

The C^k Space¹

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Summary. In this article, we formalize continuous differentiability of real-valued functions on n -dimensional real normed linear spaces. Next, we give a definition of the C^k space according to [23].

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The notation and terminology used in this paper have been introduced in the following articles: [1], [4], [10], [3], [5], [11], [17], [6], [7], [19], [18], [2], [8], [14], [12], [15], [13], [21], [22], [16], [20], and [9].

1. DEFINITION OF CONTINUOUSLY DIFFERENTIABLE FUNCTIONS AND SOME PROPERTIES

Let m be a non zero element of \mathbb{N} , f be a partial function from \mathcal{R}^m to \mathbb{R} , k be an element of \mathbb{N} , and Z be a set. We say that f is continuously differentiable up to order of k and Z if and only if

- (Def. 1) (i) $Z \subseteq \text{dom } f$, and
(ii) f is partial differentiable up to order k and Z , and
(iii) for every non empty finite sequence I of elements of \mathbb{N} such that $\text{len } I \leq k$ and $\text{rng } I \subseteq \text{Seg } m$ holds $f|{}^I Z$ is continuous on Z .

Now we state the propositions:

- (1) Let us consider a non zero element m of \mathbb{N} , a set Z , a non empty finite sequence I of elements of \mathbb{N} , and a partial function f from \mathcal{R}^m to \mathbb{R} . Suppose f is partially differentiable on Z w.r.t. I . Then $\text{dom}(f|{}^I Z) = Z$.

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- (2) Let us consider a non zero element m of \mathbb{N} , an element k of \mathbb{N} , a non empty subset X of \mathcal{R}^m , and a partial function f from \mathcal{R}^m to \mathbb{R} . Suppose
- (i) X is open, and
 - (ii) $X \subseteq \text{dom } f$.

Then f is continuously differentiable up to order of 1 and X if and only if f is differentiable on X and for every element x_0 of \mathcal{R}^m and for every real number r such that $x_0 \in X$ and $0 < r$ there exists a real number s such that $0 < s$ and for every element x_1 of \mathcal{R}^m such that $x_1 \in X$ and $|x_1 - x_0| < s$ for every element v of \mathcal{R}^m , $|f'(x_1)(v) - f'(x_0)(v)| \leq r \cdot |v|$.

- (3) Let us consider a non zero element m of \mathbb{N} , a non empty subset X of \mathcal{R}^m , and a partial function f from \mathcal{R}^m to \mathbb{R} . Suppose
- (i) X is open, and
 - (ii) $X \subseteq \text{dom } f$, and
 - (iii) f is continuously differentiable up to order of 1 and X .

Then f is continuous on X . The theorem is a consequence of (2).

- (4) Let us consider a non zero element m of \mathbb{N} , an element k of \mathbb{N} , a non empty subset X of \mathcal{R}^m , and partial functions f, g from \mathcal{R}^m to \mathbb{R} . Suppose
- (i) f is continuously differentiable up to order of k and X , and
 - (ii) g is continuously differentiable up to order of k and X , and
 - (iii) X is open.

Then $f + g$ is continuously differentiable up to order of k and X . The theorem is a consequence of (1). PROOF: For every non empty finite sequence I of elements of \mathbb{N} such that $\text{len } I \leq k$ and $\text{rng } I \subseteq \text{Seg } m$ holds $(f + g)|^I X$ is continuous on X . \square

- (5) Let us consider a non zero element m of \mathbb{N} , an element k of \mathbb{N} , a non empty subset X of \mathcal{R}^m , a real number r , and a partial function f from \mathcal{R}^m to \mathbb{R} . Suppose
- (i) f is continuously differentiable up to order of k and X , and
 - (ii) X is open.

Then $r \cdot f$ is continuously differentiable up to order of k and X . The theorem is a consequence of (1). PROOF: For every non empty finite sequence I of elements of \mathbb{N} such that $\text{len } I \leq k$ and $\text{rng } I \subseteq \text{Seg } m$ holds $r \cdot f|^I X$ is continuous on X . \square

- (6) Let us consider a non zero element m of \mathbb{N} , an element k of \mathbb{N} , a non empty subset X of \mathcal{R}^m , and partial functions f, g from \mathcal{R}^m to \mathbb{R} . Suppose
- (i) f is continuously differentiable up to order of k and X , and
 - (ii) g is continuously differentiable up to order of k and X , and

(iii) X is open.

Then $f - g$ is continuously differentiable up to order of k and X . The theorem is a consequence of (1). PROOF: For every non empty finite sequence I of elements of \mathbb{N} such that $\text{len } I \leq k$ and $\text{rng } I \subseteq \text{Seg } m$ holds $(f - g)|^I X$ is continuous on X . \square

Let us consider a non zero element m of \mathbb{N} , a non empty subset Z of \mathcal{R}^m , a partial function f from \mathcal{R}^m to \mathbb{R} , and non empty finite sequences I, G of elements of \mathbb{N} . Now we state the propositions:

$$(7) \quad f|^{G \frown I} Z = (f|^{G Z})|^{I} Z.$$

$$(8) \quad f|^{G \frown I} Z \text{ is continuous on } Z \text{ if and only if } (f|^{G Z})|^{I} Z \text{ is continuous on } Z.$$

Now we state the propositions:

(9) Let us consider a non zero element m of \mathbb{N} , a non empty subset Z of \mathcal{R}^m , a partial function f from \mathcal{R}^m to \mathbb{R} , elements i, j of \mathbb{N} , and a non empty finite sequence I of elements of \mathbb{N} . Suppose

(i) f is continuously differentiable up to order of $i + j$ and Z , and

(ii) $\text{rng } I \subseteq \text{Seg } m$, and

(iii) $\text{len } I = j$.

