

# Random Variables and Product of Probability Spaces<sup>1</sup>

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**Summary.** We have been working on the formalization of the probability and the randomness. In [15] and [16], we formalized some theorems concerning the real-valued random variables and the product of two probability spaces. In this article, we present the generalized formalization of [15] and [16]. First, we formalize the random variables of arbitrary set and prove the equivalence between random variable on  $\Sigma$ , Borel sets and a real-valued random variable on  $\Sigma$ . Next, we formalize the product of countably infinite probability spaces.

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The notation and terminology used in this paper have been introduced in the following articles: [1], [14], [12], [4], [11], [18], [7], [8], [5], [2], [3], [9], [13], [22], [15], [16], [20], [21], [17], [19], [6], and [10].

## 1. RANDOM VARIABLES

In this paper  $\Omega$ ,  $\Omega_1$ ,  $\Omega_2$  denote non empty sets,  $\Sigma$  denotes a  $\sigma$ -field of subsets of  $\Omega$ ,  $S_1$  denotes a  $\sigma$ -field of subsets of  $\Omega_1$ , and  $S_2$  denotes a  $\sigma$ -field of subsets of  $\Omega_2$ .

Now we state the proposition:

- (1) Let us consider a non empty set  $B$  and a function  $f$ . Then  $f^{-1}(\cup B) = \cup\{f^{-1}(Y) \text{ where } Y \text{ is an element of } B : \text{not contradiction}\}$ .

Let us consider a function  $f$  from  $\Omega_1$  into  $\Omega_2$ , a sequence  $B$  of subsets of  $\Omega_2$ , and a sequence  $D$  of subsets of  $\Omega_1$ . Now we state the propositions:

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- (2) If for every element  $n$  of  $\mathbb{N}$ ,  $D(n) = f^{-1}(B(n))$ , then  $f^{-1}(\cup B) = \cup D$ .
- (3) If for every element  $n$  of  $\mathbb{N}$ ,  $D(n) = f^{-1}(B(n))$ , then  $f^{-1}(\text{Intersection } B) = \text{Intersection } D$ .

Now we state the propositions:

- (4) Let us consider a function  $F$  from  $\Omega$  into  $\mathbb{R}$  and a real number  $r$ . Suppose  $F$  is a real-valued random variable on  $\Sigma$ . Then  $F^{-1}(]-\infty, r[) \in \Sigma$ . PROOF: Consider  $X$  being an element of  $\Sigma$  such that  $X = \Omega$  and  $F$  is measurable on  $X$ . For every element  $z$ ,  $z \in F^{-1}(]-\infty, r[)$  iff  $z \in \Omega_{\Sigma} \cap \text{LE-dom}(F, r)$ .  $\square$
- (5) Let us consider a function  $F$  from  $\Omega$  into  $\mathbb{R}$ . Suppose  $F$  is a real-valued random variable on  $\Sigma$ . Then  $\{x \text{ where } x \text{ is an element of the Borel sets : } F^{-1}(x) \text{ is element of } \Sigma\}$  is a  $\sigma$ -field of subsets of  $\mathbb{R}$ . The theorem is a consequence of (4) and (3). PROOF: Set  $S = \{x \text{ where } x \text{ is an element of the Borel sets : } F^{-1}(x) \text{ is an element of } \Sigma\}$ . For every element  $x$  such that  $x \in S$  holds  $x \in$  the Borel sets. Set  $r_0 =$  the element of  $\mathbb{R}$ . Reconsider  $y_0 = \text{halfline}(r_0)$  as an element of the Borel sets. For every subset  $A$  of  $\mathbb{R}$  such that  $A \in S$  holds  $A^c \in S$ . For every sequence  $A_1$  of subsets of  $\mathbb{R}$  such that  $\text{rng } A_1 \subseteq S$  holds  $\text{Intersection } A_1 \in S$ .  $\square$

Let us consider a function  $f$  from  $\Omega$  into  $\mathbb{R}$ . Now we state the propositions:

- (6) Suppose  $f$  is a real-valued random variable on  $\Sigma$ . Then  $\{x \text{ where } x \text{ is an element of the Borel sets : } f^{-1}(x) \text{ is an element of } \Sigma\} =$  the Borel sets.
- (7)  $f$  is random variable on  $\Sigma$  and the Borel sets if and only if  $f$  is a real-valued random variable on  $\Sigma$ .
- (8) The set of random variables on  $\Sigma$  and the Borel sets = the real-valued random variables set on  $\Sigma$ .

