

# The $C^k$ Space<sup>1</sup>

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**Summary.** In this article, we formalize continuous differentiability of real-valued functions on  $n$ -dimensional real normed linear spaces. Next, we give a definition of the  $C^k$  space according to [23].

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The notation and terminology used in this paper have been introduced in the following articles: [1], [4], [10], [3], [5], [11], [17], [6], [7], [19], [18], [2], [8], [14], [12], [15], [13], [21], [22], [16], [20], and [9].

## 1. DEFINITION OF CONTINUOUSLY DIFFERENTIABLE FUNCTIONS AND SOME PROPERTIES

Let  $m$  be a non zero element of  $\mathbb{N}$ ,  $f$  be a partial function from  $\mathcal{R}^m$  to  $\mathbb{R}$ ,  $k$  be an element of  $\mathbb{N}$ , and  $Z$  be a set. We say that  $f$  is continuously differentiable up to order of  $k$  and  $Z$  if and only if

- (Def. 1) (i)  $Z \subseteq \text{dom } f$ , and  
(ii)  $f$  is partial differentiable up to order  $k$  and  $Z$ , and  
(iii) for every non empty finite sequence  $I$  of elements of  $\mathbb{N}$  such that  $\text{len } I \leq k$  and  $\text{rng } I \subseteq \text{Seg } m$  holds  $f|{}^I Z$  is continuous on  $Z$ .

Now we state the propositions:

- (1) Let us consider a non zero element  $m$  of  $\mathbb{N}$ , a set  $Z$ , a non empty finite sequence  $I$  of elements of  $\mathbb{N}$ , and a partial function  $f$  from  $\mathcal{R}^m$  to  $\mathbb{R}$ . Suppose  $f$  is partially differentiable on  $Z$  w.r.t.  $I$ . Then  $\text{dom}(f|{}^I Z) = Z$ .

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- (2) Let us consider a non zero element  $m$  of  $\mathbb{N}$ , an element  $k$  of  $\mathbb{N}$ , a non empty subset  $X$  of  $\mathcal{R}^m$ , and a partial function  $f$  from  $\mathcal{R}^m$  to  $\mathbb{R}$ . Suppose
- (i)  $X$  is open, and
  - (ii)  $X \subseteq \text{dom } f$ .

Then  $f$  is continuously differentiable up to order of 1 and  $X$  if and only if  $f$  is differentiable on  $X$  and for every element  $x_0$  of  $\mathcal{R}^m$  and for every real number  $r$  such that  $x_0 \in X$  and  $0 < r$  there exists a real number  $s$  such that  $0 < s$  and for every element  $x_1$  of  $\mathcal{R}^m$  such that  $x_1 \in X$  and  $|x_1 - x_0| < s$  for every element  $v$  of  $\mathcal{R}^m$ ,  $|f'(x_1)(v) - f'(x_0)(v)| \leq r \cdot |v|$ .

- (3) Let us consider a non zero element  $m$  of  $\mathbb{N}$ , a non empty subset  $X$  of  $\mathcal{R}^m$ , and a partial function  $f$  from  $\mathcal{R}^m$  to  $\mathbb{R}$ . Suppose
- (i)  $X$  is open, and
  - (ii)  $X \subseteq \text{dom } f$ , and
  - (iii)  $f$  is continuously differentiable up to order of 1 and  $X$ .

Then  $f$  is continuous on  $X$ . The theorem is a consequence of (2).

- (4) Let us consider a non zero element  $m$  of  $\mathbb{N}$ , an element  $k$  of  $\mathbb{N}$ , a non empty subset  $X$  of  $\mathcal{R}^m$ , and partial functions  $f, g$  from  $\mathcal{R}^m$  to  $\mathbb{R}$ . Suppose
- (i)  $f$  is continuously differentiable up to order of  $k$  and  $X$ , and
  - (ii)  $g$  is continuously differentiable up to order of  $k$  and  $X$ , and
  - (iii)  $X$  is open.

