

# Categorical Pullbacks

Marco Riccardi  
Via del Pero 102  
54038 Montignoso  
Italy

**Summary.** The main purpose of this article is to introduce the categorical concept of pullback in Mizar. In the first part of this article we redefine hom-sets, monomorphisms, epimorphisms and isomorphisms [7] within a free-object category [1] and it is shown there that ordinal numbers can be considered as categories. Then the pullback is introduced in terms of its universal property and the Pullback Lemma is formalized [15]. In the last part of the article we formalize the pullback of functors [14] and it is also shown that it is not possible to write an equivalent definition in the context of the previous Mizar formalization of category theory [8].

MSC: 18A30 03B35

Keywords: category pullback; pullback lemma

MML identifier: CAT\_7, version: 8.1.03 5.29.1227

The notation and terminology used in this paper have been introduced in the following articles: [2], [8], [17], [18], [6], [13], [9], [10], [3], [11], [20], [21], [16], [19], [4], [5], and [12].

## 1. PRELIMINARIES

One can verify that every set which is ordinal is also non pair.

Let  $\mathcal{C}$  be an empty category structure. Let us note that  $\text{Mor } \mathcal{C}$  is empty.

Let  $\mathcal{C}$  be a non empty category structure. Note that  $\text{Mor } \mathcal{C}$  is non empty.

Let  $\mathcal{C}$  be an empty category structure with identities. Let us note that  $\text{Ob } \mathcal{C}$  is empty.

Let  $\mathcal{C}$  be a non empty category structure with identities. Observe that  $\text{Ob } \mathcal{C}$  is non empty.

Let  $\mathcal{C}$  be category structure with identities and  $a$  be an object of  $\mathcal{C}$ . One can check that  $\text{id}_a$  is identity.

Now we state the propositions:

- (1) Let us consider a category structure  $\mathcal{C}$ , and a morphism  $f$  of  $\mathcal{C}$ . Suppose  $\mathcal{C}$  is not empty. Then  $f \in$  the carrier of  $\mathcal{C}$ .
- (2) Let us consider category structure  $\mathcal{C}$  with identities, and an object  $a$  of  $\mathcal{C}$ . Suppose  $\mathcal{C}$  is not empty. Then  $a \in$  the carrier of  $\mathcal{C}$ .
- (3) Let us consider a composable category structure  $\mathcal{C}$ , and morphisms  $f_1, f_2, f_3$  of  $\mathcal{C}$ . Suppose  $f_1 \triangleright f_2$  and  $f_2 \triangleright f_3$  and  $f_2$  is identity. Then  $f_1 \triangleright f_3$ .
- (4) Let us consider a composable category structure  $\mathcal{C}$  with identities, and morphisms  $f_1, f_2$  of  $\mathcal{C}$ . Suppose  $f_1 \triangleright f_2$ . Then
  - (i)  $\text{dom}(f_1 \circ f_2) = \text{dom } f_2$ , and
  - (ii)  $\text{cod}(f_1 \circ f_2) = \text{cod } f_1$ .
- (5) Let us consider a non empty, composable category structure  $\mathcal{C}$  with identities, and morphisms  $f_1, f_2$  of  $\mathcal{C}$ . Then  $f_1 \triangleright f_2$  if and only if  $\text{dom } f_1 = \text{cod } f_2$ .
- (6) Let us consider a composable category structure  $\mathcal{C}$  with identities, and a morphism  $f$  of  $\mathcal{C}$ . If  $f$  is identity, then  $\text{dom } f = f$  and  $\text{cod } f = f$ .
- (7) Let us consider a composable category structure  $\mathcal{C}$  with identities, and morphisms  $f_1, f_2$  of  $\mathcal{C}$ . Suppose  $f_1 \triangleright f_2$  and  $f_1$  is identity and  $f_2$  is identity. Then  $f_1 = f_2$ .

Let us consider a non empty, composable category structure  $\mathcal{C}$  with identities and morphisms  $f_1, f_2$  of  $\mathcal{C}$ . Now we state the propositions:

- (8) If  $\text{dom } f_1 = f_2$ , then  $f_1 \triangleright f_2$  and  $f_1 \circ f_2 = f_1$ .
- (9) If  $f_1 = \text{cod } f_2$ , then  $f_1 \triangleright f_2$  and  $f_1 \circ f_2 = f_2$ .

Now we state the propositions:

- (10) Let us consider categories  $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3, \mathcal{C}_4$ , a functor  $\mathcal{F}$  from  $\mathcal{C}_1$  to  $\mathcal{C}_2$ , a functor  $\mathcal{G}$  from  $\mathcal{C}_2$  to  $\mathcal{C}_3$ , and a functor  $\mathcal{H}$  from  $\mathcal{C}_3$  to  $\mathcal{C}_4$ . Suppose  $\mathcal{F}$  is covariant and  $\mathcal{G}$  is covariant and  $\mathcal{H}$  is covariant. Then  $\mathcal{H} \circ (\mathcal{G} \circ \mathcal{F}) = (\mathcal{H} \circ \mathcal{G}) \circ \mathcal{F}$ .
- (11) Let us consider categories  $\mathcal{C}, \mathcal{D}$ , and a functor  $\mathcal{F}$  from  $\mathcal{C}$  to  $\mathcal{D}$ . Suppose  $\mathcal{F}$  is covariant. Then
  - (i)  $\mathcal{F} \circ \text{id}_{\mathcal{C}} = \mathcal{F}$ , and
  - (ii)  $\text{id}_{\mathcal{D}} \circ \mathcal{F} = \mathcal{F}$ .
- (12) Let us consider composable category structures  $\mathcal{C}, \mathcal{D}$  with identities. Then  $\mathcal{C} \cong \mathcal{D}$  if and only if there exists a functor  $\mathcal{F}$  from  $\mathcal{C}$  to  $\mathcal{D}$  such that  $\mathcal{F}$  is covariant and bijective. The theorem is a consequence of (5).

(13) Let us consider empty category structures  $\mathcal{C}$ ,  $\mathcal{D}$  with identities. Then  $\mathcal{C} \cong \mathcal{D}$ .

Let us consider category structures  $\mathcal{C}$ ,  $\mathcal{D}$  with identities. Now we state the propositions:

(14) Suppose  $\mathcal{C} \cong \mathcal{D}$ . Then

(i)  $\overline{\text{Mor } \mathcal{C}} = \overline{\text{Mor } \mathcal{D}}$ , and

(ii)  $\overline{\text{Ob } \mathcal{C}} = \overline{\text{Ob } \mathcal{D}}$ .

(15) If  $\mathcal{C} \cong \mathcal{D}$  and  $\mathcal{C}$  is empty, then  $\mathcal{D}$  is empty. The theorem is a consequence of (14).

## 2. HOM-SETS

Let  $\mathcal{C}$  be a category structure and  $a, b$  be objects of  $\mathcal{C}$ . The functor  $\text{hom}(a, b)$  yielding a subset of  $\text{Mor } \mathcal{C}$  is defined by the term

(Def. 1)  $\{f, \text{ where } f \text{ is a morphism of } \mathcal{C} : \text{there exist morphisms } f_1, f_2 \text{ of } \mathcal{C} \text{ such that } a = f_1 \text{ and } b = f_2 \text{ and } f \triangleright f_1 \text{ and } f_2 \triangleright f\}$ .

Let  $\mathcal{C}$  be a non empty, composable category structure with identities. Observe that the functor  $\text{hom}(a, b)$  yields a subset of  $\text{Mor } \mathcal{C}$  and is defined by the term

(Def. 2)  $\{f, \text{ where } f \text{ is a morphism of } \mathcal{C} : \text{dom } f = a \text{ and } \text{cod } f = b\}$ .

Let  $\mathcal{C}$  be a category structure. Assume  $\text{hom}(a, b) \neq \emptyset$ .

A morphism from  $a$  to  $b$  is a morphism of  $\mathcal{C}$  and is defined by

(Def. 3)  $it \in \text{hom}(a, b)$ .

Let  $\mathcal{C}$  be category structure with identities and  $a$  be an object of  $\mathcal{C}$ . Assume  $\text{hom}(a, a) \neq \emptyset$ . Observe that the functor  $\text{id}_a$  yields a morphism from  $a$  to  $a$ . Let  $\mathcal{C}$  be a non empty category structure with identities. Note that  $\text{hom}(a, a)$  is non empty.

