FORMALIZED MATHEMATICS
Vol. 23, No. 1, Pages 1–14, 2015
DOI: 10.2478/forma-2015-0001

DE GRUYTER OPEN
degruyter.com/view/j/forma

Categorical Pullbacks

Marco Riccardi
Via del Pero 102
54038 Montignoso
Italy

Summary. The main purpose of this article is to introduce the categorical concept of pullback in Mizar. In the first part of this article we redefine homsets, monomorphisms, epimorpshisms and isomorphisms [7] within a free-object category [1] and it is shown there that ordinal numbers can be considered as categories. Then the pullback is introduced in terms of its universal property and the Pullback Lemma is formalized [15]. In the last part of the article we formalize the pullback of functors [14] and it is also shown that it is not possible to write an equivalent definition in the context of the previous Mizar formalization of category theory [8].

MSC: 18A30 03B35

Keywords: category pullback; pullback lemma

MML identifier: `CAT_7`, version: `8.1.03 5.29.1227`

The notation and terminology used in this paper have been introduced in the following articles: [2], [8], [17], [18], [6], [13], [9], [10], [3], [11], [20], [21], [16], [19], [4], [5], and [12].

1. Preliminaries

One can verify that every set which is ordinal is also non pair.

Let $\mathscr{C}$ be an empty category structure. Let us note that $\mathrm{Mor}\,\mathscr{C}$ is empty.

Let $\mathscr{C}$ be a non empty category structure. Note that $\mathrm{Mor}\,\mathscr{C}$ is non empty.

Let $\mathscr{C}$ be an empty category structure with identities. Let us note that $\mathrm{Ob}\,\mathscr{C}$ is empty.

Let $\mathscr{C}$ be a non empty category structure with identities. Observe that $\mathrm{Ob}\,\mathscr{C}$ is non empty.

ISSN 1426–2630(Print), 1898-9934(Online)

Let $\mathscr{C}$ be category structure with identities and a be an object of $\mathscr{C}$. One can check that id-a is identity.

Now we state the propositions:

(1) Let us consider a category structure $\mathscr{C}$, and a morphism f of $\mathscr{C}$. Suppose $\mathscr{C}$ is not empty. Then $f \in$ the carrier of $\mathscr{C}$.

(2) Let us consider category structure $\mathscr{C}$ with identities, and an object a of $\mathscr{C}$. Suppose $\mathscr{C}$ is not empty. Then $a \in$ the carrier of $\mathscr{C}$.

(3) Let us consider a composable category structure $\mathscr{C}$, and morphisms f_1, f_2, f_3 of $\mathscr{C}$. Suppose $f_1 \rhd f_2$ and $f_2 \rhd f_3$ and f_2 is identity. Then $f_1 \rhd f_3$.

(4) Let us consider a composable category structure $\mathscr{C}$ with identities, and morphisms f_1, f_2 of $\mathscr{C}$. Suppose $f_1 \rhd f_2$. Then

  (i) $\operatorname{dom}(f_1 \circ f_2) = \operatorname{dom} f_2$, and

  (ii) $\operatorname{cod}(f_1 \circ f_2) = \operatorname{cod} f_1$.

(5) Let us consider a non empty, composable category structure $\mathscr{C}$ with identities, and morphisms f_1, f_2 of $\mathscr{C}$. Then $f_1 \rhd f_2$ if and only if $\operatorname{dom} f_1 = \operatorname{cod} f_2$.

(6) Let us consider a composable category structure $\mathscr{C}$ with identities, and a morphism f of $\mathscr{C}$. If f is identity, then $\operatorname{dom} f = f$ and $\operatorname{cod} f = f$.

(7) Let us consider a composable category structure $\mathscr{C}$ with identities, and morphisms f_1, f_2 of $\mathscr{C}$. Suppose $f_1 \rhd f_2$ and f_1 is identity and f_2 is identity. Then $f_1 = f_2$.

Let us consider a non empty, composable category structure $\mathscr{C}$ with identities and morphisms f_1, f_2 of $\mathscr{C}$. Now we state the propositions:

(8) If $\operatorname{dom} f_1 = f_2$, then $f_1 \rhd f_2$ and $f_1 \circ f_2 = f_1$.

(9) If $f_1 = \operatorname{cod} f_2$, then $f_1 \rhd f_2$ and $f_1 \circ f_2 = f_2$.

Now we state the propositions:

(10) Let us consider categories $\mathscr{C}_1$, $\mathscr{C}_2$, $\mathscr{C}_3$, $\mathscr{C}_4$, a functor $\mathcal{F}$ from $\mathscr{C}_1$ to $\mathscr{C}_2$, a functor $\mathcal{G}$ from $\mathscr{C}_2$ to $\mathscr{C}_3$, and a functor $\mathcal{H}$ from $\mathscr{C}_3$ to $\mathscr{C}_4$. Suppose $\mathcal{F}$ is covariant and $\mathcal{G}$ is covariant and $\mathcal{H}$ is covariant. Then $\mathcal{H} \circ (\mathcal{G} \circ \mathcal{F}) = (\mathcal{H} \circ \mathcal{G}) \circ \mathcal{F}$.

(11) Let us consider categories $\mathscr{C}$, $\mathscr{D}$, and a functor $\mathcal{F}$ from $\mathscr{C}$ to $\mathscr{D}$. Suppose $\mathcal{F}$ is covariant. Then

  (i) $\mathcal{F} \circ \mathrm{id}_{\mathscr{C}} = \mathcal{F}$, and

  (ii) $\mathrm{id}_{\mathscr{D}} \circ \mathcal{F} = \mathcal{F}$.