Then  $f + g$  is continuously differentiable up to order of  $k$  and  $X$ . The theorem is a consequence of (1). PROOF: For every non empty finite sequence  $I$  of elements of  $\mathbb{N}$  such that  $\text{len } I \leq k$  and  $\text{rng } I \subseteq \text{Seg } m$  holds  $(f + g)|^I X$  is continuous on  $X$ .  $\square$

- (5) Let us consider a non zero element  $m$  of  $\mathbb{N}$ , an element  $k$  of  $\mathbb{N}$ , a non empty subset  $X$  of  $\mathcal{R}^m$ , a real number  $r$ , and a partial function  $f$  from  $\mathcal{R}^m$  to  $\mathbb{R}$ . Suppose
- (i)  $f$  is continuously differentiable up to order of  $k$  and  $X$ , and
  - (ii)  $X$  is open.

Then  $r \cdot f$  is continuously differentiable up to order of  $k$  and  $X$ . The theorem is a consequence of (1). PROOF: For every non empty finite sequence  $I$  of elements of  $\mathbb{N}$  such that  $\text{len } I \leq k$  and  $\text{rng } I \subseteq \text{Seg } m$  holds  $r \cdot f|^I X$  is continuous on  $X$ .  $\square$

- (6) Let us consider a non zero element  $m$  of  $\mathbb{N}$ , an element  $k$  of  $\mathbb{N}$ , a non empty subset  $X$  of  $\mathcal{R}^m$ , and partial functions  $f, g$  from  $\mathcal{R}^m$  to  $\mathbb{R}$ . Suppose
- (i)  $f$  is continuously differentiable up to order of  $k$  and  $X$ , and
  - (ii)  $g$  is continuously differentiable up to order of  $k$  and  $X$ , and

(iii)  $X$  is open.

Then  $f - g$  is continuously differentiable up to order of  $k$  and  $X$ . The theorem is a consequence of (1). PROOF: For every non empty finite sequence  $I$  of elements of  $\mathbb{N}$  such that  $\text{len } I \leq k$  and  $\text{rng } I \subseteq \text{Seg } m$  holds  $(f - g)|^I X$  is continuous on  $X$ .  $\square$

Let us consider a non zero element  $m$  of  $\mathbb{N}$ , a non empty subset  $Z$  of  $\mathcal{R}^m$ , a partial function  $f$  from  $\mathcal{R}^m$  to  $\mathbb{R}$ , and non empty finite sequences  $I, G$  of elements of  $\mathbb{N}$ . Now we state the propositions:

$$(7) \quad f|^{G \frown I} Z = (f|^{G Z})|^{I} Z.$$

$$(8) \quad f|^{G \frown I} Z \text{ is continuous on } Z \text{ if and only if } (f|^{G Z})|^{I} Z \text{ is continuous on } Z.$$

Now we state the propositions:

(9) Let us consider a non zero element  $m$  of  $\mathbb{N}$ , a non empty subset  $Z$  of  $\mathcal{R}^m$ , a partial function  $f$  from  $\mathcal{R}^m$  to  $\mathbb{R}$ , elements  $i, j$  of  $\mathbb{N}$ , and a non empty finite sequence  $I$  of elements of  $\mathbb{N}$ . Suppose

(i)  $f$  is continuously differentiable up to order of  $i + j$  and  $Z$ , and

(ii)  $\text{rng } I \subseteq \text{Seg } m$ , and

(iii)  $\text{len } I = j$ .

Then  $f|^{I} Z$  is continuously differentiable up to order of  $i$  and  $Z$ . The theorem is a consequence of (1) and (7).

(10) Let us consider a non zero element  $m$  of  $\mathbb{N}$ , a non empty subset  $Z$  of  $\mathcal{R}^m$ , a partial function  $f$  from  $\mathcal{R}^m$  to  $\mathbb{R}$ , and elements  $i, j$  of  $\mathbb{N}$ . Suppose

(i)  $f$  is continuously differentiable up to order of  $i$  and  $Z$ , and

(ii)  $j \leq i$ .

