

Isomorphisms of Direct Products of Cyclic Groups of Prime Power Order

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Summary. In this paper we formalized some theorems concerning the cyclic groups of prime power order. We formalize that every commutative cyclic group of prime power order is isomorphic to a direct product of family of cyclic groups [1], [18].

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The notation and terminology used in this paper have been introduced in the following articles: [2], [20], [6], [11], [7], [8], [24], [18], [25], [26], [27], [28], [13], [23], [16], [21], [3], [4], [15], [5], [9], [22], [17], [12], [30], [31], [14], [29], and [10].

1. BASIC PROPERTIES OF CYCLIC GROUPS OF PRIME POWER ORDER

Let G be a finite group. The functor $\text{Ordset}(G)$ yielding a subset of \mathbb{N} is defined by the term

(Def. 1) the set of all $\text{ord}(a)$ where a is an element of G .

One can check that $\text{Ordset}(G)$ is finite and non empty.

Now we state the propositions:

- (1) Let us consider a finite group G . Then there exists an element g of G such that $\text{ord}(g) = \sup \text{Ordset}(G)$.

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- (2) Let us consider a strict group G and a strict normal subgroup N of G . If G is commutative, then G/N is commutative.
- (3) Let us consider a finite group G and elements a, b of G . Then $b \in \text{gr}(\{a\})$ if and only if there exists an element p of \mathbb{N} such that $b = a^p$.
- (4) Let us consider a finite group G , an element a of G , and elements n, p, s of \mathbb{N} . Suppose

(i) $\overline{\text{gr}(\{a\})} = n$, and

(ii) $n = p \cdot s$.

Then $\text{ord}(a^p) = s$.

Let us consider an element k of \mathbb{N} , a finite group G , and an element a of G . Now we state the propositions:

- (5) $\text{gr}(\{a\}) = \text{gr}(\{a^k\})$ if and only if $\text{gcd}(k, \text{ord}(a)) = 1$.
- (6) If $\text{gcd}(k, \text{ord}(a)) = 1$, then $\text{ord}(a) = \text{ord}(a^k)$.
- (7) $\text{ord}(a) \mid k \cdot \text{ord}(a^k)$.

Now we state the proposition:

- (8) Let us consider a group G and elements a, b of G . Suppose $b \in \text{gr}(\{a\})$. Then $\text{gr}(\{b\})$ is a strict subgroup of $\text{gr}(\{a\})$.

Let G be a strict commutative group and x be an element of $\text{SubGr } G$. The functor $\text{NormSp}_{\mathbb{R}}(x)$ yielding a normal strict subgroup of G is defined by the term

(Def. 2) x .

Now we state the propositions:

- (9) Let us consider groups G, H , a subgroup K of H , and a homomorphism f from G to H . Then there exists a strict subgroup J of G such that the carrier of $J = f^{-1}$ (the carrier of K). PROOF: Reconsider $I_3 = f^{-1}$ (the carrier of K) as a non empty subset of the carrier of G . For every elements g_1, g_2 of G such that $g_1, g_2 \in I_3$ holds $g_1 \cdot g_2 \in I_3$ by [8, (38)], [25, (50)]. For every element g of G such that $g \in I_3$ holds $g^{-1} \in I_3$ by [8, (38)], [25, (51)], [28, (32)]. Consider J being a strict subgroup of G such that the carrier of $J = f^{-1}$ (the carrier of K). \square
- (10) Let us consider a natural number p , a finite group G , and elements x, d of G . Suppose
- (i) $\text{ord}(d) = p$, and
- (ii) p is prime, and
- (iii) $x \in \text{gr}(\{d\})$.
- Then
- (iv) $x = \mathbf{1}_G$, or
- (v) $\text{gr}(\{x\}) = \text{gr}(\{d\})$.

The theorem is a consequence of (8). PROOF: If $\text{gr}(\{x\}) = \{\mathbf{1}\}_{\text{gr}(\{d\})}$, then $x = \mathbf{1}_G$ by [19, (2)], [25, (44)]. \square

- (11) Let us consider a group G and normal subgroups H, K of G . Suppose $(\text{the carrier of } H) \cap (\text{the carrier of } K) = \{\mathbf{1}_G\}$. Then $(\text{the canonical homomorphism onto cosets of } H) \upharpoonright (\text{the carrier of } K)$ is one-to-one. PROOF: Set $f = \text{the canonical homomorphism onto cosets of } H$. Set $g = f \upharpoonright (\text{the carrier of } K)$. For every elements x_1, x_2 such that $x_1, x_2 \in \text{dom } g$ and $g(x_1) = g(x_2)$ holds $x_1 = x_2$ by [30, (57)], [7, (49)], [25, (46), (103), (51)]. \square

Let us consider finite commutative groups G, F , an element a of G , and a homomorphism f from G to F . Now we state the propositions:

- (12) The carrier of $\text{gr}(\{f(a)\}) = f \circ \text{the carrier of } \text{gr}(\{a\})$.
 (13) $\text{ord}(f(a)) \leq \text{ord}(a)$.
 (14) If f is one-to-one, then $\text{ord}(f(a)) = \text{ord}(a)$.

