

A Test for the Stability of Networks

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Summary. A complex polynomial is called a Hurwitz polynomial, if all its roots have a real part smaller than zero. This kind of polynomial plays an all-dominant role in stability checks of electrical (analog or digital) networks. In this article we prove that a polynomial p can be shown to be Hurwitz by checking whether the rational function $e(p)/o(p)$ can be realized as a reactance of one port, that is as an electrical impedance or admittance consisting of inductors and capacitors. Here $e(p)$ and $o(p)$ denote the even and the odd part of p [25].

MML identifier: HURWITZ2, version: 8.1.01 5.8.1171

The notation and terminology used in this paper have been introduced in the following articles: [16], [14], [2], [3], [10], [4], [5], [22], [19], [21], [15], [1], [6], [17], [11], [12], [13], [18], [8], [26], [23], [20], [24], [9], [27], and [7].

1. PRELIMINARIES

Now we state the propositions:

- (1) Let us consider complex numbers x, y . If $\Im(x) = 0$ and $\Re(y) = 0$, then $\Re\left(\frac{x}{y}\right) = 0$.
- (2) Let us consider a complex number a . Then $a \cdot \bar{a} = |a|^2$.

One can check that there exists a polynomial of \mathbb{C}_F which is Hurwitz and 0 is even.

Now we state the propositions:

- (3) Let us consider an add-associative right zeroed right complementable associative distributive non empty double loop structure L , an even element k of \mathbb{N} , and an element x of L . Then $\text{power}_L(-x, k) = \text{power}_L(x, k)$.
- (4) Let us consider an add-associative right zeroed right complementable associative distributive non empty double loop structure L , an odd element k of \mathbb{N} , and an element x of L . Then $\text{power}_L(-x, k) = -\text{power}_L(x, k)$. The theorem is a consequence of (3).
- (5) Let us consider an even element k of \mathbb{N} and an element x of \mathbb{C}_F . If $\Re(x) = 0$, then $\Im(\text{power}_{\mathbb{C}_F}(x, k)) = 0$.
- (6) Let us consider an odd element k of \mathbb{N} and an element x of \mathbb{C}_F . If $\Re(x) = 0$, then $\Re(\text{power}_{\mathbb{C}_F}(x, k)) = 0$.

2. EVEN AND ODD PART OF POLYNOMIALS

Let L be a non empty zero structure and p be a sequence of L . The functors the even part of p and the odd part of p yielding sequences of L are defined by the conditions, respectively.

- (Def. 1) Let us consider an even natural number i . Then
 - (i) (the even part of p)(i) = $p(i)$, and
 - (ii) for every odd natural number i , (the even part of p)(i) = 0_L .
- (Def. 2) Let us consider an even natural number i . Then
 - (i) (the odd part of p)(i) = 0_L , and
 - (ii) for every odd natural number i , (the odd part of p)(i) = $p(i)$.

Let p be a polynomial of L . Observe that the even part of p is finite-Support and the odd part of p is finite-Support. Now we state the propositions:

- (7) Let us consider a non empty zero structure L . Then
 - (i) the even part of $\mathbf{0}.L = \mathbf{0}.L$, and
 - (ii) the odd part of $\mathbf{0}.L = \mathbf{0}.L$.
- (8) Let us consider a non empty multiplicative loop with zero structure L . Then
 - (i) the even part of $\mathbf{1}.L = \mathbf{1}.L$, and
 - (ii) the odd part of $\mathbf{1}.L = \mathbf{0}.L$.

Let us consider a left zeroed right zeroed non empty additive loop structure L and a polynomial p of L . Now we state the propositions:

- (9) (The even part of p) + (the odd part of p) = p .
- (10) (The odd part of p) + (the even part of p) = p .

Let us consider an add-associative right zeroed right complementable non empty additive loop structure L and a polynomial p of L . Now we state the propositions:

- (11) $p -$ the odd part of $p =$ the even part of p .
- (12) $p -$ the even part of $p =$ the odd part of p .

Let us consider an add-associative right zeroed right complementable Abelian non empty additive loop structure L and a polynomial p of L . Now we state the propositions:

- (13) (The even part of $p) - p = -$ the odd part of p .
- (14) (The odd part of $p) - p = -$ the even part of p .

Let us consider an add-associative right zeroed right complementable Abelian non empty additive loop structure L and polynomials p, q of L . Now we state the propositions:

- (15) The even part of $p + q =$ (the even part of $p) +$ (the even part of q).
- (16) The odd part of $p + q =$ (the odd part of $p) +$ (the odd part of q).