Then  $f$  is continuously differentiable up to order of  $j$  and  $Z$ .

(11) Let us consider a non zero element  $m$  of  $\mathbb{N}$  and a non empty subset  $Z$  of  $\mathcal{R}^m$ . Suppose  $Z$  is open. Let us consider an element  $k$  of  $\mathbb{N}$  and partial functions  $f, g$  from  $\mathcal{R}^m$  to  $\mathbb{R}$ . Suppose

(i)  $f$  is continuously differentiable up to order of  $k$  and  $Z$ , and

(ii)  $g$  is continuously differentiable up to order of  $k$  and  $Z$ .

Then  $f \cdot g$  is continuously differentiable up to order of  $k$  and  $Z$ . The theorem is a consequence of (10), (1), (3), (9), and (7). PROOF: Define  $\mathcal{P}[\text{element of } \mathbb{N}] \equiv$  for every partial functions  $f, g$  from  $\mathcal{R}^m$  to  $\mathbb{R}$  such that  $f$  is continuously differentiable up to order of  $\$1$  and  $Z$  and  $g$  is continuously differentiable up to order of  $\$1$  and  $Z$  holds  $f \cdot g$  is continuously differentiable up to order of  $\$1$  and  $Z$ . Set  $Z0 = (0 \text{ qua natural number})$ .  $\mathcal{P}[0]$ . For every element  $k$  of  $\mathbb{N}$  such that  $\mathcal{P}[k]$  holds  $\mathcal{P}[k + 1]$ .  $\square$

(12) Let us consider a non zero element  $m$  of  $\mathbb{N}$ , a partial function  $f$  from  $\mathcal{R}^m$  to  $\mathbb{R}$ , a non empty subset  $X$  of  $\mathcal{R}^m$ , and a real number  $d$ . Suppose

(i)  $X$  is open, and

(ii)  $f = X \mapsto d$ .

Let us consider an element  $x$  of  $\mathcal{R}^m$ . If  $x \in X$ , then  $f$  is differentiable in  $x$  and  $f'(x) = \mathcal{R}^m \mapsto 0$ .

(13) Let us consider a non zero element  $m$  of  $\mathbb{N}$ , a partial function  $f$  from  $\mathcal{R}^m$  to  $\mathbb{R}$ , a non empty subset  $X$  of  $\mathcal{R}^m$ , and a real number  $d$ . Suppose

(i)  $X$  is open, and

(ii)  $f = X \mapsto d$ .

Let us consider an element  $x_0$  of  $\mathcal{R}^m$  and a real number  $r$ . Suppose

(iii)  $x_0 \in X$ , and

(iv)  $0 < r$ .

Then there exists a real number  $s$  such that

(v)  $0 < s$ , and

(vi) for every element  $x_1$  of  $\mathcal{R}^m$  such that  $x_1 \in X$  and  $|x_1 - x_0| < s$  for every element  $v$  of  $\mathcal{R}^m$ ,  $|f'(x_1)(v) - f'(x_0)(v)| \leq r \cdot |v|$ .

The theorem is a consequence of (12).

(14) Let us consider a non zero element  $m$  of  $\mathbb{N}$ , a partial function  $f$  from  $\mathcal{R}^m$  to  $\mathbb{R}$ , a non empty subset  $X$  of  $\mathcal{R}^m$ , and a real number  $d$ . Suppose

(i)  $X$  is open, and

(ii)  $f = X \mapsto d$ .

Then

(iii)  $f$  is differentiable on  $X$ , and

(iv)  $\text{dom } f'_{\upharpoonright X} = X$ , and

(v) for every element  $x$  of  $\mathcal{R}^m$  such that  $x \in X$  holds  $(f'_{\upharpoonright X})_x = \mathcal{R}^m \mapsto 0$ .

The theorem is a consequence of (12).

