FORMALIZED MATHEMATICS
Vol. 21, No. 3, Pages 163–170, 2013
DOI: 10.2478/forma-2013-0018

VERSITA
degruyter.com/view/j/forma

Double Sequences and Limits[1]

Noboru Endou
Gifu National College of Thechnology
Japan

Hiroyuki Okazaki
Shinshu University
Nagano, Japan

Yasunari Shidama
Shinshu University
Nagano, Japan

Summary. Double sequences are important extension of the ordinary notion of a sequence. In this article we formalized three types of limits of double sequences and the theory of these limits.

MSC: 54A20 03B35

Keywords: formalization of basic metric space; limits of double sequences

MML identifier: DBLSEQ_1, version: 8.1.02 5.19.1189

The notation and terminology used in this paper have been introduced in the following articles: [3], [4], [13], [5], [15], [6], [7], [16], [10], [1], [2], [8], [11], [18], [12], [17], and [9].

In this paper R, R_1, R_2 denote functions from $\mathbb{N} \times \mathbb{N}$ into $\mathbb{R}$, r_1, r_2 denote convergent sequences of real numbers, n, m, N, M denote natural numbers, and e, r denote real numbers.

Let us consider R. We say that R is p-convergent if and only if

(Def. 1) There exists a real number p such that for every real number e such that $0 < e$ there exists a natural number N such that for every natural numbers n, m such that $n \geqslant N$ and $m \geqslant N$ holds $|R(n,m) - p| < e$.

Assume R is p-convergent. The functor P-lim R yielding a real number is defined by

(Def. 2) Let us consider a real number e. Suppose $0 < e$. Then there exists a natural number N such that for every natural numbers n, m such that $n \geqslant N$ and $m \geqslant N$ holds $|R(n,m) - it| < e$.

[1]This work was supported by JSPS KAKENHI 23500029.

We say that R is convergent in the first coordinate if and only if

(Def. 3) Let us consider an element m of $\mathbb{N}$. Then $\operatorname{curry}'(R, m)$ is convergent.

We say that R is convergent in the second coordinate if and only if

(Def. 4) Let us consider an element n of $\mathbb{N}$. Then $\operatorname{curry}(R, n)$ is convergent.

The lim in the first coordinate of R yielding a function from $\mathbb{N}$ into $\mathbb{R}$ is defined by

(Def. 5) Let us consider an element m of $\mathbb{N}$. Then $it(m) = \lim \operatorname{curry}'(R, m)$.

The lim in the second coordinate of R yielding a function from $\mathbb{N}$ into $\mathbb{R}$ is defined by

(Def. 6) Let us consider an element n of $\mathbb{N}$. Then $it(n) = \lim \operatorname{curry}(R, n)$.

Assume the lim in the first coordinate of R is convergent. The first coordinate major iterated lim of R yielding a real number is defined by

(Def. 7) Let us consider a real number e. Suppose $0 < e$. Then there exists a natural number M such that for every natural number m such that $m \geqslant M$ holds $|$(the lim in the first coordinate of $R)(m) - it| < e$.

Assume the lim in the second coordinate of R is convergent. The second coordinate major iterated lim of R yielding a real number is defined by

(Def. 8) Let us consider a real number e. Suppose $0 < e$. Then there exists a natural number N such that for every natural number n such that $n \geqslant N$ holds $|$(the lim in the second coordinate of $R)(n) - it| < e$.

Let R be a function from $\mathbb{N} \times \mathbb{N}$ into $\mathbb{R}$. We say that R is uniformly convergent in the first coordinate if and only if

(Def. 9) (i) R is convergent in the first coordinate, and

(ii) for every real number e such that $e > 0$ there exists a natural number M such that for every natural number m such that $m \geqslant M$ for every natural number n, $|R(n, m) -$ (the lim in the first coordinate of $R)(n)| < e$.

We say that R is uniformly convergent in the second coordinate if and only if

(Def. 10) (i) R is convergent in the second coordinate, and

(ii) for every real number e such that $e > 0$ there exists a natural number N such that for every natural number n such that $n \geqslant N$ for every natural number m, $|R(n, m) -$ (the lim in the second coordinate of $R)(m)| < e$.

Let us consider R. We say that R is non-decreasing if and only if

(Def. 11) Let us consider natural numbers n_1, m_1, n_2, m_2. If $n_1 \geqslant n_2$ and $m_1 \geqslant m_2$, then $R(n_1, m_1) \geqslant R(n_2, m_2)$.

