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Topological Properties of Real Normed Space[1]

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Summary. In this article, we formalize topological properties of real normed spaces. In the first part, open and closed, density, separability and sequence and its convergence are discussed. Then we argue properties of real normed subspace. Then we discuss linear functions between real normed speces. Several kinds of subspaces induced by linear functions such as kernel, image and inverse image are considered here. The fact that Lipschitz continuity operators preserve convergence of sequences is also refered here. Then we argue the condition when real normed subspaces become Banach's spaces. We also formalize quotient vector space. In the last session, we argue the properties of the closure of real normed space. These formalizations are based on [19](p.3-41), [2] and [34](p.3-67).

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The notation and terminology used in this paper have been introduced in the following articles: [3], [23], [14], [4], [5], [20], [25], [15], [18], [9], [1], [6], [17], [31], [32], [26], [27], [28], [8], [22], [11], [16], [30], [10], [33], [12], [13], [29], and [7].

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1. Open and Closed

Let X be a real normed space. One can check that there exists a subset of X which is open and closed.

Now we state the proposition:

(1) Let us consider a real normed space X and a subset R of X. Then R is closed if and only if $R^{\rm c}$ is open.

Let X be a real normed space and R be a closed subset of X. Let us observe that $R^{\rm c}$ is open.

Now we state the proposition:

(2) Let us consider a real normed space X and a subset R of X. Then R is open if and only if $R^{\rm c}$ is closed.

Let X be a real normed space and R be an open subset of X. Let us observe that $R^{\rm c}$ is closed and Ω_X is closed and $\emptyset_X$ is closed and Ω_X is open and $\emptyset_X$ is open.

Let P, Q be closed subsets of X. Note that $P \cap Q$ is closed as a subset of X and $P \cup Q$ is closed as a subset of X.

Let P, Q be open subsets of X. Let us observe that $P \cap Q$ is open as a subset of X and $P \cup Q$ is open as a subset of X.

Let Y be a subset of X. The functor $\overline{Y}$ yielding a subset of X is defined by

(Def. 1) there exists a subset Z of LinearTopSpaceNorm X such that $Z = Y$ and $it = \overline{Z}$.

One can verify that $\overline{Y}$ is closed.

Now we state the propositions:

(3) Let us consider a real normed space X, a subset Y of X, and a subset Z of LinearTopSpaceNorm X. If $Y = Z$, then $\overline{Y} = \overline{Z}$.

(4) Let us consider a real normed space X and a subset Z of X. Then $Z \subseteq \overline{Z}$. The theorem is a consequence of (3).

Let us consider a real normed space X, a subset Y of X, and an object v. Now we state the propositions:

(5) If $v \in$ the carrier of X, then $v \in \overline{Y}$ iff for every subset G of X such that G is open and $v \in G$ holds G meets Y.

Proof: Reconsider $Z = Y$ as a subset of LinearTopSpaceNorm X. For every subset G_0 of LinearTopSpaceNorm X such that G_0 is open and $v \in G_0$ holds G_0 meets Z by [9, (33)]. □

(6) $v \in \overline{Y}$ if and only if there exists a sequence s_2 of X such that $\operatorname{rng} s_2 \subseteq Y$ and s_2 is convergent and $\lim s_2 = v$.

PROOF: Reconsider $Z = Y$ as a subset of LinearTopSpaceNorm X. $\overline{Z} = \overline{Y}$. For every subset G of LinearTopSpaceNorm X such that G is open and $v \in G$ holds G meets Z by [9, (22)], [18, (7)], [5, (4)]. □

(7) Let us consider a real normed space X and a subset A of X. Then there exists a family F of subsets of X such that

(i) for every subset C of X, $C \in F$ iff C is closed and $A \subseteq C$, and

(ii) $\overline{A} = \bigcap F$.

PROOF: Reconsider $B = A$ as a subset of LinearTopSpaceNorm X. Consider G being a family of subsets of LinearTopSpaceNorm X such that for every subset C of LinearTopSpaceNorm X, $C \in G$ iff C is closed and $B \subseteq C$ and $\overline{B} = \bigcap G$. Reconsider $F = G$ as a family of subsets of X. For every subset C of X, $C \in F$ iff C is closed and $A \subseteq C$ by [9, (32)]. □

Let us consider a real normed space X and subsets A, B of X. Now we state the propositions:

(8) If $A \subseteq B$, then $\overline{A} \subseteq \overline{B}$. The theorem is a consequence of (3).

