

Difference of Function on Vector Space over \mathbb{F}^1

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Summary. In [11], the definitions of forward difference, backward difference, and central difference as difference operations for functions on \mathbb{R} were formalized. However, the definitions of forward difference, backward difference, and central difference for functions on vector spaces over \mathbb{F} have not been formalized. In cryptology, these definitions are very important in evaluating the security of cryptographic systems [3], [10]. Differential cryptanalysis [4] that undertakes a general purpose attack against block ciphers [13] can be formalized using these definitions. In this article, we formalize the definitions of forward difference, backward difference, and central difference for functions on vector spaces over \mathbb{F} . Moreover, we formalize some facts about these definitions.

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The notation and terminology used in this paper have been introduced in the following articles: [12], [15], [5], [6], [16], [1], [2], [7], [19], [20], [17], [14], [18], [9], [21], and [8].

From now on C denotes a non empty set, G_1 denotes a field, V denotes a vector space over G_1 , v , u denote elements of V , W denotes a subset of V , and f , f_1 , f_2 , f_3 denote partial functions from C to V .

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Let us consider $C, G_1,$ and V . Let f be a partial function from C to V and r be an element of G_1 . The functor $r \cdot f$ yielding a partial function from C to V is defined by

(Def. 1) $\text{dom } it = \text{dom } f$ and for every element c of C such that $c \in \text{dom } it$ holds $it_c = r \cdot f_c$.

Let f be a function from C into V . One can check that $r \cdot f$ is total.

Let us consider v and W . The functor $v \oplus W$ yielding a subset of V is defined by the term

(Def. 2) $\{v + u : u \in W\}$.

Let F, G be fields, V be a vector space over F, W be a vector space over G, f be a partial function from V to $W,$ and h be an element of V . The functor $\text{Shift}(f, h)$ yielding a partial function from V to W is defined by

(Def. 3) $\text{dom } it = -h \oplus \text{dom } f$ and for every element x of V such that $x \in -h \oplus \text{dom } f$ holds $it(x) = f(x + h)$.

Now we state the proposition:

(1) Let us consider an element x of V and a subset A of V . If $A =$ the carrier of $V,$ then $x \oplus A = A$.

PROOF: For every object $y, y \in x \oplus A$ iff $y \in A$ by [17, (29), (15), (13)]. \square

Let F, G be fields, V be a vector space over F, W be a vector space over G, f be a function from V into $W,$ and h be an element of V . One can verify that the functor $\text{Shift}(f, h)$ yields a function from V into W and is defined by

(Def. 4) for every element x of $V, it(x) = f(x + h)$.

Let f be a partial function from V to W . The functor $\Delta_h[f]$ yielding a partial function from V to W is defined by the term

(Def. 5) $\text{Shift}(f, h) - f$.

Let f be a function from V into W . Observe that $\Delta_h[f]$ is quasi total.

Let f be a partial function from V to W . The functor $\nabla_h[f]$ yielding a partial function from V to W is defined by the term

(Def. 6) $f - \text{Shift}(f, -h)$.

Let f be a function from V into W . Let us note that $\nabla_h[f]$ is quasi total.

Let f be a partial function from V to W . The functor $\delta_h[f]$ yielding a partial function from V to W is defined by the term

(Def. 7) $\text{Shift}(f, (2 \cdot 1_F)^{-1} \cdot h) - \text{Shift}(f, -(2 \cdot 1_F)^{-1} \cdot h)$.

Let f be a function from V into W . One can check that $\delta_h[f]$ is quasi total.

The forward difference of f and h yielding a sequence of partial functions from the carrier of V into the carrier of W is defined by

(Def. 8) $it(0) = f$ and for every natural number $n, it(n + 1) = \Delta_h[it(n)]$.

We introduce $\vec{\Delta}_h[f]$ as a synonym of the forward difference of f and h .

From now on F, G denote fields, V denotes a vector space over F , W denotes a vector space over G , f, f_1, f_2 denote functions from V into W , x, h denote elements of V , and r, r_1, r_2 denote elements of G .

Now we state the propositions:

(2) Let us consider a partial function f from V to W . If $x, x + h \in \text{dom } f$, then $(\Delta_h[f])_x = f_{x+h} - f_x$.

(3) Let us consider a natural number n . Then $(\vec{\Delta}_h[f])(n)$ is a function from V into W .

