

Double Sequences and Limits¹

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Summary. Double sequences are important extension of the ordinary notion of a sequence. In this article we formalized three types of limits of double sequences and the theory of these limits.

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The notation and terminology used in this paper have been introduced in the following articles: [3], [4], [13], [5], [15], [6], [7], [16], [10], [1], [2], [8], [11], [18], [12], [17], and [9].

In this paper R , R_1 , R_2 denote functions from $\mathbb{N} \times \mathbb{N}$ into \mathbb{R} , r_1 , r_2 denote convergent sequences of real numbers, n , m , N , M denote natural numbers, and e , r denote real numbers.

Let us consider R . We say that R is p -convergent if and only if

(Def. 1) There exists a real number p such that for every real number e such that $0 < e$ there exists a natural number N such that for every natural numbers n , m such that $n \geq N$ and $m \geq N$ holds $|R(n, m) - p| < e$.

Assume R is p -convergent. The functor $P\text{-lim } R$ yielding a real number is defined by

(Def. 2) Let us consider a real number e . Suppose $0 < e$. Then there exists a natural number N such that for every natural numbers n , m such that $n \geq N$ and $m \geq N$ holds $|R(n, m) - it| < e$.

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We say that R is convergent in the first coordinate if and only if

(Def. 3) Let us consider an element m of \mathbb{N} . Then $\text{curry}'(R, m)$ is convergent.

We say that R is convergent in the second coordinate if and only if

(Def. 4) Let us consider an element n of \mathbb{N} . Then $\text{curry}(R, n)$ is convergent.

The lim in the first coordinate of R yielding a function from \mathbb{N} into \mathbb{R} is defined by

(Def. 5) Let us consider an element m of \mathbb{N} . Then $it(m) = \lim \text{curry}'(R, m)$.

The lim in the second coordinate of R yielding a function from \mathbb{N} into \mathbb{R} is defined by

(Def. 6) Let us consider an element n of \mathbb{N} . Then $it(n) = \lim \text{curry}(R, n)$.

Assume the lim in the first coordinate of R is convergent. The first coordinate major iterated lim of R yielding a real number is defined by

(Def. 7) Let us consider a real number e . Suppose $0 < e$. Then there exists a natural number M such that for every natural number m such that $m \geq M$ holds $|(the\ lim\ in\ the\ first\ coordinate\ of\ R)(m) - it| < e$.

Assume the lim in the second coordinate of R is convergent. The second coordinate major iterated lim of R yielding a real number is defined by

(Def. 8) Let us consider a real number e . Suppose $0 < e$. Then there exists a natural number N such that for every natural number n such that $n \geq N$ holds $|(the\ lim\ in\ the\ second\ coordinate\ of\ R)(n) - it| < e$.

Let R be a function from $\mathbb{N} \times \mathbb{N}$ into \mathbb{R} . We say that R is uniformly convergent in the first coordinate if and only if

(Def. 9) (i) R is convergent in the first coordinate, and

(ii) for every real number e such that $e > 0$ there exists a natural number M such that for every natural number m such that $m \geq M$ for every natural number n , $|R(n, m) - (the\ lim\ in\ the\ first\ coordinate\ of\ R)(n)| < e$.

We say that R is uniformly convergent in the second coordinate if and only if

(Def. 10) (i) R is convergent in the second coordinate, and

(ii) for every real number e such that $e > 0$ there exists a natural number N such that for every natural number n such that $n \geq N$ for every natural number m , $|R(n, m) - (the\ lim\ in\ the\ second\ coordinate\ of\ R)(m)| < e$.

Let us consider R . We say that R is non-decreasing if and only if

(Def. 11) Let us consider natural numbers n_1, m_1, n_2, m_2 . If $n_1 \geq n_2$ and $m_1 \geq m_2$, then $R(n_1, m_1) \geq R(n_2, m_2)$.

We say that R is non-increasing if and only if

(Def. 12) Let us consider natural numbers n_1, m_1, n_2, m_2 . If $n_1 \geq n_2$ and $m_1 \geq m_2$, then $R(n_1, m_1) \leq R(n_2, m_2)$.

Now we state the proposition:

- (1) Let us consider real numbers a, b, c . If $a \leq b \leq c$, then $|b| \leq |a|$ or $|b| \leq |c|$.

Note that every function from $\mathbb{N} \times \mathbb{N}$ into \mathbb{R} which is non-decreasing and p-convergent is also lower bounded and upper bounded and every function from $\mathbb{N} \times \mathbb{N}$ into \mathbb{R} which is non-increasing and p-convergent is also lower bounded and upper bounded.

Let r be an element of \mathbb{R} . Let us note that $\mathbb{N} \times \mathbb{N} \mapsto r$ is p-convergent convergent in the first coordinate and convergent in the second coordinate as a function from $\mathbb{N} \times \mathbb{N}$ into \mathbb{R} .

