

# A Test for the Stability of Networks

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**Summary.** A complex polynomial is called a Hurwitz polynomial, if all its roots have a real part smaller than zero. This kind of polynomial plays an all-dominant role in stability checks of electrical (analog or digital) networks. In this article we prove that a polynomial  $p$  can be shown to be Hurwitz by checking whether the rational function  $e(p)/o(p)$  can be realized as a reactance of one port, that is as an electrical impedance or admittance consisting of inductors and capacitors. Here  $e(p)$  and  $o(p)$  denote the even and the odd part of  $p$  [25].

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The notation and terminology used in this paper have been introduced in the following articles: [16], [14], [2], [3], [10], [4], [5], [22], [19], [21], [15], [1], [6], [17], [11], [12], [13], [18], [8], [26], [23], [20], [24], [9], [27], and [7].

## 1. PRELIMINARIES

Now we state the propositions:

- (1) Let us consider complex numbers  $x, y$ . If  $\Im(x) = 0$  and  $\Re(y) = 0$ , then  $\Re\left(\frac{x}{y}\right) = 0$ .
- (2) Let us consider a complex number  $a$ . Then  $a \cdot \bar{a} = |a|^2$ .

One can check that there exists a polynomial of  $\mathbb{C}_F$  which is Hurwitz and 0 is even.

Now we state the propositions:

- (3) Let us consider an add-associative right zeroed right complementable associative distributive non empty double loop structure  $L$ , an even element  $k$  of  $\mathbb{N}$ , and an element  $x$  of  $L$ . Then  $\text{power}_L(-x, k) = \text{power}_L(x, k)$ .
- (4) Let us consider an add-associative right zeroed right complementable associative distributive non empty double loop structure  $L$ , an odd element  $k$  of  $\mathbb{N}$ , and an element  $x$  of  $L$ . Then  $\text{power}_L(-x, k) = -\text{power}_L(x, k)$ . The theorem is a consequence of (3).
- (5) Let us consider an even element  $k$  of  $\mathbb{N}$  and an element  $x$  of  $\mathbb{C}_F$ . If  $\Re(x) = 0$ , then  $\Im(\text{power}_{\mathbb{C}_F}(x, k)) = 0$ .
- (6) Let us consider an odd element  $k$  of  $\mathbb{N}$  and an element  $x$  of  $\mathbb{C}_F$ . If  $\Re(x) = 0$ , then  $\Re(\text{power}_{\mathbb{C}_F}(x, k)) = 0$ .

## 2. EVEN AND ODD PART OF POLYNOMIALS

Let  $L$  be a non empty zero structure and  $p$  be a sequence of  $L$ . The functors the even part of  $p$  and the odd part of  $p$  yielding sequences of  $L$  are defined by the conditions, respectively.

- (Def. 1) Let us consider an even natural number  $i$ . Then
  - (i) (the even part of  $p$ )( $i$ ) =  $p(i)$ , and
  - (ii) for every odd natural number  $i$ , (the even part of  $p$ )( $i$ ) =  $0_L$ .
- (Def. 2) Let us consider an even natural number  $i$ . Then
  - (i) (the odd part of  $p$ )( $i$ ) =  $0_L$ , and
  - (ii) for every odd natural number  $i$ , (the odd part of  $p$ )( $i$ ) =  $p(i)$ .

Let  $p$  be a polynomial of  $L$ . Observe that the even part of  $p$  is finite-Support and the odd part of  $p$  is finite-Support. Now we state the propositions:

- (7) Let us consider a non empty zero structure  $L$ . Then
  - (i) the even part of  $\mathbf{0}.L = \mathbf{0}.L$ , and
  - (ii) the odd part of  $\mathbf{0}.L = \mathbf{0}.L$ .
- (8) Let us consider a non empty multiplicative loop with zero structure  $L$ . Then
  - (i) the even part of  $\mathbf{1}.L = \mathbf{1}.L$ , and
  - (ii) the odd part of  $\mathbf{1}.L = \mathbf{0}.L$ .