We say that R is non-increasing if and only if

(Def. 12) Let us consider natural numbers n_1, m_1, n_2, m_2. If $n_1 \geqslant n_2$ and $m_1 \geqslant m_2$, then $R(n_1, m_1) \leqslant R(n_2, m_2)$.

Now we state the proposition:

(1) Let us consider real numbers a, b, c. If $a \leqslant b \leqslant c$, then $|b| \leqslant |a|$ or $|b| \leqslant |c|$.

Note that every function from $\mathbb{N} \times \mathbb{N}$ into $\mathbb{R}$ which is non-decreasing and p-convergent is also lower bounded and upper bounded and every function from $\mathbb{N} \times \mathbb{N}$ into $\mathbb{R}$ which is non-increasing and p-convergent is also lower bounded and upper bounded.

Let r be an element of $\mathbb{R}$. Let us note that $\mathbb{N} \times \mathbb{N} \longmapsto r$ is p-convergent convergent in the first coordinate and convergent in the second coordinate as a function from $\mathbb{N} \times \mathbb{N}$ into $\mathbb{R}$.

Now we state the proposition:

(2) Let us consider an element r of $\mathbb{R}$. Then $\text{P-lim}(\mathbb{N} \times \mathbb{N} \longmapsto r) = r$. Proof: Set $R = \mathbb{N} \times \mathbb{N} \longmapsto r$. For every natural numbers n, m, $R(n, m) = r$ by [15, (70)]. $\square$

Note that there exists a function from $\mathbb{N} \times \mathbb{N}$ into $\mathbb{R}$ which is p-convergent, convergent in the first coordinate, and convergent in the second coordinate.

In this paper P_1 denotes a p-convergent function from $\mathbb{N} \times \mathbb{N}$ into $\mathbb{R}$.

Let P_4 be a p-convergent convergent in the second coordinate function from $\mathbb{N} \times \mathbb{N}$ into $\mathbb{R}$. Note that the lim in the second coordinate of P_4 is convergent.

Now we state the proposition:

(3) Suppose R is p-convergent and convergent in the second coordinate. Then $\text{P-lim}\, R =$ the second coordinate major iterated lim of R. Proof: Consider z being a real number such that for every e such that $0 < e$ there exists a natural number N_1 such that for every n and m such that $n \geqslant N_1$ and $m \geqslant N_1$ holds $|R(n, m) - z| < e$. For every e such that $0 < e$ there exists N such that for every n such that $n \geqslant N$ holds $|(\text{the lim in the second coordinate of } R)(n) - z| < e$ by [4, (63), (60)]. For every e such that $0 < e$ there exists N such that for every n such that $n \geqslant N$ holds $|(\text{the lim in the second coordinate of } R)(n) - \text{P-lim}\, R| < e$ by [4, (60), (63)]. $\square$

Let P_3 be a p-convergent convergent in the first coordinate function from $\mathbb{N} \times \mathbb{N}$ into $\mathbb{R}$. Let us note that the lim in the first coordinate of P_3 is convergent.

Now we state the proposition:

(4) Suppose R is p-convergent and convergent in the first coordinate. Then $\text{P-lim}\, R =$ the first coordinate major iterated lim of R. Proof: Consider z being a real number such that for every e such that $0 < e$ there exists a natural number N_1 such that for every n and m such that $n \geqslant N_1$ and $m \geqslant N_1$ holds $|R(n, m) - z| < e$. For every e such that $0 < e$ there exists N such that for every n such that $n \geqslant N$ holds $|(\text{the lim in the first coordinate of } R)(n) - z| < e$ by [4, (63), (60)]. For every e such that $0 < e$

there exists N such that for every n such that $n \geqslant N$ holds $|$(the lim in the first coordinate of $R)(n) - \text{P-lim}\, R| < e$ by [4, (60), (63)]. □

One can verify that every function from $\mathbb{N} \times \mathbb{N}$ into $\mathbb{R}$ which is non-decreasing and upper bounded is also p-convergent convergent in the first coordinate and convergent in the second coordinate and every function from $\mathbb{N} \times \mathbb{N}$ into $\mathbb{R}$ which is non-increasing and lower bounded is also p-convergent convergent in the first coordinate and convergent in the second coordinate.

Now we state the propositions:

(5) Suppose R is uniformly convergent in the first coordinate and the lim in the first coordinate of R is convergent. Then

(i) R is p-convergent, and

(ii) $\text{P-lim}\, R =$ the first coordinate major iterated lim of R.