(9) $\overline{A \cup B} = \overline{A} \cup \overline{B}$. The theorem is a consequence of (3).

(10) $\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$. The theorem is a consequence of (3).

Let us consider a real normed space X and a subset A of X. Now we state the propositions:

(11) A is closed if and only if $\overline{A} = A$. The theorem is a consequence of (3).

(12) A is open if and only if $\overline{\Omega_X \setminus A} = \Omega_X \setminus A$. The theorem is a consequence of (3).

(13) Let us consider a real normed space X, a subspace Y of X, and a subset C_3 of X. Suppose $C_3 =$ the carrier of Y. Then $\overline{C_3}$ is linearly closed.
PROOF: For every points v, u of X such that $v, u \in \overline{C_3}$ holds $v+u \in \overline{C_3}$ by (6), [5, (11), (4)], [26, (20)]. For every real number r and for every point v of X such that $v \in \overline{C_3}$ holds $r \cdot v \in \overline{C_3}$ by (6), [18, (22), (28)], [5, (11), (4)]. □

2. DENSITY

Let X be a real normed space and A be a subset of X. We say that A is dense if and only if

(Def. 2) $\overline{A} = \Omega_X$.

One can check that Ω_X is dense and there exists a subset of X which is open, closed, and dense.

Now we state the propositions:

(14) Let us consider a real normed space X and a subset A of X. Then A is dense if and only if for every point x of X, there exists a sequence s_2 of X such that $\operatorname{rng} s_2 \subseteq A$ and s_2 is convergent and $\lim s_2 = x$. The theorem is a consequence of (6).

(15) Let us consider a real normed space X, a subset Y of X, and a subset Z of LinearTopSpaceNorm X. If $Y = Z$, then Y is dense iff Z is dense. The theorem is a consequence of (3).

(16) Let us consider a real normed space X and subsets R, S of X. If R is dense and $R \subseteq S$, then S is dense. The theorem is a consequence of (15).

(17) Let us consider a real normed space X and a subset R of X. Then R is dense if and only if for every subset S of X such that $S \neq \emptyset$ and S is open holds R meets S.

PROOF: Reconsider $R_1 = R$ as a subset of LinearTopSpaceNorm X. For every subset S_1 of LinearTopSpaceNorm X such that $S_1 \neq \emptyset$ and S_1 is open holds R_1 meets S_1 by [9, (33)]. $\square$

Let us consider a real normed space X and subsets R, S of X. Now we state the propositions:

(18) If R is dense and S is open, then $\overline{S} = \overline{S \cap R}$. The theorem is a consequence of (3) and (15).

(19) If R is dense and S is dense and open, then $R \cap S$ is dense. The theorem is a consequence of (15).

(20) Let us consider a real normed space X and a subset A of X. If A is dense, then A is not empty. The theorem is a consequence of (17).

3. Separability

Let X be a real normed space. We say that X is separable if and only if

(Def. 3) LinearTopSpaceNorm X is separable.

(21) Let us consider a real normed space X. Then X is separable if and only if there exists a sequence s_2 of X such that $\operatorname{rng} s_2$ is dense. The theorem is a consequence of (15) and (20).

4. Sequence and Convergence

(22) Let us consider real numbers x, y, z. Suppose $0 \leqslant y$ and for every real number e such that $0 < e$ holds $x \leqslant z + y \cdot e$. Then $x \leqslant z$.

(23) Let us consider a real normed space X, a point x of X, and a sequence s_2 of X. Suppose for every natural number n, $s_2(n) = x$. Then

(i) s_2 is convergent, and

(ii) $\lim s_2 = x$.

(24) Let us consider a real normed space X and a point x of X. Then $\{x\}$ is closed.
PROOF: For every sequence s_1 of X such that $\operatorname{rng} s_1 \subseteq \{x\}$ and s_1 is convergent holds $\lim s_1 \in \{x\}$ by [5, (4)], (23). $\square$

(25) Let us consider a real normed space X, a subset Y of X, and a vector v of X. Suppose Y is closed and for every real number e such that $0 < e$ there exists a vector w of X such that $w \in Y$ and $\|v - w\| \leqslant e$. Then $v \in Y$.