Let  $\mathcal{C}$  be a composable category structure with identities,  $a, b, c$  be objects of  $\mathcal{C}$ ,  $f$  be a morphism from  $a$  to  $b$ , and  $g$  be a morphism from  $b$  to  $c$ . Assume  $\text{hom}(a, b) \neq \emptyset$  and  $\text{hom}(b, c) \neq \emptyset$ . The functor  $g \cdot f$  yielding a morphism from  $a$  to  $c$  is defined by the term

(Def. 4)  $g \circ f$ .

Now we state the propositions:

(16) Let us consider a category structure  $\mathcal{C}$ , objects  $a, b$  of  $\mathcal{C}$ , and a morphism  $f$  from  $a$  to  $b$ . Suppose  $\text{hom}(a, b) \neq \emptyset$ . Then there exist morphisms  $f_1, f_2$  of  $\mathcal{C}$  such that

(i)  $a = f_1$ , and

- (ii)  $b = f_2$ , and
- (iii)  $f \triangleright f_1$ , and
- (iv)  $f_2 \triangleright f$ .

- (17) Let us consider a composable category structure  $\mathcal{C}$  with identities, objects  $a, b, c$  of  $\mathcal{C}$ , a morphism  $f_1$  from  $a$  to  $b$ , and a morphism  $f_2$  from  $b$  to  $c$ . Suppose  $\text{hom}(a, b) \neq \emptyset$  and  $\text{hom}(b, c) \neq \emptyset$ . Then  $f_2 \triangleright f_1$ . The theorem is a consequence of (16) and (3).
- (18) Let us consider a composable category structure  $\mathcal{C}$  with identities, objects  $a, b$  of  $\mathcal{C}$ , and a morphism  $f$  from  $a$  to  $b$ . Suppose  $\text{hom}(a, b) \neq \emptyset$ . Then
- (i)  $f \cdot \text{id}_a = f$ , and
  - (ii)  $\text{id}_b \cdot f = f$ .

The theorem is a consequence of (17).

- (19) Let us consider a non empty, composable category structure  $\mathcal{C}$  with identities, and a morphism  $f$  of  $\mathcal{C}$ . Then  $f \in \text{hom}(\text{dom } f, \text{cod } f)$ .
- (20) Let us consider a non empty, composable category structure  $\mathcal{C}$  with identities, objects  $a, b$  of  $\mathcal{C}$ , and a morphism  $f$  of  $\mathcal{C}$ . Then  $f \in \text{hom}(a, b)$  if and only if  $\text{dom } f = a$  and  $\text{cod } f = b$ .
- (21) Let us consider a non empty, composable category structure  $\mathcal{C}$  with identities, and an object  $a$  of  $\mathcal{C}$ . Then  $a \in \text{hom}(a, a)$ . The theorem is a consequence of (6).
- (22) Let us consider a composable category structure  $\mathcal{C}$  with identities, and objects  $a, b, c$  of  $\mathcal{C}$ . Suppose  $\text{hom}(a, b) \neq \emptyset$  and  $\text{hom}(b, c) \neq \emptyset$ . Then  $\text{hom}(a, c) \neq \emptyset$ . The theorem is a consequence of (16) and (3).
- (23) Let us consider a category  $\mathcal{C}$ , objects  $a, b, c, d$  of  $\mathcal{C}$ , a morphism  $f_1$  from  $a$  to  $b$ , a morphism  $f_2$  from  $b$  to  $c$ , and a morphism  $f_3$  from  $c$  to  $d$ . Suppose  $\text{hom}(a, b) \neq \emptyset$  and  $\text{hom}(b, c) \neq \emptyset$  and  $\text{hom}(c, d) \neq \emptyset$ . Then  $f_3 \cdot (f_2 \cdot f_1) = (f_3 \cdot f_2) \cdot f_1$ . The theorem is a consequence of (22) and (17).
- (24) Let us consider a composable category structure  $\mathcal{C}$  with identities, objects  $a, b, c$  of  $\mathcal{C}$ , a morphism  $f_1$  from  $a$  to  $b$ , and a morphism  $f_2$  from  $b$  to  $c$ . Suppose  $\text{hom}(a, b) \neq \emptyset$  and  $\text{hom}(b, c) \neq \emptyset$ . Then
- (i) if  $f_1$  is identity, then  $f_2 \cdot f_1 = f_2$ , and
  - (ii) if  $f_2$  is identity, then  $f_2 \cdot f_1 = f_1$ .

PROOF:  $f_2 \triangleright f_1$ . If  $f_1$  is identity, then  $f_2 \cdot f_1 = f_2$  by [17, (22), (23)].  $\square$

3. MONOMORPHISMS, EPIMORPHISMS AND ISOMORPHISMS

Let  $\mathcal{C}$  be a composable category structure with identities,  $a, b$  be objects of  $\mathcal{C}$ , and  $f$  be a morphism from  $a$  to  $b$ . We say that  $f$  is monomorphic if and only if

(Def. 5)  $\text{hom}(a, b) \neq \emptyset$  and for every object  $c$  of  $\mathcal{C}$  such that  $\text{hom}(c, a) \neq \emptyset$  for every morphisms  $g_1, g_2$  from  $c$  to  $a$  such that  $f \cdot g_1 = f \cdot g_2$  holds  $g_1 = g_2$ .

We say that  $f$  is epimorphic if and only if

(Def. 6)  $\text{hom}(a, b) \neq \emptyset$  and for every object  $c$  of  $\mathcal{C}$  such that  $\text{hom}(b, c) \neq \emptyset$  for every morphisms  $g_1, g_2$  from  $b$  to  $c$  such that  $g_1 \cdot f = g_2 \cdot f$  holds  $g_1 = g_2$ .

Now we state the proposition:

(25) Let us consider a composable category structure  $\mathcal{C}$  with identities, objects  $a, b$  of  $\mathcal{C}$ , and a morphism  $f_1$  from  $a$  to  $b$ . Suppose  $\text{hom}(a, b) \neq \emptyset$  and  $f_1$  is identity. Then  $f_1$  is monomorphic. The theorem is a consequence of (24).

Let us consider a category  $\mathcal{C}$ , objects  $a, b, c$  of  $\mathcal{C}$ , a morphism  $f_1$  from  $a$  to  $b$ , and a morphism  $f_2$  from  $b$  to  $c$ . Now we state the propositions:

(26) If  $f_1$  is monomorphic and  $f_2$  is monomorphic, then  $f_2 \cdot f_1$  is monomorphic. The theorem is a consequence of (22) and (23).

(27) If  $f_2 \cdot f_1$  is monomorphic and  $\text{hom}(a, b) \neq \emptyset$  and  $\text{hom}(b, c) \neq \emptyset$ , then  $f_1$  is monomorphic. The theorem is a consequence of (23).

Let  $\mathcal{C}$  be a composable category structure with identities,  $a, b$  be objects of  $\mathcal{C}$ , and  $f$  be a morphism from  $a$  to  $b$ . We say that  $f$  is a section if and only if

(Def. 7)  $\text{hom}(a, b) \neq \emptyset$  and  $\text{hom}(b, a) \neq \emptyset$  and there exists a morphism  $g$  from  $b$  to  $a$  such that  $g \cdot f = \text{id}-a$ .

We say that  $f$  is a retraction if and only if

(Def. 8)  $\text{hom}(a, b) \neq \emptyset$  and  $\text{hom}(b, a) \neq \emptyset$  and there exists a morphism  $g$  from  $b$  to  $a$  such that  $f \cdot g = \text{id}-b$ .

Now we state the propositions:

(28) Let us consider a category  $\mathcal{C}$ , objects  $a, b$  of  $\mathcal{C}$ , and a morphism  $f$  from  $a$  to  $b$ . If  $f$  is a section, then  $f$  is monomorphic. The theorem is a consequence of (23) and (18).

(29) Let us consider a composable category structure  $\mathcal{C}$  with identities, objects  $a, b$  of  $\mathcal{C}$ , and a morphism  $f_1$  from  $a$  to  $b$ . Suppose  $\text{hom}(a, b) \neq \emptyset$  and  $f_1$  is identity. Then  $f_1$  is epimorphic. The theorem is a consequence of (24).