(15) Let us consider a non zero element  $m$  of  $\mathbb{N}$ , a partial function  $f$  from  $\mathcal{R}^m$  to  $\mathbb{R}$ , a non empty subset  $X$  of  $\mathcal{R}^m$ , a real number  $d$ , and an element  $i$  of  $\mathbb{N}$ . Suppose

(i)  $X$  is open, and

(ii)  $f = X \mapsto d$ , and

(iii)  $1 \leq i \leq m$ .

Then

(iv)  $f$  is partially differentiable on  $X$  w.r.t.  $i$ , and

(v)  $f \upharpoonright^i X$  is continuous on  $X$ .

The theorem is a consequence of (14) and (13).

(16) Let us consider a non zero element  $m$  of  $\mathbb{N}$ , an element  $i$  of  $\mathbb{N}$ , a partial function  $f$  from  $\mathcal{R}^m$  to  $\mathbb{R}$ , a non empty subset  $X$  of  $\mathcal{R}^m$ , and a real number  $d$ . Suppose

- (i)  $X$  is open, and
- (ii)  $f = X \mapsto d$ , and
- (iii)  $1 \leq i \leq m$ .

Then  $f \upharpoonright^i X = X \mapsto 0$ . The theorem is a consequence of (15) and (12).

Let us consider a non zero element  $m$  of  $\mathbb{N}$ , a non empty finite sequence  $I$  of elements of  $\mathbb{N}$ , a non empty subset  $X$  of  $\mathcal{R}^m$ , a partial function  $f$  from  $\mathcal{R}^m$  to  $\mathbb{R}$ , and a real number  $d$ . Now we state the propositions:

(17) Suppose  $X$  is open and  $f = X \mapsto d$  and  $\text{rng } I \subseteq \text{Seg } m$ . Then

- (i)  $(\text{PartDiffSeq}(f, X, I))(0) = X \mapsto d$ , and
- (ii) for every element  $i$  of  $\mathbb{N}$  such that  $1 \leq i \leq \text{len } I$  holds  
 $(\text{PartDiffSeq}(f, X, I))(i) = X \mapsto 0$ .

(18) Suppose  $X$  is open and  $f = X \mapsto d$  and  $\text{rng } I \subseteq \text{Seg } m$ . Then

- (i)  $f$  is partially differentiable on  $X$  w.r.t.  $I$ , and
- (ii)  $f \upharpoonright^I X$  is continuous on  $X$ .

Now we state the proposition:

(19) Let us consider a non zero element  $m$  of  $\mathbb{N}$ , an element  $k$  of  $\mathbb{N}$ , a non empty subset  $X$  of  $\mathcal{R}^m$ , a partial function  $f$  from  $\mathcal{R}^m$  to  $\mathbb{R}$ , and a real number  $d$ . Suppose

- (i)  $X$  is open, and
- (ii)  $f = X \mapsto d$ .

Then  $f$  is continuously differentiable up to order of  $k$  and  $X$ . The theorem is a consequence of (18).

Let  $m$  be a non zero element of  $\mathbb{N}$ . Observe that there exists a non empty subset of  $\mathcal{R}^m$  which is open.

## 2. DEFINITION OF THE $C^k$ SPACE

Let  $m$  be a non zero element of  $\mathbb{N}$ ,  $k$  be an element of  $\mathbb{N}$ , and  $X$  be a non empty open subset of  $\mathcal{R}^m$ . The functor the  $C^k$  functions of  $k$  and  $X$  yielding a non empty subset of  $\text{RAlgebra } X$  is defined by the term

(Def. 2)  $\{f \text{ where } f \text{ is a partial function from } \mathcal{R}^m \text{ to } \mathbb{R} : f \text{ is continuously differentiable up to order of } k \text{ and } X \text{ and } \text{dom } f = X\}$ .

Let us note that the  $\mathbb{C}^k$  functions of  $k$  and  $X$  is additively linearly closed and multiplicatively closed.