5. Subspace

Now we state the propositions:

(26) Let us consider a real normed space V and a subreal normal space W of V. Suppose the carrier of W = the carrier of V. Then the normed structure of W = the normed structure of V.

(27) Let us consider a real normed space V. Then every subreal normal space of V is a subspace of V.

(28) Let us consider a real normed space V, a subreal normal space V_1 of V, points x, y of V, points x_1, y_1 of V_1, and a real number a. Suppose $x = x_1$ and $y = y_1$. Then

(i) $\|x\| = \|x_1\|$, and

(ii) $x + y = x_1 + y_1$, and

(iii) $a \cdot x = a \cdot x_1$.

(29) Let us consider a real normed space V, a subreal normal space V_1 of V, and a subset S of V. Suppose S = the carrier of V_1. Then S is linearly closed. The theorem is a consequence of (28).

Let X be a real normed space and X_1 be a set. Assume $X_1 \subseteq$ the carrier of X. The norm of X_1 induced by X yielding a function from X_1 into $\mathbb{R}$ is defined by the term

(Def. 4) (the norm of X)$\restriction X_1$.

Let V be a real normed space and V_1 be a subset of V. The functor $\mathrm{NLin}(V_1)$ yielding a non empty normed structure is defined by the term

(Def. 5) $\langle$the carrier of $\mathrm{Lin}(V_1)$, $0_{\mathrm{Lin}(V_1)}$, the addition of $\mathrm{Lin}(V_1)$, the external multiplication of $\mathrm{Lin}(V_1)$, the norm of the carrier of $\mathrm{Lin}(V_1)$ induced by $V\rangle$.

Now we state the proposition:

(30) Let us consider a real normed space V and a subset V_1 of V. Then $\mathrm{NLin}(V_1)$ is a subreal normal space of V.

Let V be a real normed space and V_1 be a subset of V. Let us observe that the functor $\mathrm{NLin}(V_1)$ yields a subreal normal space of V. Now we state the propositions:

(31) Let us consider a real linear space V and a subset V_1 of V. Suppose $V_1 \neq \emptyset$ and V_1 is linearly closed. Then the carrier of $\mathrm{Lin}(V_1) = V_1$.

(32) Let us consider a real normed space V, a subreal normal space W of V, and a subset V_1 of V. Suppose the carrier of $W = V_1$. Then $\mathrm{NLin}(V_1) =$ the normed structure of W. The theorem is a consequence of (31) and (29).

6. Linear Functions

Now we state the proposition:

(33) Let us consider real linear spaces X, Y and a function f from X into Y. If f is homogeneous, then $f^{-1}(\{0_Y\})$ is not empty.

Let X, Y be real linear spaces and f be a linear operator from X into Y. One can verify that $f^{-1}(\{0_Y\})$ is non empty.

Let us consider real linear spaces X, Y and a function f from X into Y.

Let us assume that f is additive and homogeneous. Now we state the propositions:

(34) $f^{-1}(\{0_Y\})$ is linearly closed.
Proof: Set $X_1 = f^{-1}(\{0_Y\})$. For every points v, u of X such that v, $u \in X_1$ holds $v + u \in X_1$ by [5, (38)], [27, (4)]. For every real number r and for every point v of X such that $v \in X_1$ holds $r \cdot v \in X_1$ by [5, (38)], [27, (10)]. □

(35) $\mathrm{rng}\, f$ is linearly closed.
Proof: Set $Y_1 = \mathrm{rng}\, f$. For every points v, u of Y such that v, $u \in Y_1$ holds $v + u \in Y_1$ by [5, (113), (4)]. For every real number r and for every point v of Y such that $v \in Y_1$ holds $r \cdot v \in Y_1$ by [5, (113), (4)]. □

Let X, Y be real linear spaces and f be a linear operator from X into Y. The functor $\mathrm{Ker}\, f$ yielding a subspace of X is defined by the term

(Def. 6) $\mathrm{Lin}(f^{-1}(\{0_Y\}))$.