Let us consider a left zeroed right zeroed non empty additive loop structure  $L$  and a polynomial  $p$  of  $L$ . Now we state the propositions:

- (9) (The even part of  $p$ ) + (the odd part of  $p$ ) =  $p$ .
- (10) (The odd part of  $p$ ) + (the even part of  $p$ ) =  $p$ .

Let us consider an add-associative right zeroed right complementable non empty additive loop structure  $L$  and a polynomial  $p$  of  $L$ . Now we state the propositions:

- (11)  $p -$  the odd part of  $p =$  the even part of  $p$ .
- (12)  $p -$  the even part of  $p =$  the odd part of  $p$ .

Let us consider an add-associative right zeroed right complementable Abelian non empty additive loop structure  $L$  and a polynomial  $p$  of  $L$ . Now we state the propositions:

- (13) (The even part of  $p) - p = -$ the odd part of  $p$ .
- (14) (The odd part of  $p) - p = -$ the even part of  $p$ .

Let us consider an add-associative right zeroed right complementable Abelian non empty additive loop structure  $L$  and polynomials  $p, q$  of  $L$ . Now we state the propositions:

- (15) The even part of  $p + q =$  (the even part of  $p) +$  (the even part of  $q$ ).
- (16) The odd part of  $p + q =$  (the odd part of  $p) +$  (the odd part of  $q$ ).

Let us consider a well unital non empty double loop structure  $L$  and a polynomial  $p$  of  $L$ . Now we state the propositions:

- (17) Suppose  $\deg p$  is even. Then the even part of Leading-Monomial  $p =$  Leading-Monomial  $p$ .
- (18) If  $\deg p$  is odd, then the even part of Leading-Monomial  $p = \mathbf{0} \cdot L$ .
- (19) If  $\deg p$  is even, then the odd part of Leading-Monomial  $p = \mathbf{0} \cdot L$ .
- (20) Suppose  $\deg p$  is odd. Then the odd part of Leading-Monomial  $p =$  Leading-Monomial  $p$ .

Now we state the proposition:

- (21) Let us consider a well unital add-associative right zeroed right complementable Abelian associative distributive non degenerated double loop structure  $L$  and a non zero polynomial  $p$  of  $L$ . Then  $\deg$  the even part of  $p \neq \deg$  the odd part of  $p$ . The theorem is a consequence of (9).

Let us consider a well unital add-associative right zeroed right complementable associative Abelian distributive non degenerated double loop structure  $L$  and a polynomial  $p$  of  $L$ . Now we state the propositions:

- (22) (i)  $\deg$  the even part of  $p \leq \deg p$ , and  
(ii)  $\deg$  the odd part of  $p \leq \deg p$ .
- (23)  $\deg p = \max(\deg$  the even part of  $p, \deg$  the odd part of  $p)$ .

### 3. EVEN AND ODD POLYNOMIALS AND RATIONAL FUNCTIONS

Let  $L$  be a non empty additive loop structure and  $f$  be a function from  $L$  into  $L$ . We say that  $f$  is even if and only if

(Def. 3) Let us consider an element  $x$  of  $L$ . Then  $f(-x) = f(x)$ .

We say that  $f$  is odd if and only if

(Def. 4) Let us consider an element  $x$  of  $L$ . Then  $f(-x) = -f(x)$ .

Let  $L$  be a well unital non empty double loop structure and  $p$  be a polynomial of  $L$ . We say that  $p$  is even if and only if

(Def. 5) Polynomial-Function( $L, p$ ) is even.

We say that  $p$  is odd if and only if

(Def. 6) Polynomial-Function( $L, p$ ) is odd.

Let  $Z$  be a rational function of  $L$ . We say that  $Z$  is odd if and only if

(Def. 7) (i)  $Z_1$  is even and  $Z_2$  is odd, or

(ii)  $Z_1$  is odd and  $Z_2$  is even.