Let us consider  $\Omega_1$ ,  $\Omega_2$ ,  $S_1$ , and  $S_2$ . Let  $F$  be a function from  $\Omega_1$  into  $\Omega_2$ . We say that  $F$  is  $(S_1, S_2)$ -random variable-like if and only if

(Def. 1)  $F$  is random variable on  $S_1$  and  $S_2$ .

Observe that there exists a function from  $\Omega_1$  into  $\Omega_2$  which is  $(S_1, S_2)$ -random variable-like.

A random variable of  $S_1$  and  $S_2$  is an  $(S_1, S_2)$ -random variable-like function from  $\Omega_1$  into  $\Omega_2$ . Now we state the proposition:

- (9) Let us consider a function  $f$  from  $\Omega$  into  $\mathbb{R}$ . Then  $f$  is a random variable of  $\Sigma$  and the Borel sets if and only if  $f$  is a real-valued random variable on  $\Sigma$ .

Let  $F$  be a function. We say that  $F$  is random variable family-like if and only if

(Def. 2) Let us consider a set  $x$ . Suppose  $x \in \text{dom } F$ . Then there exist non empty sets  $\Omega_1, \Omega_2$  and there exists a  $\sigma$ -field  $S_1$  of subsets of  $\Omega_1$  and there exists

a  $\sigma$ -field  $S_2$  of subsets of  $\Omega_2$  and there exists a random variable  $f$  of  $S_1$  and  $S_2$  such that  $F(x) = f$ .

One can verify that there exists a function which is random variable family-like.

A random variable family is a random variable family-like function. In this paper  $F$  denotes a random variable of  $S_1$  and  $S_2$ .

Let  $Y$  be a non empty set,  $S$  be a  $\sigma$ -field of subsets of  $Y$ , and  $F$  be a function.

We say that  $F$  is  $S$ -measure valued if and only if

(Def. 3) Let us consider a set  $x$ . If  $x \in \text{dom } F$ , then there exists a  $\sigma$ -measure  $M$  on  $S$  such that  $F(x) = M$ .

Note that there exists a function which is  $S$ -measure valued.

Let  $F$  be a function. We say that  $F$  is  $S$ -probability valued if and only if

(Def. 4) Let us consider a set  $x$ . If  $x \in \text{dom } F$ , then there exists a probability  $P$  on  $S$  such that  $F(x) = P$ .

Let us note that there exists a function which is  $S$ -probability valued.

Let  $X, Y$  be non empty sets. One can verify that there exists an  $S$ -probability valued function which is  $X$ -defined.

One can verify that there exists an  $X$ -defined  $S$ -probability valued function which is total.

Let  $Y$  be a non empty set. Let us note that every function which is  $S$ -probability valued is also  $S$ -measure valued.

Let  $F$  be a function. We say that  $F$  is  $S$ -random variable family if and only if

(Def. 5) Let us consider a set  $x$ . Suppose  $x \in \text{dom } F$ . Then there exists a real-valued random variable  $Z$  on  $S$  such that  $F(x) = Z$ .

Observe that there exists a function which is  $S$ -random variable family.