Let X, Y be real normed spaces. The functor $\mathrm{NKer}\, f$ yielding a subreal normal space of X is defined by the term

(Def. 7) $\mathrm{NLin}((f^{-1}(\{0_Y\})))$.

Let X, Y be real linear spaces. The functor $\Im(f)$ yielding a subspace of Y is defined by the term

(Def. 8) $\mathrm{Lin}(\mathrm{rng}\, f)$.

Let X, Y be real normed spaces. The functor $\Im(f)$ yielding a subreal normal space of Y is defined by the term

(Def. 9) $\mathrm{NLin}(\mathrm{rng}\, f)$.

Let X, Y be real linear spaces and L be a linear operator from X into Y. We say that L is isomorphism if and only if

(Def. 10) L is one-to-one and onto.

One can check that every linear operator from X into Y which is isomorphism is also one-to-one and onto and every linear operator from X into Y which is one-to-one and onto is also isomorphism.

Now we state the proposition:

(36) Let us consider real linear spaces X, Y and a linear operator L from X into Y. Suppose L is isomorphism. Then there exists a linear operator K from Y into X such that

(i) $K = L^{-1}$, and

(ii) K is isomorphism.

PROOF: Reconsider $K = L^{-1}$ as a function from Y into X. K is additive by [5, (113)], [4, (34)]. K is homogeneous by [5, (113)], [4, (34)]. □

Let X, Y be real normed spaces and L be a linear operator from X into Y. We say that L is isomorphism if and only if

(Def. 11) L is one-to-one and onto and for every point x of X, $\|x\| = \|L(x)\|$.

Let us note that every linear operator from X into Y which is isomorphism is also one-to-one and onto.

Now we state the propositions:

(37) Let us consider real normed spaces X, Y and a linear operator L from X into Y. Suppose L is isomorphism. Then there exists a Lipschitzian linear operator K from Y into X such that

(i) $K = L^{-1}$, and

(ii) K is isomorphism.

PROOF: Reconsider $K = L^{-1}$ as a function from Y into X. K is additive by [5, (113)], [4, (34)]. K is homogeneous by [5, (113)], [4, (34)]. For every point y of Y, $\|y\| = \|K(y)\|$ by [5, (113)], [4, (34)]. □

(38) Let us consider real normed spaces X, Y, a Lipschitzian linear operator L from X into Y, and a sequence s_2 of X. Suppose s_2 is convergent. Then

(i) $L \cdot s_2$ is convergent, and

(ii) $\lim(L \cdot s_2) = L(\lim s_2)$.

(39) Let us consider real normed spaces X, Y, a function L from X into Y, and a point w of Y. Suppose L is continuous on the carrier of X. Then $L^{-1}(\{w\})$ is closed.

PROOF: For every sequence s_2 of X such that $\operatorname{rng} s_2 \subseteq L^{-1}(\{w\})$ and s_2 is convergent holds $\lim s_2 \in L^{-1}(\{w\})$ by [15, (18)], [5, (4), (38), (115)]. □

(40) Let us consider real normed spaces X, Y and a Lipschitzian linear operator L from X into Y. Then

(i) the carrier of $\operatorname{Ker} L = L^{-1}(\{0_Y\})$, and

(ii) $L^{-1}(\{0_Y\})$ is closed.

Let us consider real normed spaces X, Y, a Lipschitzian linear operator L from X into Y, and a sequence s_2 of X.

Let us assume that L is isomorphism. Now we state the propositions:

(41) s_2 is convergent if and only if $L \cdot s_2$ is convergent.

PROOF: Set $L_3 = L \cdot s_2$. Consider K being a Lipschitzian linear operator from Y into X such that $K = L^{-1}$ and K is isomorphism. For every element n of $\mathbb{N}$, $(K \cdot L_3)(n) = s_2(n)$ by [4, (13), (34)]. □

(42) If s_2 is Cauchy sequence by norm, then $L \cdot s_2$ is Cauchy sequence by norm.

PROOF: Set $L_3 = L \cdot s_2$. For every real number r such that $r > 0$ there exists a natural number k such that for every natural numbers n, m such that $n \geqslant k$ and $m \geqslant k$ holds $\|L_3(n) - L_3(m)\| < r$ by [22, (8)], [4, (13)], [27, (16)]. □

(43) s_2 is Cauchy sequence by norm if and only if $L \cdot s_2$ is Cauchy sequence by norm.