(6) Suppose R is uniformly convergent in the second coordinate and the lim in the second coordinate of R is convergent. Then

(i) R is p-convergent, and

(ii) $\text{P-lim}\, R =$ the second coordinate major iterated lim of R.

Let us consider R. We say that R is Cauchy if and only if

(Def. 13) Let us consider a real number e. Suppose $e > 0$. Then there exists a natural number N such that for every natural numbers n_1, n_2, m_1, m_2 such that $N \leqslant n_1 \leqslant n_2$ and $N \leqslant m_1 \leqslant m_2$ holds $|R(n_2, m_2) - R(n_1, m_1)| < e$.

Now we state the propositions:

(7) R is p-convergent if and only if R is Cauchy. PROOF: Define $\mathcal{R}$(element of $\mathbb{N}) = R(\$_1, \$_1)$. Consider s_1 being a function from $\mathbb{N}$ into $\mathbb{R}$ such that for every element n of $\mathbb{N}$, $s_1(n) = \mathcal{R}(n)$ from [7, Sch. 4]. Reconsider $z = \lim s_1$ as a complex number. For every e such that $0 < e$ there exists N such that for every n and m such that $n \geqslant N$ and $m \geqslant N$ holds $|R(n, m) - z| < e$ by [4, (63)]. □

(8) Let us consider a function R from $\mathbb{N} \times \mathbb{N}$ into $\mathbb{R}$. Suppose

(i) R is non-decreasing, or

(ii) R is non-increasing.

Then R is p-convergent if and only if R is lower bounded and upper bounded.

Let X, Y be non empty sets, H be a binary operation on Y, and f, g be functions from X into Y. Observe that the functor $H_{f,g}$ yields a function from $X \times X$ into Y. Now we state the propositions:

(9) (i) $\cdot_{\mathbb{R}_{r_1, r_2}}$ is convergent in the first coordinate and convergent in the second coordinate, and

(ii) the lim in the first coordinate of $\cdot_{\mathbb{R}_{r_1, r_2}}$ is convergent, and

(iii) the first coordinate major iterated lim of $\cdot_{\mathbb{R}\, r_1, r_2} = \lim r_1 \cdot \lim r_2$, and

(iv) the lim in the second coordinate of $\cdot_{\mathbb{R} r_1, r_2}$ is convergent, and

(v) the second coordinate major iterated lim of $\cdot_{\mathbb{R}\, r_1, r_2} = \lim r_1 \cdot \lim r_2$, and

(vi) $\cdot_{\mathbb{R} r_1, r_2}$ is p-convergent, and

(vii) P-lim $\cdot_{\mathbb{R} r_1, r_2} = \lim r_1 \cdot \lim r_2$.

Proof: Set $R = \cdot_{\mathbb{R} r_1, r_2}$. For every n and m, $R(n, m) = r_1(n) \cdot r_2(m)$ by [5, (77)]. For every element m of $\mathbb{N}$ and for every real number e such that $0 < e$ there exists N such that for every n such that $n \geqslant N$ holds $|(\mathrm{curry}'(R, m))(n) - \lim r_1 \cdot r_2(m)| < e$ by [4, (47), (65), (44)]. For every element m of $\mathbb{N}$, $\mathrm{curry}'(R, m)$ is convergent. For every element m of $\mathbb{N}$ and for every real number e such that $0 < e$ there exists N such that for every n such that $n \geqslant N$ holds $|(\mathrm{curry}(R, m))(n) - r_1(m) \cdot \lim r_2| < e$ by [4, (47), (65), (44)]. For every element m of $\mathbb{N}$, $\mathrm{curry}(R, m)$ is convergent. For every e such that $0 < e$ there exists N such that for every n such that $n \geqslant N$ holds $|($the lim in the first coordinate of $R)(n) - \lim r_1 \cdot \lim r_2| < e$ by [4, (46), (65)]. For every e such that $0 < e$ there exists N such that for every n such that $n \geqslant N$ holds $|($the lim in the second coordinate of $R)(n) - \lim r_1 \cdot \lim r_2| < e$ by [4, (46), (65)]. For every e such that $0 < e$ there exists N such that for every n and m such that $n \geqslant N$ and $m \geqslant N$ holds $|R(n, m) - \lim r_1 \cdot \lim r_2| < e$ by [12, (3)], [4, (63), (46), (65)]. $\square$