We introduce  $Z$  is even as an antonym for  $Z$  is odd.

Observe that there exists a polynomial of  $L$  which is even.

Let  $L$  be an add-associative right zeroed right complementable well unital non empty double loop structure. Let us note that there exists a polynomial of  $L$  which is odd.

Let  $L$  be a well unital add-associative right zeroed right complementable associative non degenerated double loop structure. Observe that there exists a polynomial of  $L$  which is non zero and even.

Let  $L$  be an add-associative right zeroed right complementable Abelian well unital non degenerated double loop structure. One can verify that there exists a polynomial of  $L$  which is non zero and odd.

Now we state the propositions:

(24) Let us consider a well unital non empty double loop structure  $L$ , an even polynomial  $p$  of  $L$ , and an element  $x$  of  $L$ . Then  $\text{eval}(p, -x) = \text{eval}(p, x)$ .

(25) Let us consider an add-associative right zeroed right complementable Abelian well unital non degenerated double loop structure  $L$ , an odd polynomial  $p$  of  $L$ , and an element  $x$  of  $L$ . Then  $\text{eval}(p, -x) = -\text{eval}(p, x)$ .

Let  $L$  be a well unital non empty double loop structure. One can verify that  $\mathbf{0}.L$  is even.

Let  $L$  be an add-associative right zeroed right complementable well unital non empty double loop structure. One can verify that  $\mathbf{0}.L$  is odd.

Let  $L$  be a well unital add-associative right zeroed right complementable associative non degenerated double loop structure. Note that  $\mathbf{1}.L$  is even.

Let  $L$  be an Abelian add-associative right zeroed right complementable well unital left distributive non empty double loop structure and  $p, q$  be even polynomials of  $L$ . Let us note that  $p + q$  is even.

Let  $p, q$  be odd polynomials of  $L$ . Let us note that  $p + q$  is odd.

Let  $L$  be an Abelian add-associative right zeroed right complementable associative well unital distributive non degenerated double loop structure and  $p$

be a polynomial of  $L$ . One can check that the even part of  $p$  is even and the odd part of  $p$  is odd.

Now we state the propositions:

- (26) Let us consider an Abelian add-associative right zeroed right complementable well unital distributive non degenerated double loop structure  $L$ , an even polynomial  $p$  of  $L$ , an odd polynomial  $q$  of  $L$ , and an element  $x$  of  $L$ . If  $x$  is a common root of  $p$  and  $q$ , then  $-x$  is a root of  $p + q$ . The theorem is a consequence of (24) and (25).
- (27) Let us consider a Hurwitz polynomial  $p$  of  $\mathbb{C}_F$ . Then the even part of  $p$  and the odd part of  $p$  have no common roots. The theorem is a consequence of (9) and (26).

#### 4. REAL POSITIVE POLYNOMIALS AND RATIONAL FUNCTIONS

Let  $p$  be a polynomial of  $\mathbb{C}_F$ . We say that  $p$  is real if and only if

(Def. 8) Let us consider a natural number  $i$ . Then  $p(i)$  is a real number.

We say that  $p$  is positive if and only if

(Def. 9) Let us consider an element  $x$  of  $\mathbb{C}_F$ . If  $\Re(x) > 0$ , then  $\Re(\text{eval}(p, x)) > 0$ .

Let us note that  $\mathbf{0} \cdot \mathbb{C}_F$  is real and non positive and  $\mathbf{1} \cdot \mathbb{C}_F$  is real and positive and there exists a polynomial of  $\mathbb{C}_F$  which is non zero, real, and positive and every polynomial of  $\mathbb{C}_F$  which is real is also real-valued.

Let  $p$  be a real polynomial of  $\mathbb{C}_F$ . One can verify that the even part of  $p$  is real and the odd part of  $p$  is real.

Let  $L$  be a non empty additive loop structure and  $p$  be a polynomial of  $L$ .