PROOF: Set $L_3 = L \cdot s_2$. Consider K being a Lipschitzian linear operator from Y into X such that $K = L^{-1}$ and K is isomorphism. For every element n of $\mathbb{N}$, $(K \cdot L_3)(n) = s_2(n)$ by [4, (13), (34)]. □

Now we state the propositions:

(44) Let us consider real normed spaces X, Y. Suppose there exists a Lipschitzian linear operator L from X into Y such that L is isomorphism. Then X is complete if and only if Y is complete. The theorem is a consequence of (37), (43), and (41).

(45) Let us consider real normed spaces X, Y, a Lipschitzian linear operator L from X into Y, a subset V of X, and a subset W of Y. Suppose L is isomorphism and $W = L^{\circ}V$. Then V is closed if and only if W is closed. The theorem is a consequence of (37).

(46) Let us consider real normed spaces X, Y and a linear operator L from X into Y. Suppose L is onto. Then $\Im(L)$ = the normed structure of Y. The theorem is a consequence of (31), (35), and (26).

7. Banach Space

Now we state the propositions:

(47) Let us consider a real Banach space V and a subreal normal space V_1 of V. Suppose there exists a subset C_2 of V such that C_2 = the carrier of V_1 and C_2 is closed. Then V_1 is a real Banach space.
Proof: For every sequence s_2 of V_1 such that s_2 is Cauchy sequence by norm holds s_2 is convergent by [5, (7)], [22, (8)], [27, (16)], (28). □

(48) Let us consider a real normed space V, a subreal normal space V_1 of V, and a subset C_2 of V. Suppose V_1 is complete and C_2 = the carrier of V_1. Then C_2 is closed.
Proof: For every sequence s_1 of V such that $\operatorname{rng} s_1 \subseteq C_2$ and s_1 is convergent holds $\lim s_1 \in C_2$ by [5, (6)], [21, (4)], [27, (16)], (28). □

(49) Let us consider a real Banach space X and a non empty subset M of X. Suppose M is linearly closed and closed. Then $\mathrm{NLin}(M)$ is a real Banach space. The theorem is a consequence of (31) and (47).

8. Quotient Vector Space

Let X be a real linear space and Y be a subspace of X. Observe that the functor $\mathrm{RLSp2RVSp}\, Y$ yields a subspace of $\mathrm{RLSp2RVSp}\, X$. The functor ${}^{X}/_{Y}$ yielding a real linear space is defined by the term

(Def. 12) $\mathrm{RVSp2RLSp}({}^{\mathrm{RLSp2RVSp}\, X}/_{\mathrm{RLSp2RVSp}\, Y})$.

Now we state the propositions:

(50) Let us consider a real linear space X, an element v of X, a real number a, an element v_1 of $\mathrm{RLSp2RVSp}\, X$, and an element a_1 of $\mathbb{R}_{\mathrm{F}}$. If $v = v_1$ and $a = a_1$, then $a \cdot v = a_1 \cdot v_1$.

(51) Let us consider a vector space X over $\mathbb{R}_{\mathrm{F}}$, an element v of X, an element a of $\mathbb{R}_{\mathrm{F}}$, an element v_1 of $\mathrm{RVSp2RLSp}\, X$, and a real number a_1. If $v = v_1$ and $a = a_1$, then $a \cdot v = a_1 \cdot v_1$.

(52) Let us consider a real linear space X, a subspace Y of X, an element v of X, and an element v_1 of $\mathrm{RLSp2RVSp}\, X$. If $v = v_1$, then $v + Y = v_1 + \mathrm{RLSp2RVSp}\, Y$.

(53) Let us consider a real linear space X, a subspace Y of X, and an object x. Then x is a coset of Y if and only if x is a coset of RLSp2RVSp Y. The theorem is a consequence of (52).

Let X be a real linear space and Y be a subspace of X. The functor CosetSet(X, Y) yielding a non empty family of subsets of X is defined by the term

(Def. 13) the set of all A where A is a coset of Y.

Let V be a real linear space and W be a subspace of V. The functor zeroCoset(V, W) yielding an element of CosetSet(V, W) is defined by the term

(Def. 14) the carrier of W.