Then $f|^{I} Z$ is continuously differentiable up to order of i and Z . The theorem is a consequence of (1) and (7).

(10) Let us consider a non zero element m of \mathbb{N} , a non empty subset Z of \mathcal{R}^m , a partial function f from \mathcal{R}^m to \mathbb{R} , and elements i, j of \mathbb{N} . Suppose

(i) f is continuously differentiable up to order of i and Z , and

(ii) $j \leq i$.

Then f is continuously differentiable up to order of j and Z .

(11) Let us consider a non zero element m of \mathbb{N} and a non empty subset Z of \mathcal{R}^m . Suppose Z is open. Let us consider an element k of \mathbb{N} and partial functions f, g from \mathcal{R}^m to \mathbb{R} . Suppose

(i) f is continuously differentiable up to order of k and Z , and

(ii) g is continuously differentiable up to order of k and Z .

Then $f \cdot g$ is continuously differentiable up to order of k and Z . The theorem is a consequence of (10), (1), (3), (9), and (7). PROOF: Define $\mathcal{P}[\text{element of } \mathbb{N}] \equiv$ for every partial functions f, g from \mathcal{R}^m to \mathbb{R} such that f is continuously differentiable up to order of $\$1$ and Z and g is continuously differentiable up to order of $\$1$ and Z holds $f \cdot g$ is continuously differentiable up to order of $\$1$ and Z . Set $Z0 = (0 \text{ qua natural number})$. $\mathcal{P}[0]$. For every element k of \mathbb{N} such that $\mathcal{P}[k]$ holds $\mathcal{P}[k + 1]$. \square

(12) Let us consider a non zero element m of \mathbb{N} , a partial function f from \mathcal{R}^m to \mathbb{R} , a non empty subset X of \mathcal{R}^m , and a real number d . Suppose

- (i) X is open, and
- (ii) $f = X \mapsto d$.

Let us consider an element x of \mathcal{R}^m . If $x \in X$, then f is differentiable in x and $f'(x) = \mathcal{R}^m \mapsto 0$.

- (13) Let us consider a non zero element m of \mathbb{N} , a partial function f from \mathcal{R}^m to \mathbb{R} , a non empty subset X of \mathcal{R}^m , and a real number d . Suppose

- (i) X is open, and
- (ii) $f = X \mapsto d$.

Let us consider an element x_0 of \mathcal{R}^m and a real number r . Suppose

- (iii) $x_0 \in X$, and
- (iv) $0 < r$.

Then there exists a real number s such that

- (v) $0 < s$, and
- (vi) for every element x_1 of \mathcal{R}^m such that $x_1 \in X$ and $|x_1 - x_0| < s$ for every element v of \mathcal{R}^m , $|f'(x_1)(v) - f'(x_0)(v)| \leq r \cdot |v|$.

The theorem is a consequence of (12).

- (14) Let us consider a non zero element m of \mathbb{N} , a partial function f from \mathcal{R}^m to \mathbb{R} , a non empty subset X of \mathcal{R}^m , and a real number d . Suppose

- (i) X is open, and
- (ii) $f = X \mapsto d$.

Then

- (iii) f is differentiable on X , and
- (iv) $\text{dom } f'_{\upharpoonright X} = X$, and
- (v) for every element x of \mathcal{R}^m such that $x \in X$ holds $(f'_{\upharpoonright X})_x = \mathcal{R}^m \mapsto 0$.

The theorem is a consequence of (12).

- (15) Let us consider a non zero element m of \mathbb{N} , a partial function f from \mathcal{R}^m to \mathbb{R} , a non empty subset X of \mathcal{R}^m , a real number d , and an element i of \mathbb{N} . Suppose

- (i) X is open, and
- (ii) $f = X \mapsto d$, and
- (iii) $1 \leq i \leq m$.

Then

- (iv) f is partially differentiable on X w.r.t. i , and
- (v) $f \upharpoonright^i X$ is continuous on X .

The theorem is a consequence of (14) and (13).

(16) Let us consider a non zero element m of \mathbb{N} , an element i of \mathbb{N} , a partial function f from \mathcal{R}^m to \mathbb{R} , a non empty subset X of \mathcal{R}^m , and a real number d . Suppose

- (i) X is open, and
- (ii) $f = X \mapsto d$, and
- (iii) $1 \leq i \leq m$.

Then $f \upharpoonright^i X = X \mapsto 0$. The theorem is a consequence of (15) and (12).

