

# The Linearity of Riemann Integral on Functions from $\mathbb{R}$ into Real Banach Space<sup>1</sup>

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**Summary.** In this article, we described basic properties of Riemann integral on functions from  $\mathbb{R}$  into Real Banach Space. We proved mainly the linearity of integral operator about the integral of continuous functions on closed interval of the set of real numbers. These theorems were based on the article [10] and we referred to the former articles about Riemann integral. We applied definitions and theorems introduced in the article [9] and the article [11] to the proof. Using the definition of the article [10], we also proved some theorems on bounded functions.

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The notation and terminology used in this paper have been introduced in the following articles: [2], [12], [3], [4], [9], [10], [7], [8], [16], [1], [17], [13], [14], [5], [15], [20], [21], [18], [19], [22], and [6].

## 1. SOME PROPERTIES OF BOUNDED FUNCTIONS

In this paper  $Z$  denotes a real normed space,  $a, b, c, d, e, r$  denote real numbers, and  $A, B$  denote non empty closed interval subsets of  $\mathbb{R}$ .

Let us consider a partial function  $f$  from  $\mathbb{R}$  to the carrier of  $Z$ . Now we state the propositions:

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- (1) If  $f$  is bounded and  $A \subseteq \text{dom } f$ , then  $f \upharpoonright A$  is bounded.
- (2) If  $f \upharpoonright A$  is bounded and  $B \subseteq A$  and  $B \subseteq \text{dom}(f \upharpoonright A)$ , then  $f \upharpoonright B$  is bounded.
- (3) If  $a \leq c \leq d \leq b$  and  $f \upharpoonright [a, b]$  is bounded and  $[a, b] \subseteq \text{dom } f$ , then  $f \upharpoonright [c, d]$  is bounded.

Now we state the proposition:

- (4) Let us consider sets  $X, Y$  and partial functions  $f_1, f_2$  from  $\mathbb{R}$  to the carrier of  $Z$ . Suppose
  - (i)  $f_1 \upharpoonright X$  is bounded, and
  - (ii)  $f_2 \upharpoonright Y$  is bounded.

Then

- (iii)  $(f_1 + f_2) \upharpoonright (X \cap Y)$  is bounded, and
- (iv)  $(f_1 - f_2) \upharpoonright (X \cap Y)$  is bounded.

Let us consider a set  $X$  and a partial function  $f$  from  $\mathbb{R}$  to the carrier of  $Z$ . Now we state the propositions:

- (5) If  $f \upharpoonright X$  is bounded, then  $(r \cdot f) \upharpoonright X$  is bounded.
- (6) If  $f \upharpoonright X$  is bounded, then  $(-f) \upharpoonright X$  is bounded.

Now we state the propositions:

- (7) Let us consider a function  $f$  from  $A$  into the carrier of  $Z$ . Then  $f$  is bounded if and only if  $\|f\|$  is bounded.
- (8) Let us consider a partial function  $f$  from  $\mathbb{R}$  to the carrier of  $Z$ . Suppose  $A \subseteq \text{dom } f$ . Then  $\|f \upharpoonright A\| = \|f\| \upharpoonright A$ .
- (9) Let us consider a partial function  $g$  from  $\mathbb{R}$  to the carrier of  $Z$ . Suppose
  - (i)  $A \subseteq \text{dom } g$ , and
  - (ii)  $g \upharpoonright A$  is bounded.

Then  $\|g\| \upharpoonright A$  is bounded. The theorem is a consequence of (8) and (7).

## 2. SOME PROPERTIES OF INTEGRAL OF CONTINUOUS FUNCTIONS

In the sequel  $X, Y$  denote real Banach spaces and  $E$  denotes a point of  $Y$ .

Let us consider a real normed space  $Y$  and a continuous partial function  $f$  from  $\mathbb{R}$  to the carrier of  $Y$ . Now we state the propositions:

- (10) If  $a \leq b$  and  $[a, b] \subseteq \text{dom } f$ , then  $\|f\| \upharpoonright [a, b]$  is bounded.
- (11) If  $a \leq b$  and  $[a, b] \subseteq \text{dom } f$ , then  $f \upharpoonright [a, b]$  is bounded.
- (12) If  $a \leq b$  and  $[a, b] \subseteq \text{dom } f$ , then  $\|f\|$  is integrable on  $[a, b]$ .