Now we state the propositions:

(54) Let us consider a real linear space X and a subspace Y of X. Then CosetSet$(X, Y) =$ CosetSet(RLSp2RVSp X, RLSp2RVSp Y). The theorem is a consequence of (53).

(55) Let us consider a real linear space V and a subspace W of V. Then the carrier of ${}^{V}/_{W} =$ CosetSet(V, W). The theorem is a consequence of (54).

(56) Let us consider a real linear space V, a subspace W of V, and an object x. Then x is a point of ${}^{V}/_{W}$ if and only if there exists a point v of V such that $x = v + W$. The theorem is a consequence of (55).

(57) Let us consider a real linear space V and a subspace W of V. Then $0_{{}^{V}/_{W}} =$ zeroCoset(V, W).

Let us consider a real linear space V, a subspace W of V, a vector A of ${}^{V}/_{W}$, a vector v of V, and a real number a.

Let us assume that $A = v + W$. Now we state the propositions:

(58) $a \cdot A = a \cdot v + W$. The theorem is a consequence of (52).

(59) $-A = -v + W$. The theorem is a consequence of (58).

Let us consider a real linear space V, a subspace W of V, vectors A_1, A_2 of ${}^{V}/_{W}$, and vectors v_1, v_2 of V.

Let us assume that $A_1 = v_1 + W$ and $A_2 = v_2 + W$. Now we state the propositions:

(60) $A_1 + A_2 = v_1 + v_2 + W$. The theorem is a consequence of (52).

(61) $A_1 - A_2 = v_1 - v_2 + W$. The theorem is a consequence of (59) and (60).

Let us consider a real linear space V and a subspace W of V. Now we state the propositions:

(62) (i) $0_{{}^{V}/_{W}} =$ the carrier of W, and

(ii) $0_{{}^{V}/_{W}} = 0_V + W$.

The theorem is a consequence of (57).

(63) There exists a linear operator Q_2 from V into ${}^{V}/_{W}$ such that

(i) Q_2 is onto, and

(ii) for every vector v of V, $Q_2(v) = v + W$.

Proof: Define $\mathcal{P}[\text{vector of } V, \text{object}] \equiv \$_2 = \$_1 + W$. For every element x of the carrier of V, there exists an element y of the carrier of ${}^{V}/_{W}$ such that $\mathcal{P}[x, y]$. Consider Q_2 being a function from the carrier of V into ${}^{V}/_{W}$ such that for every element x of V, $\mathcal{P}[x, Q_2(x)]$ from [5, Sch. 3]. For every elements v, w of V, $Q_2(v + w) = Q_2(v) + Q_2(w)$. For every vector v of V and for every real number r, $Q_2(r \cdot v) = r \cdot Q_2(v)$. For every object v such that $v \in$ the carrier of ${}^{V}/_{W}$ there exists an object s such that $s \in$ the carrier of V and $v = Q_2(s)$. □

Let V be a real linear space and W be a subspace of V. The surjection induced by (V, W) yielding a linear operator from V into ${}^{V}/_{W}$ is defined by

(Def. 15) *it* is onto and for every vector v of V, $it(v) = v + W$.

Now we state the proposition:

(64) Let us consider real linear spaces V, W and a linear operator L from V into W. Then there exists a linear operator Q_2 from ${}^{V}/_{\operatorname{Ker} L}$ into $\Im(L)$ such that

(i) Q_2 is isomorphism, and

(ii) for every point z of ${}^{V}/_{\operatorname{Ker} L}$ and for every vector v of V such that $z = v + \operatorname{Ker} L$ holds $Q_2(z) = L(v)$.

