defined by the condition (Def. 18).

(Def. 18) Let z be a complex number, A be an element of $\text{CCosetSet } M$, and f be a partial function from X to \mathbb{C} . If $f \in A$, then $(\text{ImultCCoset } M)(z, A) = \text{a.e-Ceq-class}(z \cdot f, M)$.

Let X be a non empty set, let S be a σ -field of subsets of X , and let M be a σ -measure on S . The functor $\text{Pre-L-CTSpace } M$ yields a strict Abelian add-associative right zeroed right complementable vector distributive scalar distributive scalar associative scalar unital non empty CLS structure and is defined by the conditions (Def. 19).

- (Def. 19)(i) The carrier of $\text{Pre-L-CTSpace } M = \text{CCosetSet } M$,
- (ii) the addition of $\text{Pre-L-CTSpace } M = \text{addCCoset } M$,
- (iii) $0_{\text{Pre-L-CTSpace } M} = \text{zeroCCoset } M$, and
- (iv) the external multiplication of $\text{Pre-L-CTSpace } M = \text{ImultCCoset } M$.

5. COMPLEX NORMED SPACE OF INTEGRABLE FUNCTIONS

Next we state several propositions:

- (36) If $f, g \in L_1\text{CFunctions } M$ and $f \text{ a.e.cpfunc} = g$ and M , then $\int f \, dM = \int g \, dM$.
- (37) If f is integrable on M , then $\int f \, dM \in \mathbb{C}$ and $\int |f| \, dM \in \mathbb{R}$ and $|f|$ is integrable on M .
- (38) If $f, g \in L_1\text{CFunctions } M$ and $f \text{ a.e.cpfunc} = g$ and M , then $|f| \stackrel{M}{\text{a.e.}} |g|$ and $\int |f| \, dM = \int |g| \, dM$.
- (39) If there exists a vector x of $\text{Pre-L-CTSpace } M$ such that $f, g \in x$, then $f \text{ a.e.cpfunc} = g$ and M and $f, g \in L_1\text{CFunctions } M$.
- (40) There exists a function N_2 from the carrier of $\text{Pre-L-CTSpace } M$ into \mathbb{R} such that for every point x of $\text{Pre-L-CTSpace } M$ holds there exists a partial function f from X to \mathbb{C} such that $f \in x$ and $N_2(x) = \int |f| \, dM$.

In the sequel x is a point of $\text{Pre-L-CTSpace } M$.

The following two propositions are true:

- (41) If $f \in x$, then f is integrable on M and $f \in L_1\text{CFunctions } M$ and $|f|$ is integrable on M .
- (42) If $f, g \in x$, then $f \text{ a.e.cpfunc} = g$ and M and $\int f \, dM = \int g \, dM$ and $\int |f| \, dM = \int |g| \, dM$.

Let X be a non empty set, let S be a σ -field of subsets of X , and let M be a σ -measure on S . The functor $L-1\text{-CNorm } M$ yields a function from the carrier of $\text{Pre-L-CTSpace } M$ into \mathbb{R} and is defined by:

(Def. 20) For every point x of Pre-L- \mathbb{C} Space M there exists a partial function f from X to \mathbb{C} such that $f \in x$ and $(L-1-CNorm M)(x) = \int |f| dM$.

Let X be a non empty set, let S be a σ -field of subsets of X , and let M be a σ -measure on S . The functor L-1- \mathbb{C} Space M yields a non empty complex normed space structure and is defined as follows:

(Def. 21) L-1- \mathbb{C} Space $M = \langle$ the carrier of Pre-L- \mathbb{C} Space M , the zero of Pre-L- \mathbb{C} Space M , the addition of Pre-L- \mathbb{C} Space M , the external multiplication of Pre-L- \mathbb{C} Space M , L-1-CNorm M \rangle .

In the sequel x denotes a point of L-1- \mathbb{C} Space M .

Next we state several propositions:

(43)(i) There exists a partial function f from X to \mathbb{C} such that $f \in L_1\mathbb{C}Functions M$ and $x = \text{a.e-Ceq-class}(f, M)$ and $\|x\| = \int |f| dM$, and

(ii) for every partial function f from X to \mathbb{C} such that $f \in x$ holds $\int |f| dM = \|x\|$.