Let us consider a well unital non empty double loop structure L and a polynomial p of L . Now we state the propositions:

- (17) Suppose $\deg p$ is even. Then the even part of Leading-Monomial $p =$ Leading-Monomial p .
- (18) If $\deg p$ is odd, then the even part of Leading-Monomial $p = \mathbf{0} \cdot L$.
- (19) If $\deg p$ is even, then the odd part of Leading-Monomial $p = \mathbf{0} \cdot L$.
- (20) Suppose $\deg p$ is odd. Then the odd part of Leading-Monomial $p =$ Leading-Monomial p .

Now we state the proposition:

- (21) Let us consider a well unital add-associative right zeroed right complementable Abelian associative distributive non degenerated double loop structure L and a non zero polynomial p of L . Then \deg the even part of $p \neq \deg$ the odd part of p . The theorem is a consequence of (9).

Let us consider a well unital add-associative right zeroed right complementable associative Abelian distributive non degenerated double loop structure L and a polynomial p of L . Now we state the propositions:

- (22) (i) \deg the even part of $p \leq \deg p$, and
(ii) \deg the odd part of $p \leq \deg p$.
- (23) $\deg p = \max(\deg$ the even part of p, \deg the odd part of $p)$.

3. EVEN AND ODD POLYNOMIALS AND RATIONAL FUNCTIONS

Let L be a non empty additive loop structure and f be a function from L into L . We say that f is even if and only if

(Def. 3) Let us consider an element x of L . Then $f(-x) = f(x)$.

We say that f is odd if and only if

(Def. 4) Let us consider an element x of L . Then $f(-x) = -f(x)$.

Let L be a well unital non empty double loop structure and p be a polynomial of L . We say that p is even if and only if

(Def. 5) Polynomial-Function(L, p) is even.

We say that p is odd if and only if

(Def. 6) Polynomial-Function(L, p) is odd.

Let Z be a rational function of L . We say that Z is odd if and only if

(Def. 7) (i) Z_1 is even and Z_2 is odd, or

(ii) Z_1 is odd and Z_2 is even.

We introduce Z is even as an antonym for Z is odd.

Observe that there exists a polynomial of L which is even.

Let L be an add-associative right zeroed right complementable well unital non empty double loop structure. Let us note that there exists a polynomial of L which is odd.

Let L be a well unital add-associative right zeroed right complementable associative non degenerated double loop structure. Observe that there exists a polynomial of L which is non zero and even.

Let L be an add-associative right zeroed right complementable Abelian well unital non degenerated double loop structure. One can verify that there exists a polynomial of L which is non zero and odd.

Now we state the propositions:

(24) Let us consider a well unital non empty double loop structure L , an even polynomial p of L , and an element x of L . Then $\text{eval}(p, -x) = \text{eval}(p, x)$.

(25) Let us consider an add-associative right zeroed right complementable Abelian well unital non degenerated double loop structure L , an odd polynomial p of L , and an element x of L . Then $\text{eval}(p, -x) = -\text{eval}(p, x)$.

Let L be a well unital non empty double loop structure. One can verify that $\mathbf{0}.L$ is even.

Let L be an add-associative right zeroed right complementable well unital non empty double loop structure. One can verify that $\mathbf{0}.L$ is odd.

Let L be a well unital add-associative right zeroed right complementable associative non degenerated double loop structure. Note that $\mathbf{1}.L$ is even.

Let L be an Abelian add-associative right zeroed right complementable well unital left distributive non empty double loop structure and p, q be even polynomials of L . Let us note that $p + q$ is even.

Let p, q be odd polynomials of L . Let us note that $p + q$ is odd.

Let L be an Abelian add-associative right zeroed right complementable associative well unital distributive non degenerated double loop structure and p

be a polynomial of L . One can check that the even part of p is even and the odd part of p is odd.

Now we state the propositions:

- (26) Let us consider an Abelian add-associative right zeroed right complementable well unital distributive non degenerated double loop structure L , an even polynomial p of L , an odd polynomial q of L , and an element x of L . If x is a common root of p and q , then $-x$ is a root of $p + q$. The theorem is a consequence of (24) and (25).
- (27) Let us consider a Hurwitz polynomial p of \mathbb{C}_F . Then the even part of p and the odd part of p have no common roots. The theorem is a consequence of (9) and (26).

4. REAL POSITIVE POLYNOMIALS AND RATIONAL FUNCTIONS

Let p be a polynomial of \mathbb{C}_F . We say that p is real if and only if

(Def. 8) Let us consider a natural number i . Then $p(i)$ is a real number.