Proof: The carrier of $\Im(L) = \operatorname{rng} L$. The carrier of $\operatorname{Ker} L = L^{-1}(\{0_W\})$. Define $\mathcal{P}[\text{object}, \text{object}] \equiv$ there exists a vector v of V such that $\$_1 = v + \operatorname{Ker} L$ and $\$_2 = L(v)$. For every element x of the carrier of ${}^{V}/_{\operatorname{Ker} L}$, there exists an element y of the carrier of $\Im(L)$ such that $\mathcal{P}[x, y]$ by (56), [5, (4)]. Consider Q_2 being a function from the carrier of ${}^{V}/_{\operatorname{Ker} L}$ into the carrier of $\Im(L)$ such that for every element x of ${}^{V}/_{\operatorname{Ker} L}$, $\mathcal{P}[x, Q_2(x)]$ from [5, Sch. 3]. For every point z of ${}^{V}/_{\operatorname{Ker} L}$ and for every vector v of V such that $z = v + \operatorname{Ker} L$ holds $Q_2(z) = L(v)$ by [26, (54), (63)], [27, (28), (15), (4)]. For every objects x_1, x_2 such that x_1, $x_2 \in$ the carrier of ${}^{V}/_{\operatorname{Ker} L}$ and $Q_2(x_1) = Q_2(x_2)$ holds $x_1 = x_2$ by [27, (16), (15)], [5, (38)], [27, (29), (13)]. For every object v such that $v \in$ the carrier of $\Im(L)$ there exists an object s such that $s \in$ the carrier of ${}^{V}/_{\operatorname{Ker} L}$ and $v = Q_2(s)$ by (35), (31), [5, (11)], (56). For every elements v, w of ${}^{V}/_{\operatorname{Ker} L}$, $Q_2(v+w) = Q_2(v)+Q_2(w)$ by (56), (60), [26, (13)]. For every vector v of ${}^{V}/_{\operatorname{Ker} L}$ and for every real number r, $Q_2(r \cdot v) = r \cdot Q_2(v)$ by (56), (58), [26, (14)]. □

Let V, W be real linear spaces and L be a linear operator from V into W. The bijection induced by (V, W, L) yielding a linear operator from ${}^{V}/_{\mathrm{Ker}\,L}$ into $\Im(L)$ is defined by

(Def. 16) *it* is isomorphism and for every point z of ${}^{V}/_{\mathrm{Ker}\,L}$ and for every vector v of V such that $z = v + \mathrm{Ker}\,L$ holds $it(z) = L(v)$.

Now we state the proposition:

(65) Let us consider real linear spaces V, W and a linear operator L from V into W. Then $L =$ (the bijection induced by (V, W, L)) $\cdot$ (the surjection induced by $(V, \mathrm{Ker}\,L)$). The theorem is a consequence of (56).

Let V be a real normed space, W be a subspace of V, and v be a vector of V. The functor $\mathrm{NormVSets}(V, W, v)$ yielding a non empty subset of $\mathbb{R}$ is defined by the term

(Def. 17) $\{\|x\|$, where x is a vector of $V : x \in v + W\}$.

Let us observe that $\mathrm{NormVSets}(V, W, v)$ is non empty and lower bounded.

Now we state the proposition:

(66) Let us consider a real normed space V, a subspace W of V, and a vector v of V. Then $0 \leqslant \inf \mathrm{NormVSets}(V, W, v) \leqslant \|v\|$.

Let V be a real normed space and W be a subspace of V. The functor $\mathrm{NormCoset}(V, W)$ yielding a function from $\mathrm{CosetSet}(V, W)$ into $\mathbb{R}$ is defined by

(Def. 18) for every element A of $\mathrm{CosetSet}(V, W)$ and for every vector v of V such that $A = v + W$ holds $it(A) = \inf \mathrm{NormVSets}(V, W, v)$.

Let X be a real normed space and Y be a subspace of X. Assume there exists a subset C_3 of X such that $C_3 =$ the carrier of Y and C_3 is closed. The functor $\mathrm{NVectQuot}(X, Y)$ yielding a strict real normed space is defined by

(Def. 19) the RLS structure of $it = {}^{X}/_{Y}$ and the norm of $it = \mathrm{NormCoset}(X, Y)$.

Now we state the proposition:

(67) Let us consider real normed spaces V, W and a Lipschitzian linear operator L from V into W. Then there exists a Lipschitzian linear operator Q_2 from $\mathrm{NVectQuot}(V, (\mathrm{Ker}\,L))$ into $\Im(L)$ and there exists a point P_3 of the real norm space of bounded linear operators from $\mathrm{NVectQuot}(V, (\mathrm{Ker}\,L))$ into $\Im(L)$ and there exists a point P_2 of the real norm space of bounded linear operators from V into W such that Q_2 is onto and one-to-one and $L = P_2$ and $Q_2 = P_3$ and $\|P_2\| = \|P_3\|$ and for every point z of $\mathrm{NVectQuot}(V, (\mathrm{Ker}\,L))$ and for every vector v of V such that $z = v + \mathrm{Ker}\,L$ holds $Q_2(z) = L(v)$.