Let us consider a category  $\mathcal{C}$ , objects  $a, b, c$  of  $\mathcal{C}$ , a morphism  $f_1$  from  $a$  to  $b$ , and a morphism  $f_2$  from  $b$  to  $c$ . Now we state the propositions:

- (30) If  $f_1$  is epimorphic and  $f_2$  is epimorphic, then  $f_2 \cdot f_1$  is epimorphic. The theorem is a consequence of (22) and (23).
- (31) If  $f_2 \cdot f_1$  is epimorphic and  $\text{hom}(a, b) \neq \emptyset$  and  $\text{hom}(b, c) \neq \emptyset$ , then  $f_2$  is epimorphic. The theorem is a consequence of (23).
- (32) Let us consider a category  $\mathcal{C}$ , objects  $a, b$  of  $\mathcal{C}$ , and a morphism  $f$  from  $a$  to  $b$ . If  $f$  is a retraction, then  $f$  is epimorphic. The theorem is a consequence of (23) and (18).

Let  $\mathcal{C}$  be a composable category structure with identities,  $a, b$  be objects of  $\mathcal{C}$ , and  $f$  be a morphism from  $a$  to  $b$ . We say that  $f$  is isomorphism if and only if

- (Def. 9)  $\text{hom}(a, b) \neq \emptyset$  and  $\text{hom}(b, a) \neq \emptyset$  and there exists a morphism  $g$  from  $b$  to  $a$  such that  $g \cdot f = \text{id-}a$  and  $f \cdot g = \text{id-}b$ .

We say that  $a$  and  $b$  are isomorphic if and only if

- (Def. 10) there exists a morphism  $f$  from  $a$  to  $b$  such that  $f$  is isomorphism.

Note that  $a$  and  $b$  are isomorphic if and only if the condition (Def. 11) is satisfied.

- (Def. 11)  $\text{hom}(a, b) \neq \emptyset$  and  $\text{hom}(b, a) \neq \emptyset$  and there exists a morphism  $f$  from  $a$  to  $b$  and there exists a morphism  $g$  from  $b$  to  $a$  such that  $g \cdot f = \text{id-}a$  and  $f \cdot g = \text{id-}b$ .

Now we state the proposition:

- (33) Let us consider a category  $\mathcal{C}$ , objects  $a, b$  of  $\mathcal{C}$ , and a morphism  $f$  from  $a$  to  $b$ . If  $f$  is isomorphism, then  $f$  is monomorphic and epimorphic. The theorem is a consequence of (28) and (32).

#### 4. ORDINAL NUMBERS AS CATEGORIES

Let  $\mathcal{C}$  be a category structure. We say that  $\mathcal{C}$  is a preorder if and only if

- (Def. 12) for every objects  $a, b$  of  $\mathcal{C}$  and for every morphisms  $f_1, f_2$  of  $\mathcal{C}$  such that  $f_1, f_2 \in \text{hom}(a, b)$  holds  $f_1 = f_2$ .

Observe that every category structure which is empty is also a preorder and there exists a category structure which is strict and preorder and every composable category structure with identities which is a preorder is also associative.

Let  $\mathcal{C}$  be category structure with identities. The functor  $\text{RelOb } \mathcal{C}$  yielding a binary relation on  $\text{Ob } \mathcal{C}$  is defined by the term

- (Def. 13)  $\{\langle a, b \rangle, \text{ where } a, b \text{ are objects of } \mathcal{C} : \text{ there exists a morphism } f \text{ of } \mathcal{C} \text{ such that } f \in \text{hom}(a, b)\}$ .

Let  $\mathcal{C}$  be an empty category structure with identities. Let us note that  $\text{RelOb } \mathcal{C}$  is empty.

Now we state the propositions:

- (34) Let us consider a composable category structure  $\mathcal{C}$  with identities. Then
- (i)  $\text{dom RelOb } \mathcal{C} = \text{Ob } \mathcal{C}$ , and
  - (ii)  $\text{rng RelOb } \mathcal{C} = \text{Ob } \mathcal{C}$ .

The theorem is a consequence of (6) and (19).

- (35) Let us consider composable category structures  $\mathcal{C}_1, \mathcal{C}_2$  with identities. Suppose  $\mathcal{C}_1 \cong \mathcal{C}_2$ . Then  $\text{RelOb } \mathcal{C}_1$  and  $\text{RelOb } \mathcal{C}_2$  are isomorphic. The theorem is a consequence of (15), (34), and (20).

Let  $\mathcal{C}$  be a non empty, composable category structure with identities. One can verify that  $\text{RelOb } \mathcal{C}$  is non empty.

Now we state the propositions:

- (36) Let us consider preorder, composable category structure  $\mathcal{C}$  with identities. Suppose  $\mathcal{C}$  is not empty. Then there exists a function  $\mathcal{F}$  from  $\mathcal{C}$  into  $\text{RelOb } \mathcal{C}$  such that
- (i)  $\mathcal{F}$  is bijective, and
  - (ii) for every morphism  $f$  of  $\mathcal{C}$ ,  $\mathcal{F}(f) = \langle \text{dom } f, \text{cod } f \rangle$ .

PROOF: Reconsider  $\mathcal{C}_1 = \mathcal{C}$  as a non empty, composable category structure with identities. Define  $\mathcal{P}[\text{object}, \text{object}] \equiv$  for every morphism  $f$  of  $\mathcal{C}_1$  such that  $\$1 = f$  holds  $\$2 = \langle \text{dom } f, \text{cod } f \rangle$ . For every element  $x$  of the carrier of  $\mathcal{C}_1$ , there exists an element  $y$  of  $\text{RelOb } \mathcal{C}_1$  such that  $\mathcal{P}[x, y]$ . Consider  $\mathcal{F}$  being a function from the carrier of  $\mathcal{C}_1$  into  $\text{RelOb } \mathcal{C}_1$  such that for every element  $x$  of the carrier of  $\mathcal{C}_1$ ,  $\mathcal{P}[x, \mathcal{F}(x)]$  from [10, Sch. 3]. For every object  $y$  such that  $y \in \text{RelOb } \mathcal{C}$  holds  $y \in \text{rng } \mathcal{F}$  by (20), [9, (3)]. For every objects  $x_1, x_2$  such that  $x_1, x_2 \in \text{dom } \mathcal{F}$  and  $\mathcal{F}(x_1) = \mathcal{F}(x_2)$  holds  $x_1 = x_2$ .  $\square$

- (37) Let us consider an ordinal number  $O$ . Then there exists a strict, a pre-order category  $\mathcal{C}$  such that
- (i)  $\text{Ob } \mathcal{C} = O$ , and
  - (ii) for every objects  $o_1, o_2$  of  $\mathcal{C}$  such that  $o_1 \in o_2$  holds  $\text{hom}(o_1, o_2) = \{\langle o_1, o_2 \rangle\}$ , and
  - (iii)  $\text{RelOb } \mathcal{C} = \subseteq_O$ , and
  - (iv)  $\text{Mor } \mathcal{C} = O \cup \{\langle o_1, o_2 \rangle, \text{ where } o_1, o_2 \text{ are elements of } O : o_1 \in o_2\}$ .

The theorem is a consequence of (6), (20), and (21).

Let  $O$  be an ordinal number and  $\mathcal{C}$  be a composable category structure with identities. We say that  $\mathcal{C}$  is  $O$ -ordered if and only if

- (Def. 14)  $\text{RelOb } \mathcal{C}$  and  $\subseteq_O$  are isomorphic.

Let  $O$  be a non empty, ordinal number. Let us observe that every composable category structure with identities which is  $O$ -ordered is also non empty.

Let  $O$  be an ordinal number. Note that there exists a composable category structure with identities which is strict,  $O$ -ordered, and preorder.

Let  $O$  be an empty, ordinal number. Let us observe that every composable category structure with identities which is  $O$ -ordered is also empty.

Now we state the proposition:

- (38) Let us consider ordinal numbers  $O_1, O_2$ , a  $O_1$ -ordered, a preorder category  $\mathcal{C}_1$ , and a  $O_2$ -ordered, a preorder category  $\mathcal{C}_2$ . Then  $O_1 = O_2$  if and only if  $\mathcal{C}_1 \cong \mathcal{C}_2$ .

PROOF: If  $O_1 = O_2$ , then  $\mathcal{C}_1 \cong \mathcal{C}_2$  by (13), [4, (39), (41)], (36). If  $\mathcal{C}_1 \cong \mathcal{C}_2$ , then  $O_1 = O_2$  by (35), [4, (42), (40)], [5, (10)].  $\square$

Let  $O$  be an ordinal number. The functor  $\mathbf{O}$  yielding a strict,  $O$ -ordered, a preorder category is defined by the term

(Def. 15) the strict,  $O$ -ordered, a preorder category.