Let us consider a non zero element m of \mathbb{N} , a non empty finite sequence I of elements of \mathbb{N} , a non empty subset X of \mathcal{R}^m , a partial function f from \mathcal{R}^m to \mathbb{R} , and a real number d . Now we state the propositions:

(17) Suppose X is open and $f = X \mapsto d$ and $\text{rng } I \subseteq \text{Seg } m$. Then

- (i) $(\text{PartDiffSeq}(f, X, I))(0) = X \mapsto d$, and
- (ii) for every element i of \mathbb{N} such that $1 \leq i \leq \text{len } I$ holds
 $(\text{PartDiffSeq}(f, X, I))(i) = X \mapsto 0$.

(18) Suppose X is open and $f = X \mapsto d$ and $\text{rng } I \subseteq \text{Seg } m$. Then

- (i) f is partially differentiable on X w.r.t. I , and
- (ii) $f \upharpoonright^I X$ is continuous on X .

Now we state the proposition:

(19) Let us consider a non zero element m of \mathbb{N} , an element k of \mathbb{N} , a non empty subset X of \mathcal{R}^m , a partial function f from \mathcal{R}^m to \mathbb{R} , and a real number d . Suppose

- (i) X is open, and
- (ii) $f = X \mapsto d$.

Then f is continuously differentiable up to order of k and X . The theorem is a consequence of (18).

Let m be a non zero element of \mathbb{N} . Observe that there exists a non empty subset of \mathcal{R}^m which is open.

2. DEFINITION OF THE C^k SPACE

Let m be a non zero element of \mathbb{N} , k be an element of \mathbb{N} , and X be a non empty open subset of \mathcal{R}^m . The functor the \mathbb{C}^k functions of k and X yielding a non empty subset of $\text{RAlgebra } X$ is defined by the term

(Def. 2) $\{f \text{ where } f \text{ is a partial function from } \mathcal{R}^m \text{ to } \mathbb{R} : f \text{ is continuously differentiable up to order of } k \text{ and } X \text{ and } \text{dom } f = X\}$.

Let us note that the \mathbb{C}^k functions of k and X is additively linearly closed and multiplicatively closed.

The functor the \mathbb{R} algebra of \mathbb{C}^k functions of k and X yielding a subalgebra of $\mathbb{R}\text{Algebra } X$ is defined by the term

- (Def. 3) \langle the \mathbb{C}^k functions of k and X , mult (the \mathbb{C}^k functions of k and X , $\mathbb{R}\text{Algebra } X$), Add (the \mathbb{C}^k functions of k and X , $\mathbb{R}\text{Algebra } X$), Mult (the \mathbb{C}^k functions of k and X , $\mathbb{R}\text{Algebra } X$), One (the \mathbb{C}^k functions of k and X , $\mathbb{R}\text{Algebra } X$), Zero (the \mathbb{C}^k functions of k and X , $\mathbb{R}\text{Algebra } X$) \rangle .

Let us note that the \mathbb{R} algebra of \mathbb{C}^k functions of k and X is Abelian add-associative right zeroed right complementable vector distributive scalar distributive scalar associative scalar unital commutative associative right unital right distributive and vector associative.

Now we state the propositions:

- (20) Let us consider a non zero element m of \mathbb{N} , an element k of \mathbb{N} , a non empty open subset X of \mathcal{R}^m , vectors F, G, H of the \mathbb{R} algebra of \mathbb{C}^k functions of k and X , and partial functions f, g, h from \mathcal{R}^m to \mathbb{R} . Suppose

- (i) $f = F$, and
- (ii) $g = G$, and
- (iii) $h = H$.

Then $H = F + G$ if and only if for every element x of X , $h(x) = f(x) + g(x)$.

- (21) Let us consider a non zero element m of \mathbb{N} , an element k of \mathbb{N} , a non empty open subset X of \mathcal{R}^m , vectors F, G, H of the \mathbb{R} algebra of \mathbb{C}^k functions of k and X , partial functions f, g, h from \mathcal{R}^m to \mathbb{R} , and a real number a . Suppose

- (i) $f = F$, and
- (ii) $g = G$.

Then $G = a \cdot F$ if and only if for every element x of X , $g(x) = a \cdot f(x)$.

- (22) Let us consider a non zero element m of \mathbb{N} , an element k of \mathbb{N} , a non empty open subset X of \mathcal{R}^m , vectors F, G, H of the \mathbb{R} algebra of \mathbb{C}^k functions of k and X , and partial functions f, g, h from \mathcal{R}^m to \mathbb{R} . Suppose

- (i) $f = F$, and
- (ii) $g = G$, and
- (iii) $h = H$.

Then $H = F \cdot G$ if and only if for every element x of X , $h(x) = f(x) \cdot g(x)$.

Let us consider a non zero element m of \mathbb{N} , an element k of \mathbb{N} , and a non empty open subset X of \mathcal{R}^m . Now we state the propositions:

- (23) $0_\alpha = X \mapsto 0$, where α is the \mathbb{R} algebra of \mathbb{C}^k functions of k and X .
- (24) $1_\alpha = X \mapsto 1$, where α is the \mathbb{R} algebra of \mathbb{C}^k functions of k and X .

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