Now we state the propositions:

- (15) Let us consider groups G, F , a subgroup H of G , and a homomorphism f from G to F . Then $f \upharpoonright (\text{the carrier of } H)$ is a homomorphism from H to F . PROOF: Reconsider $g = f \upharpoonright (\text{the carrier of } H)$ as a function from the carrier of H into the carrier of F . For every elements a, b of H , $g(a \cdot b) = g(a) \cdot g(b)$ by [25, (40)], [7, (49)], [25, (43)]. \square
- (16) Let us consider finite commutative groups G, F , an element a of G , and a homomorphism f from G to F . Suppose $f \upharpoonright (\text{the carrier of } \text{gr}(\{a\}))$ is one-to-one. Then $\text{ord}(f(a)) = \text{ord}(a)$. The theorem is a consequence of (15) and (14).
- (17) Let us consider a finite commutative group G , a prime number p , a natural number m , and an element a of G . Suppose
- (i) $\overline{G} = p^m$, and
 - (ii) $a \neq \mathbf{1}_G$.

Then there exists a natural number n such that $\text{ord}(a) = p^{n+1}$.

- (18) Let us consider a prime number p and natural numbers j, m, k . If $m = p^k$ and $p \nmid j$, then $\text{gcd}(j, m) = 1$.

2. ISOMORPHISM OF CYCLIC GROUPS OF PRIME POWER ORDER

Let us consider a strict finite commutative group G , a prime number p , and a natural number m . Now we state the propositions:

- (19) Suppose $\overline{G} = p^m$. Then there exists a normal strict subgroup K of G and there exist natural numbers n, k and there exists an element g of G such that $\text{ord}(g) = \text{sup Ordset}(G)$ and K is finite and commutative and

(the carrier of K) \cap (the carrier of $\text{gr}(\{g\})$) = $\{\mathbf{1}_G\}$ and for every element x of G , there exist elements b_1, a_1 of G such that $b_1 \in K$ and $a_1 \in \text{gr}(\{g\})$ and $x = b_1 \cdot a_1$ and $\text{ord}(g) = p^n$ and $k = m - n$ and $n \leq m$ and $\overline{K} = p^k$ and there exists a homomorphism F from $\prod \langle K, \text{gr}(\{g\}) \rangle$ to G such that F is bijective and for every elements a, b of G such that $a \in K$ and $b \in \text{gr}(\{g\})$ holds $F(\langle a, b \rangle) = a \cdot b$.

- (20) Suppose $\overline{G} = p^m$. Then there exists a non zero natural number k and there exists a k -element finite sequence a of elements of G and there exists a k -element finite sequence I_2 of elements of \mathbb{N} and there exists an associative group-like commutative multiplicative magma family F of $\text{Seg } k$ and there exists a homomorphism H_1 from $\prod F$ to G such that for every natural number i such that $i \in \text{Seg } k$ there exists an element a_2 of G such that $a_2 = a(i)$ and $F(i) = \text{gr}(\{a_2\})$ and $\text{ord}(a_2) = p^{I_2(i)}$ and for every natural number i such that $1 \leq i \leq k - 1$ holds $I_2(i) \leq I_2(i + 1)$ and for every elements p, q of $\text{Seg } k$ such that $p \neq q$ holds (the carrier of $F(p)$) \cap (the carrier of $F(q)$) = $\{\mathbf{1}_G\}$ and H_1 is bijective and for every (the carrier of G)-valued total $\text{Seg } k$ -defined function x such that for every element p of $\text{Seg } k$, $x(p) \in F(p)$ holds $x \in \prod F$ and $H_1(x) = \prod x$.
- (21) Suppose $\overline{G} = p^m$. Then there exists a non zero natural number k and there exists a k -element finite sequence a of elements of G and there exists a k -element finite sequence I_2 of elements of \mathbb{N} and there exists an associative group-like commutative multiplicative magma family F of $\text{Seg } k$ such that for every natural number i such that $i \in \text{Seg } k$ there exists an element a_2 of G such that $a_2 = a(i)$ and $F(i) = \text{gr}(\{a_2\})$ and $\text{ord}(a_2) = p^{I_2(i)}$ and for every natural number i such that $1 \leq i \leq k - 1$ holds $I_2(i) \leq I_2(i + 1)$ and for every elements p, q of $\text{Seg } k$ such that $p \neq q$ holds (the carrier of $F(p)$) \cap (the carrier of $F(q)$) = $\{\mathbf{1}_G\}$ and for every element y of G , there exists a (the carrier of G)-valued total $\text{Seg } k$ -defined function x such that for every element p of $\text{Seg } k$, $x(p) \in F(p)$ and $y = \prod x$ and for every (the carrier of G)-valued total $\text{Seg } k$ -defined functions x_1, x_2 such that for every element p of $\text{Seg } k$, $x_1(p) \in F(p)$ and for every element p of $\text{Seg } k$, $x_2(p) \in F(p)$ and $\prod x_1 = \prod x_2$ holds $x_1 = x_2$.

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