(44) If $f \in x$, then $x = \text{a.e-Ceq-class}(f, M)$ and $\|x\| = \int |f| dM$.

(45) If $f \in x$ and $g \in y$, then $f + g \in x + y$ and if $f \in x$, then $a \cdot f \in a \cdot x$.

(46) If $f \in L_1\mathbb{C}Functions M$ and $\int |f| dM = 0$, then $f \text{ a.e.cpfunc} = X \mapsto 0_{\mathbb{C}}$ and M .

(47) If $f, g \in L_1\mathbb{C}Functions M$, then $\int |f + g| dM \leq \int |f| dM + \int |g| dM$.

Let X be a non empty set, let S be a σ -field of subsets of X , and let M be a σ -measure on S . One can check that L-1- \mathbb{C} Space M is complex normed space-like, vector distributive, scalar distributive, scalar associative, scalar unital, Abelian, add-associative, right zeroed, and right complementable.

REFERENCES

- [1] Jonathan Backer, Piotr Rudnicki, and Christoph Schwarzweiler. Ring ideals. *Formalized Mathematics*, 9(3):565–582, 2001.
- [2] Józef Białas. Series of positive real numbers. Measure theory. *Formalized Mathematics*, 2(1):173–183, 1991.
- [3] Józef Białas. The σ -additive measure theory. *Formalized Mathematics*, 2(2):263–270, 1991.
- [4] Czesław Byliński. Binary operations. *Formalized Mathematics*, 1(1):175–180, 1990.
- [5] Czesław Byliński. The complex numbers. *Formalized Mathematics*, 1(3):507–513, 1990.
- [6] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(1):55–65, 1990.
- [7] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(1):153–164, 1990.
- [8] Czesław Byliński. Partial functions. *Formalized Mathematics*, 1(2):357–367, 1990.
- [9] Czesław Byliński. Some basic properties of sets. *Formalized Mathematics*, 1(1):47–53, 1990.
- [10] Noboru Endou. Complex linear space and complex normed space. *Formalized Mathematics*, 12(2):93–102, 2004.
- [11] P. R. Halmos. *Measure Theory*. Springer-Verlag, 1974.
- [12] Jarosław Kotowicz and Yuji Sakai. Properties of partial functions from a domain to the set of real numbers. *Formalized Mathematics*, 3(2):279–288, 1992.

- [13] Keiko Narita, Noboru Endou, and Yasunari Shidama. Integral of complex-valued measurable function. *Formalized Mathematics*, 16(4):319–324, 2008, doi:10.2478/v10037-008-0039-6.
- [14] Andrzej Nędzusiak. σ -fields and probability. *Formalized Mathematics*, 1(2):401–407, 1990.
- [15] Walter Rudin. *Real and Complex Analysis*. Mc Graw-Hill, Inc., 1974.
- [16] Yasunari Shidama and Noboru Endou. Integral of real-valued measurable function. *Formalized Mathematics*, 14(4):143–152, 2006, doi:10.2478/v10037-006-0018-8.
- [17] Andrzej Trybulec. Binary operations applied to functions. *Formalized Mathematics*, 1(2):329–334, 1990.
- [18] Andrzej Trybulec. On the sets inhabited by numbers. *Formalized Mathematics*, 11(4):341–347, 2003.
- [19] Andrzej Trybulec and Agata Darmochwał. Boolean domains. *Formalized Mathematics*, 1(1):187–190, 1990.
- [20] Wojciech A. Trybulec. Vectors in real linear space. *Formalized Mathematics*, 1(2):291–296, 1990.
- [21] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(1):67–71, 1990.
- [22] Yasushige Watase, Noboru Endou, and Yasunari Shidama. On L^1 space formed by real-valued partial functions. *Formalized Mathematics*, 16(4):361–369, 2008, doi:10.2478/v10037-008-0044-9.
- [23] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1(1):73–83, 1990.
- [24] Edmund Woronowicz. Relations defined on sets. *Formalized Mathematics*, 1(1):181–186, 1990.

Received August 27, 2012