PROOF: Define $\mathcal{X}[\text{natural number}] \equiv (\vec{\Delta}_h[f])(\$1)$ is a function from V into W . For every natural number k such that $\mathcal{X}[k]$ holds $\mathcal{X}[k + 1]$. For every natural number n , $\mathcal{X}[n]$ from [1, Sch. 2]. \square

(4) $(\Delta_h[f])_x = f_{x+h} - f_x$. The theorem is a consequence of (2).

(5) $(\nabla_h[f])_x = f_x - f_{x-h}$.

(6) $(\delta_h[f])_x = f_{x+(2 \cdot 1_F)^{-1} \cdot h} - f_{x-(2 \cdot 1_F)^{-1} \cdot h}$.

From now on n, m, k denote natural numbers.

Now we state the propositions:

(7) If f is constant, then for every x , $(\vec{\Delta}_h[f])(n + 1)_x = 0_W$.

PROOF: For every x , $f_{x+h} - f_x = 0_W$ by [17, (15)]. For every x , $(\vec{\Delta}_h[f])(n + 1)_x = 0_W$ by (3), (4), [17, (15)]. \square

(8) $(\vec{\Delta}_h[r \cdot f])(n + 1)_x = r \cdot (\vec{\Delta}_h[f])(n + 1)_x$.

PROOF: Define $\mathcal{X}[\text{natural number}] \equiv$ for every x , $(\vec{\Delta}_h[r \cdot f])(\$1 + 1)_x = r \cdot (\vec{\Delta}_h[f])(\$1 + 1)_x$. For every k such that $\mathcal{X}[k]$ holds $\mathcal{X}[k + 1]$ by (3), (4), [9, (23)]. $\mathcal{X}[0]$ by (4), [9, (23)]. For every n , $\mathcal{X}[n]$ from [1, Sch. 2]. \square

(9) $(\vec{\Delta}_h[f_1 + f_2])(n + 1)_x = (\vec{\Delta}_h[f_1])(n + 1)_x + (\vec{\Delta}_h[f_2])(n + 1)_x$.

PROOF: Define $\mathcal{X}[\text{natural number}] \equiv$ for every x , $(\vec{\Delta}_h[f_1 + f_2])(\$1 + 1)_x = (\vec{\Delta}_h[f_1])(\$1 + 1)_x + (\vec{\Delta}_h[f_2])(\$1 + 1)_x$. For every k such that $\mathcal{X}[k]$ holds $\mathcal{X}[k + 1]$ by (3), (4), [17, (27), (28)]. $\mathcal{X}[0]$ by (4), [17, (27), (28)]. For every n , $\mathcal{X}[n]$ from [1, Sch. 2]. \square

(10) $(\vec{\Delta}_h[f_1 - f_2])(n + 1)_x = (\vec{\Delta}_h[f_1])(n + 1)_x - (\vec{\Delta}_h[f_2])(n + 1)_x$.

PROOF: Define $\mathcal{X}[\text{natural number}] \equiv$ for every x , $(\vec{\Delta}_h[f_1 - f_2])(\$1 + 1)_x = (\vec{\Delta}_h[f_1])(\$1 + 1)_x - (\vec{\Delta}_h[f_2])(\$1 + 1)_x$. $\mathcal{X}[0]$ by (4), [17, (29), (27)]. For every k such that $\mathcal{X}[k]$ holds $\mathcal{X}[k + 1]$ by (3), (4), [17, (29)]. For every n , $\mathcal{X}[n]$ from [1, Sch. 2]. \square

(11) $(\vec{\Delta}_h[r_1 \cdot f_1 + r_2 \cdot f_2])(n + 1)_x = r_1 \cdot (\vec{\Delta}_h[f_1])(n + 1)_x + r_2 \cdot (\vec{\Delta}_h[f_2])(n + 1)_x$.

The theorem is a consequence of (3), (9), and (8).

(12) $(\vec{\Delta}_h[f])(1)_x = (\text{Shift}(f, h))_x - f_x$. The theorem is a consequence of (4).

Let F, G be fields, V be a vector space over F , h be an element of V , W be a vector space over G , and f be a function from V into W . The backward difference of f and h yielding a sequence of partial functions from the carrier of V into the carrier of W is defined by

(Def. 9) $it(0) = f$ and for every natural number n , $it(n + 1) = \nabla_h[it(n)]$.

The backward difference of f and h yielding a sequence of partial functions from the carrier of V into the carrier of W is defined by

(Def. 10) $it(0) = f$ and for every natural number n , $it(n + 1) = \nabla_h[it(n)]$.