Now we state the proposition:

- (39) There exists a morphism  $f$  of  $\mathbf{2}$  such that

- (i)  $f$  is not identity, and
- (ii)  $\text{Ob } \mathbf{2} = \{\text{dom } f, \text{cod } f\}$ , and
- (iii)  $\text{Mor } \mathbf{2} = \{\text{dom } f, \text{cod } f, f\}$ , and
- (iv)  $\text{dom } f, \text{cod } f, f$  are mutually different.

PROOF: Consider  $\mathcal{C}$  being a strict, a preorder category such that  $\text{Ob } \mathcal{C} = 2$  and for every objects  $o_1, o_2$  of  $\mathcal{C}$  such that  $o_1 \in o_2$  holds  $\text{hom}(o_1, o_2) = \{\langle o_1, o_2 \rangle\}$  and  $\text{RelOb } \mathcal{C} = \subseteq_2$  and  $\text{Mor } \mathcal{C} = 2 \cup \{\langle o_1, o_2 \rangle\}$ , where  $o_1, o_2$  are elements of  $2 : o_1 \in o_2$ .  $\mathcal{C} \cong \mathbf{2}$ . Consider  $\mathcal{F}$  being a functor from  $\mathcal{C}$  to  $\mathbf{2}$ ,  $\mathcal{G}$  being a functor from  $\mathbf{2}$  to  $\mathcal{C}$  such that  $\mathcal{F}$  is covariant and  $\mathcal{G}$  is covariant and  $\mathcal{G} \circ \mathcal{F} = \text{id}_{\mathcal{C}}$  and  $\mathcal{F} \circ \mathcal{G} = \text{id}_{\mathbf{2}}$ . Reconsider  $g = \langle 0, 1 \rangle$  as a morphism of  $\mathcal{C}$ .  $g$  is not identity by [17, (22)]. Set  $f = \mathcal{F}(g)$ .  $f$  is not identity by [9, (18)], [17, (34)].  $\overline{\text{Ob } \mathbf{2}} = \overline{2}$ . Consider  $x, y$  being objects such that  $x \neq y$  and  $\text{Ob } \mathbf{2} = \{x, y\}$ .  $\text{dom } f \neq \text{cod } f$ . For every object  $x$ ,  $x \in \text{Mor } \mathbf{2}$  iff  $x \in \{\text{dom } f, \text{cod } f, f\}$  by [17, (22)], [9, (18)], [17, (34)], [2, (50), (49)].  $\square$

Let  $\mathcal{C}$  be a non empty category and  $f$  be a morphism of  $\mathcal{C}$ . The functor  $\mathcal{M}_f$  yielding a covariant functor from  $\mathbf{2}$  to  $\mathcal{C}$  is defined by

(Def. 16) for every morphism  $g$  of  $\mathbf{2}$  such that  $g$  is not identity holds  $it(g) = f$ .

Now we state the proposition:

- (40) Let us consider a non empty category  $\mathcal{C}$ , and a morphism  $f$  of  $\mathcal{C}$ . Suppose  $f$  is identity. Let us consider a morphism  $g$  of  $\mathbf{2}$ . Then  $(\mathcal{M}_f)(g) = f$ . The theorem is a consequence of (39) and (6).



5. PULLBACKS

Let  $\mathcal{C}$  be a category,  $c, c_1, c_2, d$  be objects of  $\mathcal{C}$ , and  $f_1$  be a morphism from  $c_1$  to  $c$ . Assume  $\text{hom}(c_1, c) \neq \emptyset$ . Let  $f_2$  be a morphism from  $c_2$  to  $c$ . Assume  $\text{hom}(c_2, c) \neq \emptyset$ . Let  $p_1$  be a morphism from  $d$  to  $c_1$ . Assume  $\text{hom}(d, c_1) \neq \emptyset$ . Let  $p_2$  be a morphism from  $d$  to  $c_2$ . Assume  $\text{hom}(d, c_2) \neq \emptyset$ . We say that  $\langle d, p_1, p_2 \rangle$  is a pullback of  $f_1, f_2$  if and only if

(Def. 17)  $f_1 \cdot p_1 = f_2 \cdot p_2$  and for every object  $d_1$  of  $\mathcal{C}$  and for every morphism  $g_1$  from  $d_1$  to  $c_1$  and for every morphism  $g_2$  from  $d_1$  to  $c_2$  such that  $\text{hom}(d_1, c_1) \neq \emptyset$  and  $\text{hom}(d_1, c_2) \neq \emptyset$  and  $f_1 \cdot g_1 = f_2 \cdot g_2$  holds  $\text{hom}(d_1, d) \neq \emptyset$  and there exists a morphism  $h$  from  $d_1$  to  $d$  such that  $p_1 \cdot h = g_1$  and  $p_2 \cdot h = g_2$  and for every morphism  $h_1$  from  $d_1$  to  $d$  such that  $p_1 \cdot h_1 = g_1$  and  $p_2 \cdot h_1 = g_2$  holds  $h = h_1$ .

Now we state the proposition:

(41) Let us consider a category  $\mathcal{C}$ , objects  $c, c_1, c_2, d, e$  of  $\mathcal{C}$ , a morphism  $f_1$  from  $c_1$  to  $c$ , a morphism  $f_2$  from  $c_2$  to  $c$ , a morphism  $p_1$  from  $d$  to  $c_1$ , a morphism  $p_2$  from  $d$  to  $c_2$ , a morphism  $q_1$  from  $e$  to  $c_1$ , and a morphism  $q_2$  from  $e$  to  $c_2$ . Suppose  $\text{hom}(c_1, c) \neq \emptyset$  and  $\text{hom}(c_2, c) \neq \emptyset$  and  $\text{hom}(d, c_1) \neq \emptyset$  and  $\text{hom}(d, c_2) \neq \emptyset$  and  $\text{hom}(e, c_1) \neq \emptyset$  and  $\text{hom}(e, c_2) \neq \emptyset$  and  $\langle d, p_1, p_2 \rangle$  is a pullback of  $f_1, f_2$  and  $\langle e, q_1, q_2 \rangle$  is a pullback of  $f_1, f_2$ . Then  $d$  and  $e$  are isomorphic. The theorem is a consequence of (23) and (18).

Let us consider a category  $\mathcal{C}$ , objects  $c, c_1, c_2, d$  of  $\mathcal{C}$ , a morphism  $f_1$  from  $c_1$  to  $c$ , a morphism  $f_2$  from  $c_2$  to  $c$ , a morphism  $p_1$  from  $d$  to  $c_1$ , and a morphism  $p_2$  from  $d$  to  $c_2$ . Now we state the propositions:

(42) Suppose  $\text{hom}(c_1, c) \neq \emptyset$  and  $\text{hom}(c_2, c) \neq \emptyset$  and  $\text{hom}(d, c_1) \neq \emptyset$  and  $\text{hom}(d, c_2) \neq \emptyset$  and  $\langle d, p_1, p_2 \rangle$  is a pullback of  $f_1, f_2$ .

Then  $\langle d, p_2, p_1 \rangle$  is a pullback of  $f_2, f_1$ .

(43) Suppose  $\text{hom}(c_1, c) \neq \emptyset$  and  $\text{hom}(c_2, c) \neq \emptyset$  and  $\text{hom}(d, c_1) \neq \emptyset$  and  $\text{hom}(d, c_2) \neq \emptyset$  and  $\langle d, p_1, p_2 \rangle$  is a pullback of  $f_1, f_2$  and  $f_1$  is monomorphic. Then  $p_2$  is monomorphic. The theorem is a consequence of (22) and (23).

(44) Suppose  $\text{hom}(c_1, c) \neq \emptyset$  and  $\text{hom}(c_2, c) \neq \emptyset$  and  $\text{hom}(d, c_1) \neq \emptyset$  and  $\text{hom}(d, c_2) \neq \emptyset$  and  $\langle d, p_1, p_2 \rangle$  is a pullback of  $f_1, f_2$  and  $f_1$  is isomorphism. Then  $p_2$  is isomorphism. The theorem is a consequence of (22), (23), and (18).