Now we state the proposition:

- (2) Let us consider an element r of \mathbb{R} . Then $P\text{-lim}(\mathbb{N} \times \mathbb{N} \mapsto r) = r$. PROOF: Set $R = \mathbb{N} \times \mathbb{N} \mapsto r$. For every natural numbers n, m , $R(n, m) = r$ by [15, (70)]. \square

Note that there exists a function from $\mathbb{N} \times \mathbb{N}$ into \mathbb{R} which is p-convergent, convergent in the first coordinate, and convergent in the second coordinate.

In this paper P_1 denotes a p-convergent function from $\mathbb{N} \times \mathbb{N}$ into \mathbb{R} .

Let P_4 be a p-convergent convergent in the second coordinate function from $\mathbb{N} \times \mathbb{N}$ into \mathbb{R} . Note that the lim in the second coordinate of P_4 is convergent.

Now we state the proposition:

- (3) Suppose R is p-convergent and convergent in the second coordinate. Then $P\text{-lim } R =$ the second coordinate major iterated lim of R . PROOF: Consider z being a real number such that for every e such that $0 < e$ there exists a natural number N_1 such that for every n and m such that $n \geq N_1$ and $m \geq N_1$ holds $|R(n, m) - z| < e$. For every e such that $0 < e$ there exists N such that for every n such that $n \geq N$ holds $|(\text{the lim in the second coordinate of } R)(n) - z| < e$ by [4, (63), (60)]. For every e such that $0 < e$ there exists N such that for every n such that $n \geq N$ holds $|(\text{the lim in the second coordinate of } R)(n) - P\text{-lim } R| < e$ by [4, (60), (63)]. \square

Let P_3 be a p-convergent convergent in the first coordinate function from $\mathbb{N} \times \mathbb{N}$ into \mathbb{R} . Let us note that the lim in the first coordinate of P_3 is convergent.

Now we state the proposition:

- (4) Suppose R is p-convergent and convergent in the first coordinate. Then $P\text{-lim } R =$ the first coordinate major iterated lim of R . PROOF: Consider z being a real number such that for every e such that $0 < e$ there exists a natural number N_1 such that for every n and m such that $n \geq N_1$ and $m \geq N_1$ holds $|R(n, m) - z| < e$. For every e such that $0 < e$ there exists N such that for every n such that $n \geq N$ holds $|(\text{the lim in the first coordinate of } R)(n) - z| < e$ by [4, (63), (60)]. For every e such that $0 < e$

there exists N such that for every n such that $n \geq N$ holds |(the lim in the first coordinate of R)(n) - P-lim R | $< e$ by [4, (60), (63)]. \square

One can verify that every function from $\mathbb{N} \times \mathbb{N}$ into \mathbb{R} which is non-decreasing and upper bounded is also p-convergent convergent in the first coordinate and convergent in the second coordinate and every function from $\mathbb{N} \times \mathbb{N}$ into \mathbb{R} which is non-increasing and lower bounded is also p-convergent convergent in the first coordinate and convergent in the second coordinate.

Now we state the propositions:

- (5) Suppose R is uniformly convergent in the first coordinate and the lim in the first coordinate of R is convergent. Then
 - (i) R is p-convergent, and
 - (ii) P-lim R = the first coordinate major iterated lim of R .
- (6) Suppose R is uniformly convergent in the second coordinate and the lim in the second coordinate of R is convergent. Then
 - (i) R is p-convergent, and
 - (ii) P-lim R = the second coordinate major iterated lim of R .

Let us consider R . We say that R is Cauchy if and only if

- (Def. 13) Let us consider a real number e . Suppose $e > 0$. Then there exists a natural number N such that for every natural numbers n_1, n_2, m_1, m_2 such that $N \leq n_1 \leq n_2$ and $N \leq m_1 \leq m_2$ holds $|R(n_2, m_2) - R(n_1, m_1)| < e$.

Now we state the propositions:

- (7) R is p-convergent if and only if R is Cauchy. PROOF: Define \mathcal{R} (element of \mathbb{N}) = $R(\$_1, \$_1)$. Consider s_1 being a function from \mathbb{N} into \mathbb{R} such that for every element n of \mathbb{N} , $s_1(n) = \mathcal{R}(n)$ from [7, Sch. 4]. Reconsider $z = \lim s_1$ as a complex number. For every e such that $0 < e$ there exists N such that for every n and m such that $n \geq N$ and $m \geq N$ holds $|R(n, m) - z| < e$ by [4, (63)]. \square
- (8) Let us consider a function R from $\mathbb{N} \times \mathbb{N}$ into \mathbb{R} . Suppose
 - (i) R is non-decreasing, or
 - (ii) R is non-increasing.