We introduce $\vec{\nabla}_h[f]$ as a synonym of the backward difference of f and h .

Now we state the propositions:

(13) Let us consider a natural number n . Then $(\vec{\nabla}_h[f])(n)$ is a function from V into W .

PROOF: Define $\mathcal{X}[\text{natural number}] \equiv (\vec{\nabla}_h[f])(\$_1)$ is a function from V into W . For every natural number k such that $\mathcal{X}[k]$ holds $\mathcal{X}[k + 1]$. For every natural number n , $\mathcal{X}[n]$ from [1, Sch. 2]. \square

(14) If f is constant, then for every x , $(\vec{\nabla}_h[f])(n + 1)_x = 0_W$.

PROOF: For every x , $f_x - f_{x-h} = 0_W$ by [17, (15)]. For every x , $(\vec{\nabla}_h[f])(n + 1)_x = 0_W$ by (13), (5), [17, (15)]. \square

(15) $(\vec{\nabla}_h[r \cdot f])(n + 1)_x = r \cdot (\vec{\nabla}_h[f])(n + 1)_x$.

PROOF: Define $\mathcal{X}[\text{natural number}] \equiv$ for every x , $(\vec{\nabla}_h[r \cdot f])(\$_1 + 1)_x = r \cdot (\vec{\nabla}_h[f])(\$_1 + 1)_x$. For every k such that $\mathcal{X}[k]$ holds $\mathcal{X}[k + 1]$ by (13), (5), [9, (23)]. $\mathcal{X}[0]$ by (5), [9, (23)]. For every n , $\mathcal{X}[n]$ from [1, Sch. 2]. \square

(16) $(\vec{\nabla}_h[f_1 + f_2])(n + 1)_x = (\vec{\nabla}_h[f_1])(n + 1)_x + (\vec{\nabla}_h[f_2])(n + 1)_x$.

PROOF: Define $\mathcal{X}[\text{natural number}] \equiv$ for every x , $(\vec{\nabla}_h[f_1 + f_2])(\$_1 + 1)_x = (\vec{\nabla}_h[f_1])(\$_1 + 1)_x + (\vec{\nabla}_h[f_2])(\$_1 + 1)_x$. For every k such that $\mathcal{X}[k]$ holds $\mathcal{X}[k + 1]$ by (13), (5), [17, (27), (28)]. $\mathcal{X}[0]$ by (5), [17, (27), (28)]. For every n , $\mathcal{X}[n]$ from [1, Sch. 2]. \square

(17) $(\vec{\nabla}_h[f_1 - f_2])(n + 1)_x = (\vec{\nabla}_h[f_1])(n + 1)_x - (\vec{\nabla}_h[f_2])(n + 1)_x$.

PROOF: Define $\mathcal{X}[\text{natural number}] \equiv$ for every x , $(\vec{\nabla}_h[f_1 - f_2])(\$_1 + 1)_x = (\vec{\nabla}_h[f_1])(\$_1 + 1)_x - (\vec{\nabla}_h[f_2])(\$_1 + 1)_x$. $\mathcal{X}[0]$ by (5), [17, (29), (27)]. For every k such that $\mathcal{X}[k]$ holds $\mathcal{X}[k + 1]$ by (13), (5), [17, (29), (27)]. For every n , $\mathcal{X}[n]$ from [1, Sch. 2]. \square

(18) $(\vec{\nabla}_h[r_1 \cdot f_1 + r_2 \cdot f_2])(n + 1)_x = r_1 \cdot (\vec{\nabla}_h[f_1])(n + 1)_x + r_2 \cdot (\vec{\nabla}_h[f_2])(n + 1)_x$.

The theorem is a consequence of (16) and (15).

(19) $(\vec{\nabla}_h[f])(1)_x = f_x - (\text{Shift}(f, -h))_x$. The theorem is a consequence of (5).

Let F, G be fields, V be a vector space over F , h be an element of V , W be a vector space over G , and f be a partial function from V to W . The central

difference of f and h yielding a sequence of partial functions from the carrier of V into the carrier of W is defined by

(Def. 11) $it(0) = f$ and for every natural number n , $it(n + 1) = \delta_h[it(n)]$.

We introduce $\vec{\delta}_h[f]$ as a synonym of the central difference of f and h .

Now we state the propositions:

(20) Let us consider a natural number n . Then $(\vec{\delta}_h[f])(n)$ is a function from V into W .