(45) Let us consider a category  $\mathcal{C}$ , objects  $c_1, c_1, c_2, c_3, c_4, c_5, c_6$  of  $\mathcal{C}$ , a morphism  $f_1$  from  $c_1$  to  $c_2$ , a morphism  $f_2$  from  $c_2$  to  $c_3$ , a morphism  $f_3$  from  $c_1$  to  $c_4$ , a morphism  $f_4$  from  $c_2$  to  $c_5$ , a morphism  $f_5$  from

$c_3$  to  $c_6$ , a morphism  $f_6$  from  $c_4$  to  $c_5$ , and a morphism  $f_7$  from  $c_5$  to  $c_6$ . Suppose  $\text{hom}(c_1, c_2) \neq \emptyset$  and  $\text{hom}(c_2, c_3) \neq \emptyset$  and  $\text{hom}(c_1, c_4) \neq \emptyset$  and  $\text{hom}(c_2, c_5) \neq \emptyset$  and  $\text{hom}(c_3, c_6) \neq \emptyset$  and  $\text{hom}(c_4, c_5) \neq \emptyset$  and  $\text{hom}(c_5, c_6) \neq \emptyset$  and  $\langle c_2, f_2, f_4 \rangle$  is a pullback of  $f_5, f_7$ . Then  $\langle c_1, f_1, f_3 \rangle$  is a pullback of  $f_4, f_6$  if and only if  $\langle c_1, f_2 \cdot f_1, f_3 \rangle$  is a pullback of  $f_5, f_7 \cdot f_6$  and  $f_4 \cdot f_1 = f_6 \cdot f_3$ . The theorem is a consequence of (22) and (23).

## 6. PULLBACKS OF FUNCTORS

Let  $\mathcal{C}, \mathcal{D}$  be categories and  $\mathcal{F}$  be a functor from  $\mathcal{C}$  to  $\mathcal{D}$ . We say that  $\mathcal{F}$  is monomorphic if and only if

(Def. 18)  $\mathcal{F}$  is covariant and for every category  $\mathcal{B}$  and for every functors  $\mathcal{G}_1, \mathcal{G}_2$  from  $\mathcal{B}$  to  $\mathcal{C}$  such that  $\mathcal{G}_1$  is covariant and  $\mathcal{G}_2$  is covariant and  $\mathcal{F} \circ \mathcal{G}_1 = \mathcal{F} \circ \mathcal{G}_2$  holds  $\mathcal{G}_1 = \mathcal{G}_2$ .

We say that  $\mathcal{F}$  is isomorphism if and only if

(Def. 19)  $\mathcal{F}$  is covariant and there exists a functor  $\mathcal{G}$  from  $\mathcal{D}$  to  $\mathcal{C}$  such that  $\mathcal{G}$  is covariant and  $\mathcal{G} \circ \mathcal{F} = \text{id}_{\mathcal{C}}$  and  $\mathcal{F} \circ \mathcal{G} = \text{id}_{\mathcal{D}}$ .

Let  $\mathcal{C}, \mathcal{C}_1, \mathcal{C}_2, \mathcal{D}$  be categories and  $\mathcal{F}_1$  be a functor from  $\mathcal{C}_1$  to  $\mathcal{C}$ . Assume  $\mathcal{F}_1$  is covariant. Let  $\mathcal{F}_2$  be a functor from  $\mathcal{C}_2$  to  $\mathcal{C}$ . Assume  $\mathcal{F}_2$  is covariant. Let  $\mathcal{P}_1$  be a functor from  $\mathcal{D}$  to  $\mathcal{C}_1$ . Assume  $\mathcal{P}_1$  is covariant. Let  $\mathcal{P}_2$  be a functor from  $\mathcal{D}$  to  $\mathcal{C}_2$ . Assume  $\mathcal{P}_2$  is covariant. We say that  $\langle \mathcal{D}, \mathcal{P}_1, \mathcal{P}_2 \rangle$  is a pullback of  $\mathcal{F}_1, \mathcal{F}_2$  if and only if

(Def. 20)  $\mathcal{F}_1 \circ \mathcal{P}_1 = \mathcal{F}_2 \circ \mathcal{P}_2$  and for every category  $\mathcal{D}_1$  and for every functor  $\mathcal{G}_1$  from  $\mathcal{D}_1$  to  $\mathcal{C}_1$  and for every functor  $\mathcal{G}_2$  from  $\mathcal{D}_1$  to  $\mathcal{C}_2$  such that  $\mathcal{G}_1$  is covariant and  $\mathcal{G}_2$  is covariant and  $\mathcal{F}_1 \circ \mathcal{G}_1 = \mathcal{F}_2 \circ \mathcal{G}_2$  there exists a functor  $\mathcal{H}$  from  $\mathcal{D}_1$  to  $\mathcal{D}$  such that  $\mathcal{H}$  is covariant and  $\mathcal{P}_1 \circ \mathcal{H} = \mathcal{G}_1$  and  $\mathcal{P}_2 \circ \mathcal{H} = \mathcal{G}_2$  and for every functor  $\mathcal{H}_1$  from  $\mathcal{D}_1$  to  $\mathcal{D}$  such that  $\mathcal{H}_1$  is covariant and  $\mathcal{P}_1 \circ \mathcal{H}_1 = \mathcal{G}_1$  and  $\mathcal{P}_2 \circ \mathcal{H}_1 = \mathcal{G}_2$  holds  $\mathcal{H} = \mathcal{H}_1$ .

Now we state the proposition:

(46) Let us consider categories  $\mathcal{C}, \mathcal{C}_1, \mathcal{C}_2, \mathcal{D}, \mathcal{E}$ , a functor  $\mathcal{F}_1$  from  $\mathcal{C}_1$  to  $\mathcal{C}$ , a functor  $\mathcal{F}_2$  from  $\mathcal{C}_2$  to  $\mathcal{C}$ , a functor  $\mathcal{P}_1$  from  $\mathcal{D}$  to  $\mathcal{C}_1$ , a functor  $\mathcal{P}_2$  from  $\mathcal{D}$  to  $\mathcal{C}_2$ , a functor  $\mathcal{Q}_1$  from  $\mathcal{E}$  to  $\mathcal{C}_1$ , and a functor  $\mathcal{Q}_2$  from  $\mathcal{E}$  to  $\mathcal{C}_2$ . Suppose  $\mathcal{F}_1$  is covariant and  $\mathcal{F}_2$  is covariant and  $\mathcal{P}_1$  is covariant and  $\mathcal{P}_2$  is covariant and  $\mathcal{Q}_1$  is covariant and  $\mathcal{Q}_2$  is covariant and  $\langle \mathcal{D}, \mathcal{P}_1, \mathcal{P}_2 \rangle$  is a pullback of  $\mathcal{F}_1, \mathcal{F}_2$  and  $\langle \mathcal{E}, \mathcal{Q}_1, \mathcal{Q}_2 \rangle$  is a pullback of  $\mathcal{F}_1, \mathcal{F}_2$ . Then  $\mathcal{D} \cong \mathcal{E}$ .

PROOF: There exists a functor  $\mathcal{F}_8$  from  $\mathcal{D}$  to  $\mathcal{E}$  and there exists a functor  $\mathcal{G}_3$  from  $\mathcal{E}$  to  $\mathcal{D}$  such that  $\mathcal{F}_8$  is covariant and  $\mathcal{G}_3$  is covariant and  $\mathcal{G}_3 \circ \mathcal{F}_8 = \text{id}_{\mathcal{D}}$  and  $\mathcal{F}_8 \circ \mathcal{G}_3 = \text{id}_{\mathcal{E}}$  by (10), (11), [17, (35)].  $\square$

Let us consider categories  $\mathcal{C}, \mathcal{C}_1, \mathcal{C}_2, \mathcal{D}$ , a functor  $\mathcal{F}_1$  from  $\mathcal{C}_1$  to  $\mathcal{C}$ , a functor  $\mathcal{F}_2$  from  $\mathcal{C}_2$  to  $\mathcal{C}$ , a functor  $\mathcal{P}_1$  from  $\mathcal{D}$  to  $\mathcal{C}_1$ , and a functor  $\mathcal{P}_2$  from  $\mathcal{D}$  to  $\mathcal{C}_2$ . Now we state the propositions:

(47) Suppose  $\mathcal{F}_1$  is covariant and  $\mathcal{F}_2$  is covariant and  $\mathcal{P}_1$  is covariant and  $\mathcal{P}_2$  is covariant and  $\langle \mathcal{D}, \mathcal{P}_1, \mathcal{P}_2 \rangle$  is a pullback of  $\mathcal{F}_1, \mathcal{F}_2$ .  
Then  $\langle \mathcal{D}, \mathcal{P}_2, \mathcal{P}_1 \rangle$  is a pullback of  $\mathcal{F}_2, \mathcal{F}_1$ .