Now we state the propositions:

(10) Let us consider an element  $y$  of  $S_2$ . Suppose  $y \neq \emptyset$ . Then  $\{z \text{ where } z \text{ is an element of } \Omega_1 : F(z) \text{ is an element of } y\} = F^{-1}(y)$ . PROOF: Set  $D = \{z \text{ where } z \text{ is an element of } \Omega_1 : F(z) \text{ is an element of } y\}$ . For every element  $x$ ,  $x \in D$  iff  $x \in F^{-1}(y)$ .  $\square$

(11) Let us consider a random variable  $F$  of  $S_1$  and  $S_2$ . Then

- (i)  $\{x \text{ where } x \text{ is a subset of } \Omega_1 : \text{there exists an element } y \text{ of } S_2 \text{ such that } x = F^{-1}(y)\} \subseteq S_1$ , and
- (ii)  $\{x \text{ where } x \text{ is a subset of } \Omega_1 : \text{there exists an element } y \text{ of } S_2 \text{ such that } x = F^{-1}(y)\}$  is a  $\sigma$ -field of subsets of  $\Omega_1$ .

The theorem is a consequence of (3). PROOF: Set  $S = \{x \text{ where } x \text{ is a subset of } \Omega_1 : \text{there exists an element } y \text{ of } S_2 \text{ such that } x = F^{-1}(y)\}$ . For every element  $x$  such that  $x \in S$  holds  $x \in S_1$ . For every subset  $A$  of

$\Omega_1$  such that  $A \in S$  holds  $A^c \in S$ . For every sequence  $A_1$  of subsets of  $\Omega_1$  such that  $\text{rng } A_1 \subseteq S$  holds  $\text{Intersection } A_1 \in S$ .  $\square$

Let us consider  $\Omega_1, \Omega_2, S_1$ , and  $S_2$ . Let  $M$  be a measure on  $S_1$  and  $F$  be a random variable of  $S_1$  and  $S_2$ . The functor the image measure of  $F$  and  $M$  yielding a measure on  $S_2$  is defined by

(Def. 6) Let us consider an element  $y$  of  $S_2$ . Then  $it(y) = M(F^{-1}(y))$ .

Let  $M$  be a  $\sigma$ -measure on  $S_1$ . Note that the image measure of  $F$  and  $M$  is  $\sigma$ -additive.

Now we state the proposition:

(12) Let us consider a probability  $P$  on  $S_1$  and a random variable  $F$  of  $S_1$  and  $S_2$ . Then (the image measure of  $F$  and  $\text{P2M } P$ )( $\Omega_2$ ) = 1.

Let us consider  $\Omega_1, \Omega_2, S_1$ , and  $S_2$ . Let  $P$  be a probability on  $S_1$  and  $F$  be a random variable of  $S_1$  and  $S_2$ . The functor  $\text{probability}(F, P)$  yielding a probability on  $S_2$  is defined by the term

(Def. 7)  $\text{M2P}$  the image measure of  $F$  and  $\text{P2M } P$ .

Now we state the propositions:

(13) Let us consider a probability  $P$  on  $S_1$  and a random variable  $F$  of  $S_1$  and  $S_2$ . Then  $\text{probability}(F, P) = \text{the image measure of } F \text{ and } \text{P2M } P$ . The theorem is a consequence of (12).

(14) Let us consider a probability  $P$  on  $S_1$ , a random variable  $F$  of  $S_1$  and  $S_2$ , and a set  $y$ . If  $y \in S_2$ , then  $(\text{probability}(F, P))(y) = P(F^{-1}(y))$ . The theorem is a consequence of (13).

(15) Every function from  $\Omega_1$  into  $\Omega_2$  is a random variable of the trivial  $\sigma$ -field of  $\Omega_1$  and the trivial  $\sigma$ -field of  $\Omega_2$ .

(16) Let us consider a non empty set  $S$ . Then every non empty finite sequence of elements of  $S$  is a random variable of the trivial  $\sigma$ -field of  $\text{Seg len } F$  and the trivial  $\sigma$ -field of  $S$ . The theorem is a consequence of (15).