PROOF: Define $\mathcal{X}[\text{natural number}] \equiv (\vec{\delta}_h[f])(\$_1)$ is a function from V into W . For every natural number k such that $\mathcal{X}[k]$ holds $\mathcal{X}[k + 1]$. For every natural number n , $\mathcal{X}[n]$ from [1, Sch. 2]. \square

(21) If f is constant, then for every x , $(\vec{\delta}_h[f])(n + 1)_x = 0_W$.

PROOF: Define $\mathcal{X}[\text{natural number}] \equiv$ for every x , $(\vec{\delta}_h[f])(\$_1 + 1)_x = 0_W$. For every x , $f_{x+(2 \cdot 1_F)^{-1} \cdot h} - f_{x-(2 \cdot 1_F)^{-1} \cdot h} = 0_W$ by [17, (15)]. $\mathcal{X}[0]$. For every k such that $\mathcal{X}[k]$ holds $\mathcal{X}[k + 1]$ by (20), (6), [17, (13)]. For every n , $\mathcal{X}[n]$ from [1, Sch. 2]. \square

(22) $(\vec{\delta}_h[r \cdot f])(n + 1)_x = r \cdot (\vec{\delta}_h[f])(n + 1)_x$.

PROOF: Define $\mathcal{X}[\text{natural number}] \equiv$ for every x , $(\vec{\delta}_h[r \cdot f])(\$_1 + 1)_x = r \cdot (\vec{\delta}_h[f])(\$_1 + 1)_x$. For every k such that $\mathcal{X}[k]$ holds $\mathcal{X}[k + 1]$ by (20), (6), [9, (23)]. $\mathcal{X}[0]$ by (6), [9, (23)]. For every n , $\mathcal{X}[n]$ from [1, Sch. 2]. \square

(23) $(\vec{\delta}_h[f_1 + f_2])(n + 1)_x = (\vec{\delta}_h[f_1])(n + 1)_x + (\vec{\delta}_h[f_2])(n + 1)_x$.

PROOF: Define $\mathcal{X}[\text{natural number}] \equiv$ for every x , $(\vec{\delta}_h[f_1 + f_2])(\$_1 + 1)_x = (\vec{\delta}_h[f_1])(\$_1 + 1)_x + (\vec{\delta}_h[f_2])(\$_1 + 1)_x$. For every k such that $\mathcal{X}[k]$ holds $\mathcal{X}[k + 1]$ by (20), (6), [17, (27), (28)]. $\mathcal{X}[0]$ by (6), [17, (27), (28)]. For every n , $\mathcal{X}[n]$ from [1, Sch. 2]. \square

(24) $(\vec{\delta}_h[f_1 - f_2])(n + 1)_x = (\vec{\delta}_h[f_1])(n + 1)_x - (\vec{\delta}_h[f_2])(n + 1)_x$.

PROOF: Define $\mathcal{X}[\text{natural number}] \equiv$ for every x , $(\vec{\delta}_h[f_1 - f_2])(\$_1 + 1)_x = (\vec{\delta}_h[f_1])(\$_1 + 1)_x - (\vec{\delta}_h[f_2])(\$_1 + 1)_x$. $\mathcal{X}[0]$ by (6), [17, (29), (27), (28)]. For every k such that $\mathcal{X}[k]$ holds $\mathcal{X}[k + 1]$ by (20), (6), [17, (29), (27), (28)]. For every n , $\mathcal{X}[n]$ from [1, Sch. 2]. \square

(25) $(\vec{\delta}_h[r_1 \cdot f_1 + r_2 \cdot f_2])(n + 1)_x = r_1 \cdot (\vec{\delta}_h[f_1])(n + 1)_x + r_2 \cdot (\vec{\delta}_h[f_2])(n + 1)_x$.
The theorem is a consequence of (23) and (22).

(26) $(\vec{\delta}_h[f])(1)_x = (\text{Shift}(f, (2 \cdot 1_F)^{-1} \cdot h))_x - (\text{Shift}(f, -(2 \cdot 1_F)^{-1} \cdot h))_x$. The theorem is a consequence of (6).

(27) $(\vec{\Delta}_h[f])(n)_x = (\vec{\nabla}_h[f])(n)_{x+n \cdot h}$.