PROOF: the carrier of $\mathrm{Ker}\,L = L^{-1}(\{0_W\})$ and $L^{-1}(\{0_W\})$ is closed. Reconsider $V_1 = V$ as a real linear space. Reconsider $W_1 = W$ as a real linear space. Reconsider $L_1 = L$ as a linear operator from V_1 into W_1.

The carrier of ${}^{V}/_{\mathrm{Ker}\,L}$ = CosetSet(V, Ker L). Consider Q_3 being a linear operator from ${}^{V_1}/_{\mathrm{Ker}\,L_1}$ into $\Im(L_1)$ such that Q_3 is isomorphism and for every point z of ${}^{V_1}/_{\mathrm{Ker}\,L_1}$ and for every vector v of V_1 such that $z = v + \mathrm{Ker}\,L_1$ holds $Q_3(z) = L_1(v)$. Reconsider $Q_2 = Q_3$ as a function from NVectQuot(V, (Ker L)) into $\Im(L)$. For every elements v, w of NVectQuot(V, (Ker L)), $Q_2(v+w) = Q_2(v) + Q_2(w)$. For every vector v of NVectQuot(V, (Ker L)) and for every real number r, $Q_2(r \cdot v) = r \cdot Q_2(v)$. Reconsider $P_2 = L$ as a point of the real norm space of bounded linear operators from V into W. For every point v of NVectQuot(V, (Ker L)), $\|Q_2(v)\| \leqslant \|P_2\| \cdot \|v\|$ by (56), [20, (31)], [24, (7)], (28). Reconsider $P_3 = Q_2$ as a point of the real norm space of bounded linear operators from NVectQuot(V, (Ker L)) into $\Im(L)$. $\|P_2\| \leqslant \|P_3\|$. □

9. Closure

Let X be a real normed space and Y be a subset of X. The functor ClNLin(Y) yielding a non empty normed structure is defined by

(Def. 20) there exists a subset Z of X such that Z = the carrier of Lin(Y) and *it* = $\langle \overline{Z}, \mathrm{Zero}(\overline{Z}, X), \mathrm{Add}(\overline{Z}, X), \mathrm{Mult}(\overline{Z}, X)$, the norm of $\overline{Z}$ induced by $X\rangle$.

Now we state the propositions:

(68) Let us consider a real normed space X, a subset V_1 of X, and a subset C_1 of X. Suppose C_1 = the carrier of ClNLin(V_1). Then $\langle C_1, \mathrm{Zero}(C_1, X), \mathrm{Add}(C_1, X), \mathrm{Mult}(C_1, X)\rangle$ is a subspace of X. The theorem is a consequence of (13).

(69) Let us consider a real normed space X, a subset Y of X, points f, g of ClNLin(Y), and a real number a. Then

(i) $\|f\| = 0$ iff $f = 0_{\mathrm{ClNLin}(Y)}$, and

(ii) $\|a \cdot f\| = |a| \cdot \|f\|$, and

(iii) $\|f + g\| \leqslant \|f\| + \|g\|$.

The theorem is a consequence of (13).

Let X be a real normed space and Y be a subset of X. Let us observe that ClNLin(Y) is reflexive, discernible, and real normed space-like.

Now we state the proposition:

(70) Let us consider a real normed space V and a subset V_1 of V. Then ClNLin(V_1) is a real normed space. The theorem is a consequence of (68).

Let X be a real normed space and Y be a subset of X. Let us observe that ClNLin(Y) is reflexive, discernible, real normed space-like, vector distributive, scalar distributive, scalar associative, scalar unital, Abelian, add-associative, right zeroed, and right complementable.

Now we state the proposition:

(71) Let us consider a real normed space V and a subset V_1 of V. Then ClNLin(V_1) is a subreal normal space of V. The theorem is a consequence of (13).

Let V be a real normed space and V_1 be a subset of V. One can verify that the functor ClNLin(V_1) yields a subreal normal space of V.

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