(17) Let us consider finite non empty sets  $V, S$ , a random variable  $G$  of the trivial  $\sigma$ -field of  $V$  and the trivial  $\sigma$ -field of  $S$ , and a set  $y$ . Suppose  $y \in \text{the trivial } \sigma\text{-field of } S$ . Then  $(\text{probability}(G, \text{the trivial probability of } V))(y) = \frac{\overline{G^{-1}(y)}}{\overline{V}}$ . The theorem is a consequence of (14).

(18) Let us consider a finite non empty set  $S$ , a non empty finite sequence  $s$  of elements of  $S$ , and a set  $x$ . Suppose  $x \in S$ . Then there exists a random variable  $G$  of the trivial  $\sigma$ -field of  $\text{Seg len } s$  and the trivial  $\sigma$ -field of  $S$  such that

(i)  $G = s$ , and

(ii)  $(\text{probability}(G, \text{the trivial probability of } \text{Seg len } s))(\{x\}) = \text{Prob}_D(x, s)$ .

The theorem is a consequence of (16) and (17).

## 2. PRODUCT OF PROBABILITY SPACES

Let  $D$  be a non-empty many sorted set indexed by  $\mathbb{N}$  and  $n$  be a natural number. One can check that  $D(n)$  is non empty.

Let  $S, F$  be many sorted sets indexed by  $\mathbb{N}$ . We say that  $F$  is  $\sigma$ -field  $S$ -sequence-like if and only if

(Def. 8) Let us consider a natural number  $n$ . Then  $F(n)$  is a  $\sigma$ -field of subsets of  $S(n)$ .

Let  $S$  be a many sorted set indexed by  $\mathbb{N}$ . Let us observe that there exists a many sorted set indexed by  $\mathbb{N}$  which is  $\sigma$ -field  $S$ -sequence-like.

Let  $D$  be a many sorted set indexed by  $\mathbb{N}$ . A  $\sigma$ -field sequence of  $D$  is a  $\sigma$ -field  $D$ -sequence-like many sorted set indexed by  $\mathbb{N}$ . Let  $S$  be a  $\sigma$ -field sequence of  $D$  and  $n$  be a natural number. Note that the functor  $S(n)$  yields a  $\sigma$ -field of subsets of  $D(n)$ . Let  $D$  be a non-empty many sorted set indexed by  $\mathbb{N}$ . Let  $M$  be a many sorted set indexed by  $\mathbb{N}$ . We say that  $M$  is  $S$ -probability sequence-like if and only if

(Def. 9) Let us consider a natural number  $n$ . Then  $M(n)$  is a probability on  $S(n)$ .

Observe that there exists a many sorted set indexed by  $\mathbb{N}$  which is  $S$ -probability sequence-like.

A probability sequence of  $S$  is an  $S$ -probability sequence-like many sorted set indexed by  $\mathbb{N}$ . Let  $P$  be a probability sequence of  $S$  and  $n$  be a natural number. One can verify that the functor  $P(n)$  yields a probability on  $S(n)$ . Let  $D$  be a many sorted set indexed by  $\mathbb{N}$ . The functor the product domain  $D$  yielding a many sorted set indexed by  $\mathbb{N}$  is defined by

(Def. 10) (i)  $it(0) = D(0)$ , and

(ii) for every natural number  $i$ ,  $it(i + 1) = it(i) \times D(i + 1)$ .

Now we state the proposition:

(19) Let us consider a many sorted set  $D$  indexed by  $\mathbb{N}$ . Then

(i) (the product domain  $D$ )(0) =  $D(0)$ , and

(ii) (the product domain  $D$ )(1) =  $D(0) \times D(1)$ , and

(iii) (the product domain  $D$ )(2) =  $D(0) \times D(1) \times D(2)$ , and

(iv) (the product domain  $D$ )(3) =  $D(0) \times D(1) \times D(2) \times D(3)$ .

Let  $D$  be a non-empty many sorted set indexed by  $\mathbb{N}$ . Let us note that the product domain  $D$  is non-empty.

Let  $D$  be a finite-yielding many sorted set indexed by  $\mathbb{N}$ . One can check that the product domain  $D$  is finite-yielding.