(10) (i) $+_{\mathbb{R} r_1, r_2}$ is convergent in the first coordinate and convergent in the second coordinate, and

(ii) the lim in the first coordinate of $+_{\mathbb{R} r_1, r_2}$ is convergent, and

(iii) the first coordinate major iterated lim of $+_{\mathbb{R}\, r_1, r_2} = \lim r_1 + \lim r_2$, and

(iv) the lim in the second coordinate of $+_{\mathbb{R} r_1, r_2}$ is convergent, and

(v) the second coordinate major iterated lim of $+_{\mathbb{R}\, r_1, r_2} = \lim r_1 + \lim r_2$, and

(vi) $+_{\mathbb{R} r_1, r_2}$ is p-convergent, and

(vii) P-lim $+_{\mathbb{R} r_1, r_2} = \lim r_1 + \lim r_2$.

Proof: Set $R = +_{\mathbb{R} r_1, r_2}$. For every n and m, $R(n, m) = r_1(n) + r_2(m)$ by [5, (77)]. For every element m of $\mathbb{N}$ and for every real number e such that $0 < e$ there exists a natural number N such that for every natural number n such that $n \geqslant N$ holds $|(\mathrm{curry}'(R, m))(n) - (\lim r_1 + r_2(m))| < e$. For every element m of $\mathbb{N}$, $\mathrm{curry}'(R, m)$ is convergent. For every element m of $\mathbb{N}$ and for every real number e such that $0 < e$ there exists N such that for every n such that $n \geqslant N$ holds $|(\mathrm{curry}(R, m))(n) - (r_1(m) + \lim r_2)| < e$. For every element m of $\mathbb{N}$, $\mathrm{curry}(R, m)$ is convergent. For every e such that $0 < e$ there exists N such that for every n such that $n \geqslant N$ holds $|(\text{the lim in the first coordinate of } R)(n) - (\lim r_1 + \lim r_2)| < e$. For every e such that $0 < e$ there exists N such that for every n such that $n \geqslant N$ holds $|(\text{the lim in the second coordinate of } R)(n) - (\lim r_1 + \lim r_2)| < e$. For every e such that $0 < e$ there exists N such that for every n and m such that $n \geqslant N$ and $m \geqslant N$ holds $|R(n,m) - (\lim r_1 + \lim r_2)| < e$ by [4, (56)]. $\square$

(11) Suppose R_1 is p-convergent and R_2 is p-convergent. Then

(i) $R_1 + R_2$ is p-convergent, and

(ii) $\text{P-lim}(R_1 + R_2) = \text{P-lim}\, R_1 + \text{P-lim}\, R_2$.

(12) Suppose R_1 is p-convergent and R_2 is p-convergent. Then

(i) $R_1 - R_2$ is p-convergent, and

(ii) $\text{P-lim}(R_1 - R_2) = \text{P-lim}\, R_1 - \text{P-lim}\, R_2$.

(13) Let us consider a function R from $\mathbb{N} \times \mathbb{N}$ into $\mathbb{R}$ and a real number r. Suppose R is p-convergent. Then

(i) $r \cdot R$ is p-convergent, and

(ii) $\text{P-lim}(r \cdot R) = r \cdot \text{P-lim}\, R$.

(14) If R is p-convergent and for every natural numbers n, m, $R(n,m) \geqslant r$, then $\text{P-lim}\, R \geqslant r$.

(15) Suppose R_1 is p-convergent and R_2 is p-convergent and for every natural numbers n, m, $R_1(n,m) \leqslant R_2(n,m)$. Then $\text{P-lim}\, R_1 \leqslant \text{P-lim}\, R_2$. The theorem is a consequence of (12) and (14).

(16) Suppose R_1 is p-convergent and R_2 is p-convergent and $\text{P-lim}\, R_1 = \text{P-lim}\, R_2$ and for every natural numbers n, m, $R_1(n,m) \leqslant R(n,m) \leqslant R_2(n,m)$. Then

(i) R is p-convergent, and

(ii) $\text{P-lim}\, R = \text{P-lim}\, R_1$.

PROOF: For every e such that $0 < e$ there exists N such that for every n and m such that $n \geqslant N$ and $m \geqslant N$ holds $|R(n,m) - \text{P-lim}\, R_1| < e$ by [14, (4), (5), (1)]. $\square$

Let X be a non empty set and s_1 be a function from $\mathbb{N} \times \mathbb{N}$ into X. A subsequence of s_1 is a function from $\mathbb{N} \times \mathbb{N}$ into X and is defined by

(Def. 14) There exist increasing sequences N, M of $\mathbb{N}$ such that for every natural numbers n, m, $it(n,m) = s_1(N(n), M(m))$.