(48) Suppose  $\mathcal{F}_1$  is covariant and  $\mathcal{F}_2$  is covariant and  $\mathcal{P}_1$  is covariant and  $\mathcal{P}_2$  is covariant and  $\langle \mathcal{D}, \mathcal{P}_1, \mathcal{P}_2 \rangle$  is a pullback of  $\mathcal{F}_1, \mathcal{F}_2$  and  $\mathcal{F}_1$  is monomorphic. Then  $\mathcal{P}_2$  is monomorphic.

PROOF: For every category  $\mathcal{D}_1$  and for every functors  $\mathcal{Q}_1, \mathcal{Q}_2$  from  $\mathcal{D}_1$  to  $\mathcal{D}$  such that  $\mathcal{Q}_1$  is covariant and  $\mathcal{Q}_2$  is covariant and  $\mathcal{P}_2 \circ \mathcal{Q}_1 = \mathcal{P}_2 \circ \mathcal{Q}_2$  holds  $\mathcal{Q}_1 = \mathcal{Q}_2$  by [17, (35)], (10).  $\square$

(49) Suppose  $\mathcal{F}_1$  is covariant and  $\mathcal{F}_2$  is covariant and  $\mathcal{P}_1$  is covariant and  $\mathcal{P}_2$  is covariant and  $\langle \mathcal{D}, \mathcal{P}_1, \mathcal{P}_2 \rangle$  is a pullback of  $\mathcal{F}_1, \mathcal{F}_2$  and  $\mathcal{F}_1$  is isomorphism. Then  $\mathcal{P}_2$  is isomorphism. The theorem is a consequence of (10) and (11).

(50) Let us consider categories  $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3, \mathcal{C}_4, \mathcal{C}_5, \mathcal{C}_6$ , a functor  $\mathcal{F}_1$  from  $\mathcal{C}_1$  to  $\mathcal{C}_2$ , a functor  $\mathcal{F}_2$  from  $\mathcal{C}_2$  to  $\mathcal{C}_3$ , a functor  $\mathcal{F}_3$  from  $\mathcal{C}_1$  to  $\mathcal{C}_4$ , a functor  $\mathcal{F}_4$  from  $\mathcal{C}_2$  to  $\mathcal{C}_5$ , a functor  $\mathcal{F}_5$  from  $\mathcal{C}_3$  to  $\mathcal{C}_6$ , a functor  $\mathcal{F}_6$  from  $\mathcal{C}_4$  to  $\mathcal{C}_5$ , and a functor  $\mathcal{F}_7$  from  $\mathcal{C}_5$  to  $\mathcal{C}_6$ . Suppose  $\mathcal{F}_1$  is covariant and  $\mathcal{F}_2$  is covariant and  $\mathcal{F}_3$  is covariant and  $\mathcal{F}_4$  is covariant and  $\mathcal{F}_5$  is covariant and  $\mathcal{F}_6$  is covariant and  $\mathcal{F}_7$  is covariant and  $\langle \mathcal{C}_2, \mathcal{F}_2, \mathcal{F}_4 \rangle$  is a pullback of  $\mathcal{F}_5, \mathcal{F}_7$ . Then  $\langle \mathcal{C}_1, \mathcal{F}_1, \mathcal{F}_3 \rangle$  is a pullback of  $\mathcal{F}_4, \mathcal{F}_6$  if and only if  $\langle \mathcal{C}_1, \mathcal{F}_2 \circ \mathcal{F}_1, \mathcal{F}_3 \rangle$  is a pullback of  $\mathcal{F}_5, \mathcal{F}_7 \circ \mathcal{F}_6$  and  $\mathcal{F}_4 \circ \mathcal{F}_1 = \mathcal{F}_6 \circ \mathcal{F}_3$ .

PROOF: For every category  $\mathcal{D}_1$  and for every functor  $\mathcal{G}_1$  from  $\mathcal{D}_1$  to  $\mathcal{C}_2$  and for every functor  $\mathcal{G}_2$  from  $\mathcal{D}_1$  to  $\mathcal{C}_4$  such that  $\mathcal{G}_1$  is covariant and  $\mathcal{G}_2$  is covariant and  $\mathcal{F}_4 \circ \mathcal{G}_1 = \mathcal{F}_6 \circ \mathcal{G}_2$  there exists a functor  $\mathcal{H}$  from  $\mathcal{D}_1$  to  $\mathcal{C}_1$  such that  $\mathcal{H}$  is covariant and  $\mathcal{F}_1 \circ \mathcal{H} = \mathcal{G}_1$  and  $\mathcal{F}_3 \circ \mathcal{H} = \mathcal{G}_2$  and for every functor  $\mathcal{H}_1$  from  $\mathcal{D}_1$  to  $\mathcal{C}_1$  such that  $\mathcal{H}_1$  is covariant and  $\mathcal{F}_1 \circ \mathcal{H}_1 = \mathcal{G}_1$  and  $\mathcal{F}_3 \circ \mathcal{H}_1 = \mathcal{G}_2$  holds  $\mathcal{H} = \mathcal{H}_1$  by [17, (35)], (10).  $\square$

(51) Let us consider categories  $\mathcal{C}, \mathcal{C}_1, \mathcal{C}_2$ , a functor  $\mathcal{F}_1$  from  $\mathcal{C}_1$  to  $\mathcal{C}$ , and a functor  $\mathcal{F}_2$  from  $\mathcal{C}_2$  to  $\mathcal{C}$ . Suppose  $\mathcal{F}_1$  is covariant and  $\mathcal{F}_2$  is covariant. Then there exists a strict category  $\mathcal{D}$  and there exists a functor  $\mathcal{P}_1$  from  $\mathcal{D}$  to  $\mathcal{C}_1$  and there exists a functor  $\mathcal{P}_2$  from  $\mathcal{D}$  to  $\mathcal{C}_2$  such that the carrier of  $\mathcal{D} = \{ \langle f_1, f_2 \rangle, \text{ where } f_1 \text{ is a morphism of } \mathcal{C}_1, f_2 \text{ is a morphism of } \mathcal{C}_2 : f_1 \in \text{the carrier of } \mathcal{C}_1 \text{ and } f_2 \in \text{the carrier of } \mathcal{C}_2 \text{ and } \mathcal{F}_1(f_1) = \mathcal{F}_2(f_2) \}$  and the composition of  $\mathcal{D} = \{ \langle \langle f_1, f_2 \rangle, f_3 \rangle, \text{ where } f_1, f_2, f_3 \text{ are morphisms of } \mathcal{D} : f_1, f_2, f_3 \in \text{the carrier of } \mathcal{D} \text{ and for every morphisms } f_{11}, f_{12}, f_{13} \text{ of } \mathcal{C}_1 \text{ and for every morphisms } f_{21}, f_{22}, f_{23} \text{ of } \mathcal{C}_2 \text{ such that } f_1 = \langle f_{11}, f_{21} \rangle \text{ and } f_2 = \langle f_{12}, f_{22} \rangle \text{ and } f_3 = \langle f_{13}, f_{23} \rangle \}$  holds

$f_{11} \triangleright f_{12}$  and  $f_{21} \triangleright f_{22}$  and  $f_{13} = f_{11} \circ f_{12}$  and  $f_{23} = f_{21} \circ f_{22}$  and  $\mathcal{P}_1$  is covariant and  $\mathcal{P}_2$  is covariant and  $\langle \mathcal{D}, \mathcal{P}_1, \mathcal{P}_2 \rangle$  is a pullback of  $\mathcal{F}_1, \mathcal{F}_2$ .