We say that p is positive if and only if

(Def. 9) Let us consider an element x of \mathbb{C}_F . If $\Re(x) > 0$, then $\Re(\text{eval}(p, x)) > 0$.

Let us note that $\mathbf{0} \cdot \mathbb{C}_F$ is real and non positive and $\mathbf{1} \cdot \mathbb{C}_F$ is real and positive and there exists a polynomial of \mathbb{C}_F which is non zero, real, and positive and every polynomial of \mathbb{C}_F which is real is also real-valued.

Let p be a real polynomial of \mathbb{C}_F . One can verify that the even part of p is real and the odd part of p is real.

Let L be a non empty additive loop structure and p be a polynomial of L .

We say that p has all coefficients if and only if

(Def. 10) Let us consider a natural number i . If $i \leq \deg p$, then $p(i) \neq 0$.

Let p be a real polynomial of \mathbb{C}_F . We say that p has positive coefficients if and only if

(Def. 11) Let us consider a natural number i . If $i \leq \deg p$, then $p(i) > 0$.

We say that p is negative coefficients if and only if

(Def. 12) Let us consider a natural number i . If $i \leq \deg p$, then $p(i) < 0$.

One can check that every real polynomial of \mathbb{C}_F which has positive coefficients has also all coefficients and every real polynomial of \mathbb{C}_F which is negative coefficients has also all coefficients and there exists a real polynomial of \mathbb{C}_F which is non constant and has positive coefficients.

Let p be a non zero real polynomial of \mathbb{C}_F with all coefficients. Let us note that the even part of p is non zero. Note that the odd part of p is non zero.

Let Z be a rational function of \mathbb{C}_F . We say that Z is real if and only if

(Def. 13) Let us consider a natural number i . Then

- (i) $Z_1(i)$ is a real number, and
- (ii) $Z_2(i)$ is a real number.

We say that Z is positive if and only if

(Def. 14) Let us consider an element x of \mathbb{C}_F . Suppose

- (i) $\Re(x) > 0$, and
- (ii) $\text{eval}(Z_2, x) \neq 0$.

Then $\Re(\text{eval}(Z, x)) > 0$.

One can check that there exists a rational function of \mathbb{C}_F which is non zero, odd, real, and positive.

Let p_1 be a real polynomial of \mathbb{C}_F and p_2 be a non zero real polynomial of \mathbb{C}_F . Let us note that $\langle p_1, p_2 \rangle$ is real as a rational function of \mathbb{C}_F .

5. THE ROUTH-SCHUR STABILITY CRITERION

A one port function is a real positive rational function of \mathbb{C}_F . A reactance one port function is an odd real positive rational function of \mathbb{C}_F .

Let us consider a real polynomial p of \mathbb{C}_F and an element x of \mathbb{C}_F . Now we state the propositions:

- (28) If $\Re(x) = 0$, then $\Im(\text{eval}(\text{the even part of } p, x)) = 0$.
- (29) If $\Re(x) = 0$, then $\Re(\text{eval}(\text{the odd part of } p, x)) = 0$.

Now we state the proposition:

- (30) Let us consider a non constant real polynomial p of \mathbb{C}_F with positive coefficients. Suppose
 - (i) $\langle \text{the even part of } p, \text{ the odd part of } p \rangle$ is positive, and
 - (ii) the even part of p and the odd part of p have no common roots.

Then

- (iii) for every element x of \mathbb{C}_F such that $\Re(x) = 0$ and $\text{eval}(\text{the odd part of } p, x) \neq 0$ holds $\Re(\text{eval}(\langle \text{the even part of } p, \text{ the odd part of } p \rangle, x)) \geq 0$, and
- (iv) $(\text{the even part of } p) + (\text{the odd part of } p)$ is Hurwitz.

The theorem is a consequence of (28), (29), and (1).

Now we state the proposition:

- (31) ROUTH-SCHUR STABILITY CRITERION (FOR A SINGLE-INPUT, SINGLE-OUTPUT (SISO), LINEAR TIME INVARIANT (LTI) CONTROL SYSTEM):
Let us consider a non constant real polynomial p of \mathbb{C}_F with positive coefficients. Suppose
 - (i) $\langle \text{the even part of } p, \text{ the odd part of } p \rangle$ is a one port function, and

(ii) $\text{degree}(\langle \text{the even part of } p, \text{ the odd part of } p \rangle) = \text{degree}(p)$.

Then p is Hurwitz. The theorem is a consequence of (23), (30), and (9).

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Received January 17, 2013