Then R is p-convergent if and only if R is lower bounded and upper bounded.

Let X, Y be non empty sets, H be a binary operation on Y , and f, g be functions from X into Y . Observe that the functor $H_{f,g}$ yields a function from $X \times X$ into Y . Now we state the propositions:

- (9) (i) $\cdot_{\mathbb{R}_{r_1, r_2}}$ is convergent in the first coordinate and convergent in the second coordinate, and
 - (ii) the lim in the first coordinate of $\cdot_{\mathbb{R}_{r_1, r_2}}$ is convergent, and

- (iii) the first coordinate major iterated lim of $\cdot_{\mathbb{R} r_1, r_2} = \lim r_1 \cdot \lim r_2$, and
- (iv) the lim in the second coordinate of $\cdot_{\mathbb{R} r_1, r_2}$ is convergent, and
- (v) the second coordinate major iterated lim of $\cdot_{\mathbb{R} r_1, r_2} = \lim r_1 \cdot \lim r_2$, and
- (vi) $\cdot_{\mathbb{R} r_1, r_2}$ is p-convergent, and
- (vii) P-lim $\cdot_{\mathbb{R} r_1, r_2} = \lim r_1 \cdot \lim r_2$.

PROOF: Set $R = \cdot_{\mathbb{R} r_1, r_2}$. For every n and m , $R(n, m) = r_1(n) \cdot r_2(m)$ by [5, (77)]. For every element m of \mathbb{N} and for every real number e such that $0 < e$ there exists N such that for every n such that $n \geq N$ holds $|(\text{curry}'(R, m))(n) - \lim r_1 \cdot r_2(m)| < e$ by [4, (47), (65), (44)]. For every element m of \mathbb{N} , $\text{curry}'(R, m)$ is convergent. For every element m of \mathbb{N} and for every real number e such that $0 < e$ there exists N such that for every n such that $n \geq N$ holds $|(\text{curry}(R, m))(n) - r_1(m) \cdot \lim r_2| < e$ by [4, (47), (65), (44)]. For every element m of \mathbb{N} , $\text{curry}(R, m)$ is convergent. For every e such that $0 < e$ there exists N such that for every n such that $n \geq N$ holds $|(\text{the lim in the first coordinate of } R)(n) - \lim r_1 \cdot \lim r_2| < e$ by [4, (46), (65)]. For every e such that $0 < e$ there exists N such that for every n such that $n \geq N$ holds $|(\text{the lim in the second coordinate of } R)(n) - \lim r_1 \cdot \lim r_2| < e$ by [4, (46), (65)]. For every e such that $0 < e$ there exists N such that for every n and m such that $n \geq N$ and $m \geq N$ holds $|R(n, m) - \lim r_1 \cdot \lim r_2| < e$ by [12, (3)], [4, (63), (46), (65)]. \square

- (10) (i) $+_{\mathbb{R} r_1, r_2}$ is convergent in the first coordinate and convergent in the second coordinate, and
- (ii) the lim in the first coordinate of $+_{\mathbb{R} r_1, r_2}$ is convergent, and
- (iii) the first coordinate major iterated lim of $+_{\mathbb{R} r_1, r_2} = \lim r_1 + \lim r_2$, and
- (iv) the lim in the second coordinate of $+_{\mathbb{R} r_1, r_2}$ is convergent, and
- (v) the second coordinate major iterated lim of $+_{\mathbb{R} r_1, r_2} = \lim r_1 + \lim r_2$, and
- (vi) $+_{\mathbb{R} r_1, r_2}$ is p-convergent, and
- (vii) P-lim $+_{\mathbb{R} r_1, r_2} = \lim r_1 + \lim r_2$.

PROOF: Set $R = +_{\mathbb{R} r_1, r_2}$. For every n and m , $R(n, m) = r_1(n) + r_2(m)$ by [5, (77)]. For every element m of \mathbb{N} and for every real number e such that $0 < e$ there exists a natural number N such that for every natural number n such that $n \geq N$ holds $|(\text{curry}'(R, m))(n) - (\lim r_1 + r_2(m))| < e$. For every element m of \mathbb{N} , $\text{curry}'(R, m)$ is convergent. For every element m of \mathbb{N} and for every real number e such that $0 < e$ there exists N such that for every n such that $n \geq N$ holds $|(\text{curry}(R, m))(n) - (r_1(m) + \lim r_2)| < e$. For every element m of \mathbb{N} , $\text{curry}(R, m)$ is convergent. For every e such

that $0 < e$ there exists N such that for every n such that $n \geq N$ holds $|(the\ lim\ in\ the\ first\ coordinate\ of\ R)(n) - (\lim r_1 + \lim r_2)| < e$. For every e such that $0 < e$ there exists N such that for every n such that $n \geq N$ holds $|(the\ lim\ in\ the\ second\ coordinate\ of\ R)(n) - (\lim r_1 + \lim r_2)| < e$. For every e such that $0 < e$ there exists N such that for every n and m such that $n \geq N$ and $m \geq N$ holds $|R(n, m) - (\lim r_1 + \lim r_2)| < e$ by [4, (56)]. \square