Now we state the propositions:

- (13) Let us consider a continuous partial function  $f$  from  $\mathbb{R}$  to the carrier of  $Y$ . Suppose

- (i)  $a \leq c \leq d \leq b$ , and
- (ii)  $[a, b] \subseteq \text{dom } f$ .

Then  $f$  is integrable on  $[c, d]$ .

- (14) Let us consider a partial function  $f$  from  $\mathbb{R}$  to the carrier of  $Y$ . Suppose
- (i)  $a \leq b$ , and
  - (ii)  $[a, b] \subseteq \text{dom } f$ .

$$\text{Then } \int_b^a f(x)dx = - \int_a^b f(x)dx.$$

- (15) Let us consider a continuous partial function  $f$  from  $\mathbb{R}$  to the carrier of  $Y$ . Suppose
- (i)  $a \leq b$ , and
  - (ii)  $[a, b] \subseteq \text{dom } f$ , and
  - (iii)  $c \in [a, b]$ .

Then

- (iv)  $f$  is integrable on  $[a, c]$ , and
- (v)  $f$  is integrable on  $[c, b]$ , and
- (vi)  $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$ .

The theorem is a consequence of (13).

- (16) Let us consider continuous partial functions  $f, g$  from  $\mathbb{R}$  to the carrier of  $Y$ . Suppose
- (i)  $a \leq c \leq d \leq b$ , and
  - (ii)  $[a, b] \subseteq \text{dom } f$ , and
  - (iii)  $[a, b] \subseteq \text{dom } g$ .

Then

- (iv)  $f + g$  is integrable on  $[c, d]$ , and
- (v)  $(f + g)|_{[c, d]}$  is bounded.

The theorem is a consequence of (13), (11), (3), and (4).

Let us consider a continuous partial function  $f$  from  $\mathbb{R}$  to the carrier of  $Y$ . Now we state the propositions:

- (17) If  $a \leq c \leq d \leq b$  and  $[a, b] \subseteq \text{dom } f$ , then  $r \cdot f$  is integrable on  $[c, d]$  and  $(r \cdot f)|_{[c, d]}$  is bounded.
- (18) Suppose  $a \leq c \leq d \leq b$  and  $f$  is integrable on  $[a, b]$  and  $f|_{[a, b]}$  is bounded and  $[a, b] \subseteq \text{dom } f$ . Then

- (i)  $-f$  is integrable on  $[c, d]$ , and
- (ii)  $(-f) \upharpoonright [c, d]$  is bounded.

Now we state the proposition:

- (19) Let us consider continuous partial functions  $f, g$  from  $\mathbb{R}$  to the carrier of  $Y$ . Suppose

- (i)  $a \leq c \leq d \leq b$ , and
- (ii)  $[a, b] \subseteq \text{dom } f$ , and
- (iii)  $[a, b] \subseteq \text{dom } g$ .

Then

- (iv)  $f - g$  is integrable on  $[c, d]$ , and
- (v)  $(f - g) \upharpoonright [c, d]$  is bounded.

The theorem is a consequence of (11), (13), (3), and (4).

Let us consider a partial function  $f$  from  $\mathbb{R}$  to the carrier of  $Y$ . Now we state the propositions:

- (20) Suppose  $A \subseteq \text{dom } f$  and  $f \upharpoonright A$  is bounded and  $f$  is integrable on  $A$  and  $\|f\|$  is integrable on  $A$ . Then  $\|\int_A f(x)dx\| \leq \int_A \|f\|(x)dx$ .

- (21) Suppose  $a \leq b$  and  $[a, b] \subseteq \text{dom } f$  and  $f$  is integrable on  $[a, b]$  and  $\|f\|$  is integrable on  $[a, b]$  and  $f \upharpoonright [a, b]$  is bounded. Then  $\|\int_a^b f(x)dx\| \leq$

$$\int_a^b \|f\|(x)dx.$$

Let us consider a continuous partial function  $f$  from  $\mathbb{R}$  to the carrier of  $Y$ . Now we state the propositions:

- (22) Suppose  $a \leq b$  and  $[a, b] \subseteq \text{dom } f$  and  $c, d \in [a, b]$ . Then

- (i)  $\|f\|$  is integrable on  $[\min(c, d), \max(c, d)]$ , and
- (ii)  $\|f\| \upharpoonright [\min(c, d), \max(c, d)]$  is bounded, and

$$(iii) \left\| \int_c^d f(x)dx \right\| \leq \int_{\min(c,d)}^{\max(c,d)} \|f\|(x)dx.$$