Let us consider P_1. Observe that every subsequence of P_1 is p-convergent.

Now we state the proposition:

(17) Let us consider a subsequence P_2 of P_1. Then $\text{P-lim}\, P_2 = \text{P-lim}\, P_1$.

Let R be a convergent in the first coordinate function from $\mathbb{N} \times \mathbb{N}$ into $\mathbb{R}$. Note that every subsequence of R is convergent in the first coordinate.

Now we state the proposition:

(18) Let us consider a subsequence R_1 of R. Suppose

(i) R is convergent in the first coordinate, and

(ii) the lim in the first coordinate of R is convergent.

Then

(iii) the lim in the first coordinate of R_1 is convergent, and

(iv) the first coordinate major iterated lim of $R_1 =$ the first coordinate major iterated lim of R.

Proof: Consider I_1, I_2 being increasing sequences of $\mathbb{N}$ such that for every natural numbers n, m, $R_1(n, m) = R(I_1(n), I_2(m))$. For every e such that $0 < e$ there exists N such that for every m such that $m \geqslant N$ holds $|$(the lim in the first coordinate of $R_1)(m) -$ the first coordinate major iterated lim of $R| < e$. $\square$

Let R be a convergent in the second coordinate function from $\mathbb{N} \times \mathbb{N}$ into $\mathbb{R}$. One can check that every subsequence of R is convergent in the second coordinate.

Now we state the proposition:

(19) Let us consider a subsequence R_1 of R. Suppose

(i) R is convergent in the second coordinate, and

(ii) the lim in the second coordinate of R is convergent.

Then

(iii) the lim in the second coordinate of R_1 is convergent, and

(iv) the second coordinate major iterated lim of $R_1 =$ the second coordinate major iterated lim of R.

Proof: Consider I_1, I_2 being increasing sequences of $\mathbb{N}$ such that for every n and m, $R_1(n, m) = R(I_1(n), I_2(m))$. For every e such that $0 < e$ there exists N such that for every m such that $m \geqslant N$ holds $|$(the lim in the second coordinate of $R_1)(m) -$ the second coordinate major iterated lim of $R| < e$. $\square$

References

[1] Grzegorz Bancerek. The fundamental properties of natural numbers. *Formalized Mathematics*, 1(**1**):41–46, 1990.

[2] Grzegorz Bancerek. The ordinal numbers. *Formalized Mathematics*, 1(**1**):91–96, 1990.

[3] Czesław Byliński. Binary operations. *Formalized Mathematics*, 1(**1**):175–180, 1990.

[4] Czesław Byliński. The complex numbers. *Formalized Mathematics*, 1(**3**):507–513, 1990.

[5] Czesław Byliński. Binary operations applied to finite sequences. *Formalized Mathematics*, 1(**4**):643–649, 1990.
[6] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**): 55–65, 1990.
[7] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.
[8] Czesław Byliński. Partial functions. *Formalized Mathematics*, 1(**2**):357–367, 1990.
[9] Czesław Byliński. Some basic properties of sets. *Formalized Mathematics*, 1(**1**):47–53, 1990.
[10] Noboru Endou, Keiko Narita, and Yasunari Shidama. The Lebesgue monotone convergence theorem. *Formalized Mathematics*, 16(**2**):167–175, 2008. doi:10.2478/v10037-008-0023-1.
[11] Andrzej Kondracki. Basic properties of rational numbers. *Formalized Mathematics*, 1(**5**): 841–845, 1990.
[12] Jarosław Kotowicz. Convergent sequences and the limit of sequences. *Formalized Mathematics*, 1(**2**):273–275, 1990.
[13] Adam Naumowicz. Conjugate sequences, bounded complex sequences and convergent complex sequences. *Formalized Mathematics*, 6(**2**):265–268, 1997.
[14] Jan Popiołek. Some properties of functions modul and signum. *Formalized Mathematics*, 1(**2**):263–264, 1990.
[15] Andrzej Trybulec. Binary operations applied to functions. *Formalized Mathematics*, 1 (**2**):329–334, 1990.
[16] Michał J. Trybulec. Integers. *Formalized Mathematics*, 1(**3**):501–505, 1990.
[17] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(**1**):67–71, 1990.
[18] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1 (**1**):73–83, 1990.

Received August 31, 2013