The functor the  $\mathbb{R}$  algebra of  $\mathbb{C}^k$  functions of  $k$  and  $X$  yielding a subalgebra of  $\mathbb{R}\text{Algebra } X$  is defined by the term

- (Def. 3)  $\langle$ the  $\mathbb{C}^k$  functions of  $k$  and  $X$ ,  $\text{mult}$ (the  $\mathbb{C}^k$  functions of  $k$  and  $X$ ,  $\mathbb{R}\text{Algebra } X$ ),  $\text{Add}$ (the  $\mathbb{C}^k$  functions of  $k$  and  $X$ ,  $\mathbb{R}\text{Algebra } X$ ),  $\text{Mult}$ (the  $\mathbb{C}^k$  functions of  $k$  and  $X$ ,  $\mathbb{R}\text{Algebra } X$ ),  $\text{One}$ (the  $\mathbb{C}^k$  functions of  $k$  and  $X$ ,  $\mathbb{R}\text{Algebra } X$ ),  $\text{Zero}$ (the  $\mathbb{C}^k$  functions of  $k$  and  $X$ ,  $\mathbb{R}\text{Algebra } X$ ) $\rangle$ .

Let us note that the  $\mathbb{R}$  algebra of  $\mathbb{C}^k$  functions of  $k$  and  $X$  is Abelian add-associative right zeroed right complementable vector distributive scalar distributive scalar associative scalar unital commutative associative right unital right distributive and vector associative.

Now we state the propositions:

- (20) Let us consider a non zero element  $m$  of  $\mathbb{N}$ , an element  $k$  of  $\mathbb{N}$ , a non empty open subset  $X$  of  $\mathcal{R}^m$ , vectors  $F, G, H$  of the  $\mathbb{R}$  algebra of  $\mathbb{C}^k$  functions of  $k$  and  $X$ , and partial functions  $f, g, h$  from  $\mathcal{R}^m$  to  $\mathbb{R}$ . Suppose

- (i)  $f = F$ , and
- (ii)  $g = G$ , and
- (iii)  $h = H$ .

Then  $H = F + G$  if and only if for every element  $x$  of  $X$ ,  $h(x) = f(x) + g(x)$ .

- (21) Let us consider a non zero element  $m$  of  $\mathbb{N}$ , an element  $k$  of  $\mathbb{N}$ , a non empty open subset  $X$  of  $\mathcal{R}^m$ , vectors  $F, G, H$  of the  $\mathbb{R}$  algebra of  $\mathbb{C}^k$  functions of  $k$  and  $X$ , partial functions  $f, g, h$  from  $\mathcal{R}^m$  to  $\mathbb{R}$ , and a real number  $a$ . Suppose

- (i)  $f = F$ , and
- (ii)  $g = G$ .

Then  $G = a \cdot F$  if and only if for every element  $x$  of  $X$ ,  $g(x) = a \cdot f(x)$ .

- (22) Let us consider a non zero element  $m$  of  $\mathbb{N}$ , an element  $k$  of  $\mathbb{N}$ , a non empty open subset  $X$  of  $\mathcal{R}^m$ , vectors  $F, G, H$  of the  $\mathbb{R}$  algebra of  $\mathbb{C}^k$  functions of  $k$  and  $X$ , and partial functions  $f, g, h$  from  $\mathcal{R}^m$  to  $\mathbb{R}$ . Suppose

- (i)  $f = F$ , and
- (ii)  $g = G$ , and
- (iii)  $h = H$ .

Then  $H = F \cdot G$  if and only if for every element  $x$  of  $X$ ,  $h(x) = f(x) \cdot g(x)$ .

Let us consider a non zero element  $m$  of  $\mathbb{N}$ , an element  $k$  of  $\mathbb{N}$ , and a non empty open subset  $X$  of  $\mathcal{R}^m$ . Now we state the propositions:

- (23)  $0_\alpha = X \mapsto 0$ , where  $\alpha$  is the  $\mathbb{R}$  algebra of  $\mathbb{C}^k$  functions of  $k$  and  $X$ .
- (24)  $1_\alpha = X \mapsto 1$ , where  $\alpha$  is the  $\mathbb{R}$  algebra of  $\mathbb{C}^k$  functions of  $k$  and  $X$ .

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