Let us consider  $\Omega$  and  $\Sigma$ . Let  $P$  be a set. Assume  $P$  is a probability on  $\Sigma$ . The functor  $\text{modetrans}(P, \Sigma)$  yielding a probability on  $\Sigma$  is defined by the term

(Def. 11)  $P$ .

Let  $D$  be a finite-yielding non-empty many sorted set indexed by  $\mathbb{N}$ . The functor the trivial  $\sigma$ -field sequence  $D$  yielding a  $\sigma$ -field sequence of  $D$  is defined by

(Def. 12) Let us consider a natural number  $n$ . Then  $it(n) =$  the trivial  $\sigma$ -field of  $D(n)$ .

Let  $P$  be a probability sequence of the trivial  $\sigma$ -field sequence  $D$  and  $n$  be a natural number. One can check that the functor  $P(n)$  yields a probability on the trivial  $\sigma$ -field of  $D(n)$ . The functor  $\text{ProductProbability}(P, D)$  yielding a many sorted set indexed by  $\mathbb{N}$  is defined by

(Def. 13) (i)  $it(0) = P(0)$ , and  
(ii) for every natural number  $i$ ,  $it(i + 1) =$   
 $\text{Product-Probability}((\text{the product domain } D)(i), D(i + 1), \text{modetrans}$   
 $(it(i), \text{the trivial } \sigma\text{-field of } (\text{the product domain } D)(i)), P(i + 1))$ .

Let us consider a finite-yielding non-empty many sorted set  $D$  indexed by  $\mathbb{N}$ , a probability sequence  $P$  of the trivial  $\sigma$ -field sequence  $D$ , and a natural number  $n$ . Now we state the propositions:

(20)  $(\text{ProductProbability}(P, D))(n)$  is a probability on the trivial  $\sigma$ -field of  $(\text{the product domain } D)(n)$ .

(21) There exists a probability  $P_4$  on the trivial  $\sigma$ -field of  $(\text{the product domain } D)(n)$  such that

- (i)  $P_4 = (\text{ProductProbability}(P, D))(n)$ , and
- (ii)  $(\text{ProductProbability}(P, D))(n + 1) = \text{Product-Probability}((\text{the product domain } D)(n), D(n + 1), P_4, P(n + 1))$ .

Now we state the proposition:

(22) Let us consider a finite-yielding non-empty many sorted set  $D$  indexed by  $\mathbb{N}$  and a probability sequence  $P$  of the trivial  $\sigma$ -field sequence  $D$ . Then

- (i)  $(\text{ProductProbability}(P, D))(0) = P(0)$ , and
- (ii)  $(\text{ProductProbability}(P, D))(1) =$   
 $\text{Product-Probability}(D(0), D(1), P(0), P(1))$ , and
- (iii) there exists a probability  $P_1$  on the trivial  $\sigma$ -field of  $D(0) \times D(1)$  such that  $P_1 = (\text{ProductProbability}(P, D))(1)$  and  $(\text{ProductProbability}(P, D))(2) = \text{Product-Probability}(D(0) \times D(1), D(2), P_1, P(2))$ , and
- (iv) there exists a probability  $P_2$  on the trivial  $\sigma$ -field of  $D(0) \times D(1) \times D(2)$  such that  $P_2 = (\text{ProductProbability}(P, D))(2)$  and  $(\text{ProductProbability}(P, D))(3) = \text{Product-Probability}(D(0) \times D(1) \times D(2), D(3), P_2, P(3))$ , and
- (v) there exists a probability  $P_3$  on the trivial  $\sigma$ -field of  $D(0) \times D(1) \times D(2) \times D(3)$  such that  $P_3 = (\text{ProductProbability}(P, D))(3)$  and

$$(\text{ProductProbability}(P, D))(4) = \text{Product-Probability}(D(0) \times D(1) \times D(2) \times D(3), D(4), P_3, P(4)).$$

The theorem is a consequence of (19) and (21).

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