PROOF: Reconsider  $c_7 = \{\langle f_1, f_2 \rangle\}$ , where  $f_1$  is a morphism of  $\mathcal{C}_1, f_2$  is a morphism of  $\mathcal{C}_2 : f_1 \in$  the carrier of  $\mathcal{C}_1$  and  $f_2 \in$  the carrier of  $\mathcal{C}_2$  and  $\mathcal{F}_1(f_1) = \mathcal{F}_2(f_2)$  as a set. Set  $c_8 = \{\langle \langle x_1, x_2 \rangle, x_3 \rangle\}$ , where  $x_1, x_2, x_3$  are elements of  $c_7 : x_1, x_2, x_3 \in c_7$  and for every morphisms  $f_{11}, f_{12}, f_{13}$  of  $\mathcal{C}_1$  and for every morphisms  $f_{21}, f_{22}, f_{23}$  of  $\mathcal{C}_2$  such that  $x_1 = \langle f_{11}, f_{21} \rangle$

and  $x_2 = \langle f_{12}, f_{22} \rangle$  and  $x_3 = \langle f_{13}, f_{23} \rangle$  holds  $f_{11} \triangleright f_{12}$  and  $f_{21} \triangleright f_{22}$  and  $f_{13} = f_{11} \circ f_{12}$  and  $f_{23} = f_{21} \circ f_{22}$ . For every object  $x$  such that  $x \in c_8$  holds  $x \in (c_7 \times c_7) \times c_7$ . For every objects  $x, y_1, y_2$  such that  $\langle x, y_1 \rangle, \langle x, y_2 \rangle \in c_8$  holds  $y_1 = y_2$ . Set  $\mathcal{D} = \langle c_7, c_8 \rangle$ . For every morphisms  $g_1, g_2$  of  $\mathcal{D}$  such that  $g_1 \triangleright g_2$  there exist morphisms  $f_{11}, f_{12}, f_{13}$  of  $\mathcal{C}_1$  and there exist morphisms  $f_{21}, f_{22}, f_{23}$  of  $\mathcal{C}_2$  such that  $g_1 = \langle f_{11}, f_{21} \rangle$  and  $g_2 = \langle f_{12}, f_{22} \rangle$  and  $\mathcal{F}_1(f_{11}) = \mathcal{F}_2(f_{21})$  and  $\mathcal{F}_1(f_{12}) = \mathcal{F}_2(f_{22})$  and  $f_{11} \triangleright f_{12}$  and  $f_{21} \triangleright f_{22}$  and  $f_{13} = f_{11} \circ f_{12}$  and  $f_{23} = f_{21} \circ f_{22}$  and  $g_1 \circ g_2 = \langle f_{13}, f_{23} \rangle$  by (1), [17, (1)], [9, (1)]. For every morphisms  $g_1, g_2$  of  $\mathcal{D}$  such that there exist morphisms  $f_{11}, f_{12}$  of  $\mathcal{C}_1$  and there exist morphisms  $f_{21}, f_{22}$  of  $\mathcal{C}_2$  such that  $g_1 = \langle f_{11}, f_{21} \rangle$  and  $g_2 = \langle f_{12}, f_{22} \rangle$  and  $\mathcal{F}_1(f_{11}) = \mathcal{F}_2(f_{21})$  and  $\mathcal{F}_1(f_{12}) = \mathcal{F}_2(f_{22})$  and  $f_{11} \triangleright f_{12}$  and  $f_{21} \triangleright f_{22}$  holds  $g_1 \triangleright g_2$  by (1), [17, (1)]. For every morphisms  $g, g_1, g_2$  of  $\mathcal{D}$  such that  $g_1 \triangleright g_2$  holds  $g_1 \circ g_2 \triangleright g$  iff  $g_2 \triangleright g$ . For every morphisms  $g, g_1, g_2$  of  $\mathcal{D}$  such that  $g_1 \triangleright g_2$  holds  $g \triangleright g_1 \circ g_2$  iff  $g \triangleright g_1$ . For every morphism  $g_1$  of  $\mathcal{D}$  such that  $g_1 \in$  the carrier of  $\mathcal{D}$  there exists a morphism  $g$  of  $\mathcal{D}$  such that  $g \triangleright g_1$  and  $g$  is left identity by (2), [17, (31), (32)]. For every morphism  $g_1$  of  $\mathcal{D}$  such that  $g_1 \in$  the carrier of  $\mathcal{D}$  there exists a morphism  $g$  of  $\mathcal{D}$  such that  $g_1 \triangleright g$  and  $g$  is right identity by (2), [17, (31), (32)]. For every morphisms  $g_1, g_2, g_3$  of  $\mathcal{D}$  such that  $g_1 \triangleright g_2$  and  $g_2 \triangleright g_3$  and  $g_1 \circ g_2 \triangleright g_3$  and  $g_1 \triangleright g_2 \circ g_3$  holds  $g_1 \circ (g_2 \circ g_3) = (g_1 \circ g_2) \circ g_3$ . For every object  $x, x \in c_8$  iff  $x \in \{\langle \langle f_1, f_2 \rangle, f_3 \rangle\}$ , where  $f_1, f_2, f_3$  are morphisms of  $\mathcal{D} : f_1, f_2, f_3 \in$  the carrier of  $\mathcal{D}$  and for every morphisms  $f_{11}, f_{12}, f_{13}$  of  $\mathcal{C}_1$  and for every morphisms  $f_{21}, f_{22}, f_{23}$  of  $\mathcal{C}_2$  such that  $f_1 = \langle f_{11}, f_{21} \rangle$  and  $f_2 = \langle f_{12}, f_{22} \rangle$  and  $f_3 = \langle f_{13}, f_{23} \rangle$  holds  $f_{11} \triangleright f_{12}$  and  $f_{21} \triangleright f_{22}$  and  $f_{13} = f_{11} \circ f_{12}$  and  $f_{23} = f_{21} \circ f_{22}$ . There exists a functor  $\mathcal{P}_1$  from  $\mathcal{D}$  to  $\mathcal{C}_1$  and there exists a functor  $\mathcal{P}_2$  from  $\mathcal{D}$  to  $\mathcal{C}_2$  such that  $\mathcal{P}_1$  is covariant and  $\mathcal{P}_2$  is covariant and  $\mathcal{F}_1 \circ \mathcal{P}_1 = \mathcal{F}_2 \circ \mathcal{P}_2$  and for every category  $\mathcal{D}_1$  and for every functor  $\mathcal{G}_1$  from  $\mathcal{D}_1$  to  $\mathcal{C}_1$  and for every functor  $\mathcal{G}_2$  from  $\mathcal{D}_1$  to  $\mathcal{C}_2$  such that  $\mathcal{G}_1$  is covariant and  $\mathcal{G}_2$  is covariant and  $\mathcal{F}_1 \circ \mathcal{G}_1 = \mathcal{F}_2 \circ \mathcal{G}_2$  there exists a functor  $\mathcal{H}$  from  $\mathcal{D}_1$  to  $\mathcal{D}$  such that  $\mathcal{H}$  is covariant and  $\mathcal{P}_1 \circ \mathcal{H} = \mathcal{G}_1$  and  $\mathcal{P}_2 \circ \mathcal{H} = \mathcal{G}_2$  and for every functor  $\mathcal{H}_1$  from  $\mathcal{D}_1$  to  $\mathcal{D}$  such that  $\mathcal{H}_1$  is covariant and  $\mathcal{P}_1 \circ \mathcal{H}_1 = \mathcal{G}_1$  and  $\mathcal{P}_2 \circ \mathcal{H}_1 = \mathcal{G}_2$  holds  $\mathcal{H} = \mathcal{H}_1$  by [17, (31)], [9, (13)], (1), [17, (32), (34)]. Consider  $\mathcal{P}_1$

being a functor from  $\mathcal{D}$  to  $\mathcal{C}_1$ ,  $\mathcal{P}_2$  being a functor from  $\mathcal{D}$  to  $\mathcal{C}_2$  such that  $\mathcal{P}_1$  is covariant and  $\mathcal{P}_2$  is covariant and  $\mathcal{F}_1 \circ \mathcal{P}_1 = \mathcal{F}_2 \circ \mathcal{P}_2$  and for every category  $\mathcal{D}_1$  and for every functor  $\mathcal{G}_1$  from  $\mathcal{D}_1$  to  $\mathcal{C}_1$  and for every functor  $\mathcal{G}_2$  from  $\mathcal{D}_1$  to  $\mathcal{C}_2$  such that  $\mathcal{G}_1$  is covariant and  $\mathcal{G}_2$  is covariant and  $\mathcal{F}_1 \circ \mathcal{G}_1 = \mathcal{F}_2 \circ \mathcal{G}_2$  there exists a functor  $\mathcal{H}$  from  $\mathcal{D}_1$  to  $\mathcal{D}$  such that  $\mathcal{H}$  is covariant and  $\mathcal{P}_1 \circ \mathcal{H} = \mathcal{G}_1$  and  $\mathcal{P}_2 \circ \mathcal{H} = \mathcal{G}_2$  and for every functor  $\mathcal{H}_1$  from  $\mathcal{D}_1$  to  $\mathcal{D}$  such that  $\mathcal{H}_1$  is covariant and  $\mathcal{P}_1 \circ \mathcal{H}_1 = \mathcal{G}_1$  and  $\mathcal{P}_2 \circ \mathcal{H}_1 = \mathcal{G}_2$  holds  $\mathcal{H} = \mathcal{H}_1$ .  $\square$

Let  $\mathcal{C}$ ,  $\mathcal{C}_1$ ,  $\mathcal{C}_2$  be categories and  $\mathcal{F}_1$  be a functor from  $\mathcal{C}_1$  to  $\mathcal{C}$ . Assume  $\mathcal{F}_1$  is covariant. Let  $\mathcal{F}_2$  be a functor from  $\mathcal{C}_2$  to  $\mathcal{C}$ . Assume  $\mathcal{F}_2$  is covariant.