- (11) Suppose R_1 is p-convergent and R_2 is p-convergent. Then
- (i) $R_1 + R_2$ is p-convergent, and
 - (ii) $P\text{-lim}(R_1 + R_2) = P\text{-lim } R_1 + P\text{-lim } R_2$.
- (12) Suppose R_1 is p-convergent and R_2 is p-convergent. Then
- (i) $R_1 - R_2$ is p-convergent, and
 - (ii) $P\text{-lim}(R_1 - R_2) = P\text{-lim } R_1 - P\text{-lim } R_2$.
- (13) Let us consider a function R from $\mathbb{N} \times \mathbb{N}$ into \mathbb{R} and a real number r . Suppose R is p-convergent. Then
- (i) $r \cdot R$ is p-convergent, and
 - (ii) $P\text{-lim}(r \cdot R) = r \cdot P\text{-lim } R$.
- (14) If R is p-convergent and for every natural numbers n, m , $R(n, m) \geq r$, then $P\text{-lim } R \geq r$.
- (15) Suppose R_1 is p-convergent and R_2 is p-convergent and for every natural numbers n, m , $R_1(n, m) \leq R_2(n, m)$. Then $P\text{-lim } R_1 \leq P\text{-lim } R_2$. The theorem is a consequence of (12) and (14).
- (16) Suppose R_1 is p-convergent and R_2 is p-convergent and $P\text{-lim } R_1 = P\text{-lim } R_2$ and for every natural numbers n, m , $R_1(n, m) \leq R(n, m) \leq R_2(n, m)$. Then
- (i) R is p-convergent, and
 - (ii) $P\text{-lim } R = P\text{-lim } R_1$.

PROOF: For every e such that $0 < e$ there exists N such that for every n and m such that $n \geq N$ and $m \geq N$ holds $|R(n, m) - P\text{-lim } R_1| < e$ by [14, (4), (5), (1)]. \square

Let X be a non empty set and s_1 be a function from $\mathbb{N} \times \mathbb{N}$ into X . A subsequence of s_1 is a function from $\mathbb{N} \times \mathbb{N}$ into X and is defined by

- (Def. 14) There exist increasing sequences N, M of \mathbb{N} such that for every natural numbers n, m , $it(n, m) = s_1(N(n), M(m))$.

Let us consider P_1 . Observe that every subsequence of P_1 is p-convergent. Now we state the proposition:

- (17) Let us consider a subsequence P_2 of P_1 . Then $P\text{-lim } P_2 = P\text{-lim } P_1$.

Let R be a convergent in the first coordinate function from $\mathbb{N} \times \mathbb{N}$ into \mathbb{R} . Note that every subsequence of R is convergent in the first coordinate.

Now we state the proposition:

- (18) Let us consider a subsequence R_1 of R . Suppose
- (i) R is convergent in the first coordinate, and
 - (ii) the lim in the first coordinate of R is convergent.

Then

- (iii) the lim in the first coordinate of R_1 is convergent, and
- (iv) the first coordinate major iterated lim of $R_1 =$ the first coordinate major iterated lim of R .

PROOF: Consider I_1, I_2 being increasing sequences of \mathbb{N} such that for every natural numbers n, m , $R_1(n, m) = R(I_1(n), I_2(m))$. For every e such that $0 < e$ there exists N such that for every m such that $m \geq N$ holds $|(\text{the lim in the first coordinate of } R_1)(m) - \text{the first coordinate major iterated lim of } R| < e$. \square

Let R be a convergent in the second coordinate function from $\mathbb{N} \times \mathbb{N}$ into \mathbb{R} . One can check that every subsequence of R is convergent in the second coordinate.

Now we state the proposition:

- (19) Let us consider a subsequence R_1 of R . Suppose
- (i) R is convergent in the second coordinate, and
 - (ii) the lim in the second coordinate of R is convergent.

Then

- (iii) the lim in the second coordinate of R_1 is convergent, and
- (iv) the second coordinate major iterated lim of $R_1 =$ the second coordinate major iterated lim of R .

PROOF: Consider I_1, I_2 being increasing sequences of \mathbb{N} such that for every n and m , $R_1(n, m) = R(I_1(n), I_2(m))$. For every e such that $0 < e$ there exists N such that for every m such that $m \geq N$ holds $|(\text{the lim in the second coordinate of } R_1)(m) - \text{the second coordinate major iterated lim of } R| < e$. \square

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