- (23) If  $a \leq b$  and  $[a, b] \subseteq \text{dom } f$  and  $c, d \in [a, b]$ , then  $\int_c^d (r \cdot f)(x)dx =$

$$r \cdot \int_c^d f(x)dx.$$

- (24) Suppose  $a \leq b$  and  $[a, b] \subseteq \text{dom } f$  and  $c, d \in [a, b]$ . Then  $\int_c^d -f(x)dx = -\int_c^d f(x)dx$ .
- (25) Suppose  $a \leq b$  and  $[a, b] \subseteq \text{dom } f$  and  $c, d \in [a, b]$  and for every real number  $x$  such that  $x \in [\min(c, d), \max(c, d)]$  holds  $\|f_x\| \leq e$ . Then  $\|\int_c^d f(x)dx\| \leq e \cdot |d - c|$ .

Now we state the propositions:

- (26) Let us consider a real normed space  $Y$ , a non empty closed interval subset  $A$  of  $\mathbb{R}$ , a function  $f$  from  $A$  into the carrier of  $Y$ , and a point  $E$  of  $Y$ . Suppose  $\text{rng } f = \{E\}$ . Then
- (i)  $f$  is integrable, and
  - (ii)  $\text{integral } f = \text{vol}(A) \cdot E$ .
- PROOF: Reconsider  $I = \text{vol}(A) \cdot E$  as a point of  $Y$ . For every division sequence  $T$  of  $A$  and for every middle volume sequence  $S$  of  $f$  and  $T$  such that  $\delta_T$  is convergent and  $\lim \delta_T = 0$  holds middle sum( $f, S$ ) is convergent and  $\lim \text{middle sum}(f, S) = I$  by [11, (6)], [20, (70)], [11, (7)].  $\square$
- (27) Let us consider a partial function  $f$  from  $\mathbb{R}$  to the carrier of  $Y$  and a point  $E$  of  $Y$ . Suppose
- (i)  $a \leq b$ , and
  - (ii)  $[a, b] \subseteq \text{dom } f$ , and
  - (iii) for every real number  $x$  such that  $x \in [a, b]$  holds  $f_x = E$ .

Then

- (iv)  $f$  is integrable on  $[a, b]$ , and
- (v)  $\int_a^b f(x)dx = (b - a) \cdot E$ .

The theorem is a consequence of (26). PROOF: Reconsider  $A = [a, b]$  as a non empty closed interval subset of  $\mathbb{R}$ . Reconsider  $g = f \upharpoonright A$  as a function from  $A$  into the carrier of  $Y$ .  $\{E\} \subseteq \text{rng } g$  by [19, (4)], [3, (49), (3)].  $\text{rng } g \subseteq \{E\}$  by [5, (3)], [3, (49)].  $\square$

- (28) Let us consider a partial function  $f$  from  $\mathbb{R}$  to the carrier of  $Y$ . Suppose
- (i)  $a \leq b$ , and
  - (ii)  $c, d \in [a, b]$ , and
  - (iii)  $[a, b] \subseteq \text{dom } f$ , and

(iv) for every real number  $x$  such that  $x \in [a, b]$  holds  $f_x = E$ .

Then  $\int_c^d f(x)dx = (d - c) \cdot E$ . The theorem is a consequence of (27) and (14).

(29) Let us consider a continuous partial function  $f$  from  $\mathbb{R}$  to the carrier of  $Y$ . Suppose

- (i)  $a \leq b$ , and
- (ii)  $[a, b] \subseteq \text{dom } f$ , and
- (iii)  $c, d \in [a, b]$ .

Then  $\int_a^d f(x)dx = \int_a^c f(x)dx + \int_c^d f(x)dx$ . The theorem is a consequence of (14).

(30) Let us consider continuous partial functions  $f, g$  from  $\mathbb{R}$  to the carrier of  $Y$ . Suppose

- (i)  $a \leq b$ , and
- (ii)  $[a, b] \subseteq \text{dom } f$ , and
- (iii)  $[a, b] \subseteq \text{dom } g$ , and
- (iv)  $c, d \in [a, b]$ .

Then  $\int_c^d (f - g)(x)dx = \int_c^d f(x)dx - \int_c^d g(x)dx$ . The theorem is a consequence of (14).

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