We say that  $p$  has all coefficients if and only if

(Def. 10) Let us consider a natural number  $i$ . If  $i \leq \deg p$ , then  $p(i) \neq 0$ .

Let  $p$  be a real polynomial of  $\mathbb{C}_F$ . We say that  $p$  has positive coefficients if and only if

(Def. 11) Let us consider a natural number  $i$ . If  $i \leq \deg p$ , then  $p(i) > 0$ .

We say that  $p$  is negative coefficients if and only if

(Def. 12) Let us consider a natural number  $i$ . If  $i \leq \deg p$ , then  $p(i) < 0$ .

One can check that every real polynomial of  $\mathbb{C}_F$  which has positive coefficients has also all coefficients and every real polynomial of  $\mathbb{C}_F$  which is negative coefficients has also all coefficients and there exists a real polynomial of  $\mathbb{C}_F$  which is non constant and has positive coefficients.

Let  $p$  be a non zero real polynomial of  $\mathbb{C}_F$  with all coefficients. Let us note that the even part of  $p$  is non zero. Note that the odd part of  $p$  is non zero.

Let  $Z$  be a rational function of  $\mathbb{C}_F$ . We say that  $Z$  is real if and only if

(Def. 13) Let us consider a natural number  $i$ . Then

- (i)  $Z_1(i)$  is a real number, and
- (ii)  $Z_2(i)$  is a real number.

We say that  $Z$  is positive if and only if

(Def. 14) Let us consider an element  $x$  of  $\mathbb{C}_F$ . Suppose

- (i)  $\Re(x) > 0$ , and
- (ii)  $\text{eval}(Z_2, x) \neq 0$ .

Then  $\Re(\text{eval}(Z, x)) > 0$ .

One can check that there exists a rational function of  $\mathbb{C}_F$  which is non zero, odd, real, and positive.

Let  $p_1$  be a real polynomial of  $\mathbb{C}_F$  and  $p_2$  be a non zero real polynomial of  $\mathbb{C}_F$ . Let us note that  $\langle p_1, p_2 \rangle$  is real as a rational function of  $\mathbb{C}_F$ .

## 5. THE ROUTH-SCHUR STABILITY CRITERION

A one port function is a real positive rational function of  $\mathbb{C}_F$ . A reactance one port function is an odd real positive rational function of  $\mathbb{C}_F$ .

Let us consider a real polynomial  $p$  of  $\mathbb{C}_F$  and an element  $x$  of  $\mathbb{C}_F$ . Now we state the propositions:

- (28) If  $\Re(x) = 0$ , then  $\Im(\text{eval}(\text{the even part of } p, x)) = 0$ .
- (29) If  $\Re(x) = 0$ , then  $\Re(\text{eval}(\text{the odd part of } p, x)) = 0$ .

Now we state the proposition:

- (30) Let us consider a non constant real polynomial  $p$  of  $\mathbb{C}_F$  with positive coefficients. Suppose
  - (i)  $\langle \text{the even part of } p, \text{the odd part of } p \rangle$  is positive, and
  - (ii) the even part of  $p$  and the odd part of  $p$  have no common roots.

Then

- (iii) for every element  $x$  of  $\mathbb{C}_F$  such that  $\Re(x) = 0$  and  $\text{eval}(\text{the odd part of } p, x) \neq 0$  holds  $\Re(\text{eval}(\langle \text{the even part of } p, \text{the odd part of } p \rangle, x)) \geq 0$ , and
- (iv)  $(\text{the even part of } p) + (\text{the odd part of } p)$  is Hurwitz.

The theorem is a consequence of (28), (29), and (1).

Now we state the proposition:

- (31) ROUTH-SCHUR STABILITY CRITERION (FOR A SINGLE-INPUT, SINGLE-OUTPUT (SISO), LINEAR TIME INVARIANT (LTI) CONTROL SYSTEM):  
Let us consider a non constant real polynomial  $p$  of  $\mathbb{C}_F$  with positive coefficients. Suppose
  - (i)  $\langle \text{the even part of } p, \text{the odd part of } p \rangle$  is a one port function, and

(ii)  $\text{degree}(\langle \text{the even part of } p, \text{ the odd part of } p \rangle) = \text{degree}(p)$ .