PROOF: Define $\mathcal{X}[\text{natural number}] \equiv$ for every x , $(\vec{\Delta}_h[f])(\$_1)_x = (\vec{\nabla}_h[f])(\$_1)_{x+\$_1 \cdot h}$. For every k such that $\mathcal{X}[k]$ holds $\mathcal{X}[k + 1]$ by (3), [15, (13), (15)], [17, (4), (15), (28)]. $\mathcal{X}[0]$ by [17, (4)], [15, (12)]. For every n , $\mathcal{X}[n]$ from [1, Sch. 2]. \square

Let us assume that $1_F \neq -1_F$. Now we state the propositions:

$$(28) \quad (\vec{\Delta}_h[f])(2 \cdot n)_x = (\vec{\delta}_h[f])(2 \cdot n)_{x+n \cdot h}.$$

PROOF: Define $\mathcal{X}[\text{natural number}] \equiv$ for every x , $(\vec{\Delta}_h[f])(2 \cdot \$1)_x = (\vec{\delta}_h[f])(2 \cdot \$1)_{x+\$1 \cdot h}$. For every k such that $\mathcal{X}[k]$ holds $\mathcal{X}[k+1]$ by [15, (13), (15)], [17, (27), (28), (15)]. $\mathcal{X}[0]$ by [17, (4)], [15, (12)]. For every n , $\mathcal{X}[n]$ from [1, Sch. 2]. \square

$$(29) \quad (\vec{\Delta}_h[f])(2 \cdot n + 1)_x = (\vec{\delta}_h[f])(2 \cdot n + 1)_{x+n \cdot h+(2 \cdot 1_F)^{-1} \cdot h}.$$

PROOF: $2 \cdot 1_F \neq 0_F$ by [15, (13), (15)]. $(\vec{\delta}_h[f])(2 \cdot n)$ is a function from V into W . $(\vec{\Delta}_h[f])(2 \cdot n)$ is a function from V into W . \square

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REFERENCES

- [1] Grzegorz Bancerek. The fundamental properties of natural numbers. *Formalized Mathematics*, 1(1):41–46, 1990.
- [2] Grzegorz Bancerek. The ordinal numbers. *Formalized Mathematics*, 1(1):91–96, 1990.
- [3] E. Biham and A. Shamir. Differential cryptanalysis of DES-like cryptosystems. *Lecture Notes in Computer Science*, 537:2–21, 1991.
- [4] E. Biham and A. Shamir. Differential cryptanalysis of the full 16-round DES. *Lecture Notes in Computer Science*, 740:487–496, 1993.
- [5] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(1):55–65, 1990.
- [6] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(1):153–164, 1990.
- [7] Czesław Byliński. Partial functions. *Formalized Mathematics*, 1(2):357–367, 1990.
- [8] Czesław Byliński. Some basic properties of sets. *Formalized Mathematics*, 1(1):47–53, 1990.
- [9] Eugeniusz Kusak, Wojciech Leończuk, and Michał Muzalewski. Abelian groups, fields and vector spaces. *Formalized Mathematics*, 1(2):335–342, 1990.
- [10] X. Lai. Higher order derivatives and differential cryptoanalysis. *Communications and Cryptography*, pages 227–233, 1994.
- [11] Bo Li, Yan Zhang, and Xiquan Liang. Difference and difference quotient. *Formalized Mathematics*, 14(3):115–119, 2006. doi:10.2478/v10037-006-0014-z.
- [12] Michał Muzalewski and Wojciech Skaba. From loops to Abelian multiplicative groups with zero. *Formalized Mathematics*, 1(5):833–840, 1990.
- [13] Hiroyuki Okazaki and Yasunari Shidama. Formalization of the data encryption standard. *Formalized Mathematics*, 20(2):125–146, 2012. doi:10.2478/v10037-012-0016-y.
- [14] Beata Perkowska. Functional sequence from a domain to a domain. *Formalized Mathematics*, 3(1):17–21, 1992.
- [15] Christoph Schwarzweiler. The binomial theorem for algebraic structures. *Formalized Mathematics*, 9(3):559–564, 2001.
- [16] Wojciech A. Trybulec. Groups. *Formalized Mathematics*, 1(5):821–827, 1990.
- [17] Wojciech A. Trybulec. Vectors in real linear space. *Formalized Mathematics*, 1(2):291–296, 1990.
- [18] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(1):67–71, 1990.
- [19] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1(1):73–83, 1990.

- [20] Edmund Woronowicz. Relations defined on sets. *Formalized Mathematics*, 1(1):181–186, 1990.
- [21] Hiroshi Yamazaki and Yasunari Shidama. Algebra of vector functions. *Formalized Mathematics*, 3(2):171–175, 1992.

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