A pullback of  $\mathcal{F}_1, \mathcal{F}_2$  is a triple object and is defined by

(Def. 21) there exists a strict category  $\mathcal{D}$  and there exists a functor  $\mathcal{P}_1$  from  $\mathcal{D}$  to  $\mathcal{C}_1$  and there exists a functor  $\mathcal{P}_2$  from  $\mathcal{D}$  to  $\mathcal{C}_2$  such that  $it = \langle \mathcal{D}, \mathcal{P}_1, \mathcal{P}_2 \rangle$  and  $\mathcal{P}_1$  is covariant and  $\mathcal{P}_2$  is covariant and  $\langle \mathcal{D}, \mathcal{P}_1, \mathcal{P}_2 \rangle$  is a pullback of  $\mathcal{F}_1, \mathcal{F}_2$ .

Assume  $\mathcal{F}_1$  is covariant. Assume  $\mathcal{F}_2$  is covariant. The functor  $[[\mathcal{F}_1, \mathcal{F}_2]]$  yielding a strict category is defined by the term

(Def. 22) the pullback of  $\mathcal{F}_1, \mathcal{F}_{21,3}$ .

Assume  $\mathcal{F}_1$  is covariant. Assume  $\mathcal{F}_2$  is covariant. The functor  $\pi_1(\mathcal{F}_1 \boxtimes \mathcal{F}_2)$  yielding a functor from  $[[\mathcal{F}_1, \mathcal{F}_2]]$  to  $\mathcal{C}_1$  is defined by the term

(Def. 23) the pullback of  $\mathcal{F}_1, \mathcal{F}_{22,3}$ .

The functor  $\pi_2(\mathcal{F}_1 \boxtimes \mathcal{F}_2)$  yielding a functor from  $[[\mathcal{F}_1, \mathcal{F}_2]]$  to  $\mathcal{C}_2$  is defined by the term

(Def. 24) the pullback of  $\mathcal{F}_1, \mathcal{F}_{23,3}$ .

Let us consider categories  $\mathcal{C}, \mathcal{C}_1, \mathcal{C}_2$ , a functor  $\mathcal{F}_1$  from  $\mathcal{C}_1$  to  $\mathcal{C}$ , and a functor  $\mathcal{F}_2$  from  $\mathcal{C}_2$  to  $\mathcal{C}$ . Let us assume that  $\mathcal{F}_1$  is covariant and  $\mathcal{F}_2$  is covariant. Now we state the propositions:

(52) (i)  $\pi_1(\mathcal{F}_1 \boxtimes \mathcal{F}_2)$  is covariant, and

(ii)  $\pi_2(\mathcal{F}_1 \boxtimes \mathcal{F}_2)$  is covariant, and

(iii)  $\langle [[\mathcal{F}_1, \mathcal{F}_2]], \pi_1(\mathcal{F}_1 \boxtimes \mathcal{F}_2), \pi_2(\mathcal{F}_1 \boxtimes \mathcal{F}_2) \rangle$  is a pullback of  $\mathcal{F}_1, \mathcal{F}_2$ .

(53)  $[[\mathcal{F}_1, \mathcal{F}_2]] \cong [[\mathcal{F}_2, \mathcal{F}_1]]$ . The theorem is a consequence of (52), (47), and (46).

(54) There exist object-categories  $\mathcal{C}, \mathcal{C}_1, \mathcal{C}_2$  and there exists a functor  $\mathcal{F}_1$  from  $\mathcal{C}_1$  to  $\mathcal{C}$  and there exists a functor  $\mathcal{F}_2$  from  $\mathcal{C}_2$  to  $\mathcal{C}$  such that there exists no object-category  $\mathcal{D}$  and there exists a functor  $\mathcal{P}_1$  from  $\mathcal{D}$  to  $\mathcal{C}_1$  and there exists a functor  $\mathcal{P}_2$  from  $\mathcal{D}$  to  $\mathcal{C}_2$  such that  $\mathcal{F}_1 \cdot \mathcal{P}_1 = \mathcal{F}_2 \cdot \mathcal{P}_2$  and for every object-category  $\mathcal{D}_1$  and for every functor  $\mathcal{G}_1$  from  $\mathcal{D}_1$  to  $\mathcal{C}_1$  and for every functor  $\mathcal{G}_2$  from  $\mathcal{D}_1$  to  $\mathcal{C}_2$  such that  $\mathcal{F}_1 \cdot \mathcal{G}_1 = \mathcal{F}_2 \cdot \mathcal{G}_2$  there exists a functor  $\mathcal{H}$  from  $\mathcal{D}_1$  to  $\mathcal{D}$  such that  $\mathcal{P}_1 \cdot \mathcal{H} = \mathcal{G}_1$  and  $\mathcal{P}_2 \cdot \mathcal{H} = \mathcal{G}_2$  and for

every functor  $\mathcal{H}_1$  from  $\mathcal{D}_1$  to  $\mathcal{D}$  such that  $\mathcal{P}_1 \cdot \mathcal{H}_1 = \mathcal{G}_1$  and  $\mathcal{P}_2 \cdot \mathcal{H}_1 = \mathcal{G}_2$  holds  $\mathcal{H} = \mathcal{H}_1$ . The theorem is a consequence of (39) and (40).

## REFERENCES

- [1] Jiri Adamek, Horst Herrlich, and George E. Strecker. *Abstract and Concrete Categories: The Joy of Cats*. Dover Publication, New York, 2009.
- [2] Grzegorz Bancerek. Cardinal numbers. *Formalized Mathematics*, 1(2):377–382, 1990.
- [3] Grzegorz Bancerek. The ordinal numbers. *Formalized Mathematics*, 1(1):91–96, 1990.
- [4] Grzegorz Bancerek. The well ordering relations. *Formalized Mathematics*, 1(1):123–129, 1990.
- [5] Grzegorz Bancerek. Zermelo theorem and axiom of choice. *Formalized Mathematics*, 1(2):265–267, 1990.
- [6] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Formalized Mathematics*, 1(1):107–114, 1990.
- [7] Francis Borceaux. *Handbook of Categorical Algebra I. Basic Category Theory*, volume 50 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1994.
- [8] Czesław Byliński. Introduction to categories and functors. *Formalized Mathematics*, 1(2):409–420, 1990.
- [9] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(1):55–65, 1990.
- [10] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(1):153–164, 1990.
- [11] Czesław Byliński. Partial functions. *Formalized Mathematics*, 1(2):357–367, 1990.
- [12] Czesław Byliński. Some basic properties of sets. *Formalized Mathematics*, 1(1):47–53, 1990.
- [13] Agata Darmochwał. Finite sets. *Formalized Mathematics*, 1(1):165–167, 1990.
- [14] F. William Lawvere. Functorial semantics of algebraic theories and some algebraic problems in the context of functorial semantics of algebraic theories. *Reprints in Theory and Applications of Categories*, 5:1–121, 2004.
- [15] Saunders Mac Lane. *Categories for the Working Mathematician*, volume 5 of *Graduate Texts in Mathematics*. Springer Verlag, New York, Heidelberg, Berlin, 1971.
- [16] Beata Padlewska. Families of sets. *Formalized Mathematics*, 1(1):147–152, 1990.
- [17] Marco Riccardi. Object-free definition of categories. *Formalized Mathematics*, 21(3):193–205, 2013. doi:10.2478/forma-2013-0021.
- [18] Andrzej Trybulec. Enumerated sets. *Formalized Mathematics*, 1(1):25–34, 1990.
- [19] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(1):67–71, 1990.
- [20] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1(1):73–83, 1990.
- [21] Edmund Woronowicz. Relations defined on sets. *Formalized Mathematics*, 1(1):181–186, 1990.

*Received December 31, 2014*