Then  $p$  is Hurwitz. The theorem is a consequence of (23), (30), and (9).

## REFERENCES

- [1] Grzegorz Bancerek. The ordinal numbers. *Formalized Mathematics*, 1(1):91–96, 1990.
- [2] Czesław Byliński. Binary operations. *Formalized Mathematics*, 1(1):175–180, 1990.
- [3] Czesław Byliński. The complex numbers. *Formalized Mathematics*, 1(3):507–513, 1990.
- [4] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(1):55–65, 1990.
- [5] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(1):153–164, 1990.
- [6] Czesław Byliński. Partial functions. *Formalized Mathematics*, 1(2):357–367, 1990.
- [7] Czesław Byliński. Some basic properties of sets. *Formalized Mathematics*, 1(1):47–53, 1990.
- [8] Andrzej Kondracki. Basic properties of rational numbers. *Formalized Mathematics*, 1(5):841–845, 1990.
- [9] Eugeniusz Kusak, Wojciech Leończuk, and Michał Muzalewski. Abelian groups, fields and vector spaces. *Formalized Mathematics*, 1(2):335–342, 1990.
- [10] Anna Justyna Milewska. The field of complex numbers. *Formalized Mathematics*, 9(2):265–269, 2001.
- [11] Robert Milewski. The ring of polynomials. *Formalized Mathematics*, 9(2):339–346, 2001.
- [12] Robert Milewski. The evaluation of polynomials. *Formalized Mathematics*, 9(2):391–395, 2001.
- [13] Robert Milewski. Fundamental theorem of algebra. *Formalized Mathematics*, 9(3):461–470, 2001.
- [14] Michał Muzalewski and Wojciech Skaba. From loops to abelian multiplicative groups with zero. *Formalized Mathematics*, 1(5):833–840, 1990.
- [15] Jan Popiołek. Real normed space. *Formalized Mathematics*, 2(1):111–115, 1991.
- [16] Piotr Rudnicki and Andrzej Trybulec. Abian’s fixed point theorem. *Formalized Mathematics*, 6(3):335–338, 1997.
- [17] Piotr Rudnicki and Andrzej Trybulec. Multivariate polynomials with arbitrary number of variables. *Formalized Mathematics*, 9(1):95–110, 2001.
- [18] Christoph Schwarzweiler. Introduction to rational functions. *Formalized Mathematics*, 20(2):181–191, 2012. doi:10.2478/v10037-012-0021-1.
- [19] Christoph Schwarzweiler and Agnieszka Rowińska-Schwarzweiler. Schur’s theorem on the stability of networks. *Formalized Mathematics*, 14(4):135–142, 2006. doi:10.2478/v10037-006-0017-9.
- [20] Andrzej Trybulec and Czesław Byliński. Some properties of real numbers. *Formalized Mathematics*, 1(3):445–449, 1990.
- [21] Michał J. Trybulec. Integers. *Formalized Mathematics*, 1(3):501–505, 1990.
- [22] Wojciech A. Trybulec. Groups. *Formalized Mathematics*, 1(5):821–827, 1990.
- [23] Wojciech A. Trybulec. Vectors in real linear space. *Formalized Mathematics*, 1(2):291–296, 1990.
- [24] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(1):67–71, 1990.
- [25] Rolf Unbehauen. *Netzwerk- und Filtersynthese: Grundlagen und Anwendungen*. Oldenbourg-Verlag, fourth edition, 1993.
- [26] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1(1):73–83, 1990.
- [27] Hiroshi Yamazaki and Yasunari Shidama. Algebra of vector functions. *Formalized Mathematics*, 3(2):171–175, 1992.

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