(12) Let us consider composable category structures $\mathscr{C}$, $\mathscr{D}$ with identities. Then $\mathscr{C} \cong \mathscr{D}$ if and only if there exists a functor $\mathcal{F}$ from $\mathscr{C}$ to $\mathscr{D}$ such that $\mathcal{F}$ is covariant and bijective. The theorem is a consequence of (5).

(13) Let us consider empty category structures $\mathscr{C}$, $\mathscr{D}$ with identities. Then $\mathscr{C} \cong \mathscr{D}$.

Let us consider category structures $\mathscr{C}$, $\mathscr{D}$ with identities. Now we state the propositions:

(14) Suppose $\mathscr{C} \cong \mathscr{D}$. Then

(i) $\overline{\overline{\mathrm{Mor}\,\mathscr{C}}} = \overline{\overline{\mathrm{Mor}\,\mathscr{D}}}$, and

(ii) $\overline{\overline{\mathrm{Ob}\,\mathscr{C}}} = \overline{\overline{\mathrm{Ob}\,\mathscr{D}}}$.

(15) If $\mathscr{C} \cong \mathscr{D}$ and $\mathscr{C}$ is empty, then $\mathscr{D}$ is empty. The theorem is a consequence of (14).

2. Hom-sets

Let $\mathscr{C}$ be a category structure and a, b be objects of $\mathscr{C}$. The functor $\hom(a, b)$ yielding a subset of $\mathrm{Mor}\,\mathscr{C}$ is defined by the term

(Def. 1) $\{f$, where f is a morphism of $\mathscr{C}$: there exist morphisms f_1, f_2 of $\mathscr{C}$ such that $a = f_1$ and $b = f_2$ and $f \triangleright f_1$ and $f_2 \triangleright f\}$.

Let $\mathscr{C}$ be a non empty, composable category structure with identities. Observe that the functor $\hom(a, b)$ yields a subset of $\mathrm{Mor}\,\mathscr{C}$ and is defined by the term

(Def. 2) $\{f$, where f is a morphism of $\mathscr{C}$: $\mathrm{dom}\, f = a$ and $\mathrm{cod}\, f = b\}$.

Let $\mathscr{C}$ be a category structure. Assume $\hom(a, b) \neq \emptyset$.

A morphism from a to b is a morphism of $\mathscr{C}$ and is defined by

(Def. 3) $it \in \hom(a, b)$.

Let $\mathscr{C}$ be category structure with identities and a be an object of $\mathscr{C}$. Assume $\hom(a, a) \neq \emptyset$. Observe that the functor id-a yields a morphism from a to a. Let $\mathscr{C}$ be a non empty category structure with identities. Note that $\hom(a, a)$ is non empty.

Let $\mathscr{C}$ be a composable category structure with identities, a, b, c be objects of $\mathscr{C}$, f be a morphism from a to b, and g be a morphism from b to c. Assume $\hom(a, b) \neq \emptyset$ and $\hom(b, c) \neq \emptyset$. The functor $g \cdot f$ yielding a morphism from a to c is defined by the term

(Def. 4) $g \circ f$.

Now we state the propositions:

(16) Let us consider a category structure $\mathscr{C}$, objects a, b of $\mathscr{C}$, and a morphism f from a to b. Suppose $\hom(a, b) \neq \emptyset$. Then there exist morphisms f_1, f_2 of $\mathscr{C}$ such that

(i) $a = f_1$, and

(ii) $b = f_2$, and

(iii) $f \triangleright f_1$, and

(iv) $f_2 \triangleright f$.

(17) Let us consider a composable category structure $\mathscr{C}$ with identities, objects a, b, c of $\mathscr{C}$, a morphism f_1 from a to b, and a morphism f_2 from b to c. Suppose $\hom(a, b) \neq \emptyset$ and $\hom(b, c) \neq \emptyset$. Then $f_2 \triangleright f_1$. The theorem is a consequence of (16) and (3).

(18) Let us consider a composable category structure $\mathscr{C}$ with identities, objects a, b of $\mathscr{C}$, and a morphism f from a to b. Suppose $\hom(a, b) \neq \emptyset$. Then

(i) $f \cdot \text{id-}a = f$, and

(ii) $\text{id-}b \cdot f = f$.

The theorem is a consequence of (17).

(19) Let us consider a non empty, composable category structure $\mathscr{C}$ with identities, and a morphism f of $\mathscr{C}$. Then $f \in \hom(\operatorname{dom} f, \operatorname{cod} f)$.

(20) Let us consider a non empty, composable category structure $\mathscr{C}$ with identities, objects a, b of $\mathscr{C}$, and a morphism f of $\mathscr{C}$. Then $f \in \hom(a, b)$ if and only if $\operatorname{dom} f = a$ and $\operatorname{cod} f = b$.

(21) Let us consider a non empty, composable category structure $\mathscr{C}$ with identities, and an object a of $\mathscr{C}$. Then $a \in \hom(a, a)$. The theorem is a consequence of (6).

(22) Let us consider a composable category structure $\mathscr{C}$ with identities, and objects a, b, c of $\mathscr{C}$. Suppose $\hom(a, b) \neq \emptyset$ and $\hom(b, c) \neq \emptyset$. Then $\hom(a, c) \neq \emptyset$. The theorem is a consequence of (16) and (3).

(23) Let us consider a category $\mathscr{C}$, objects a, b, c, d of $\mathscr{C}$, a morphism f_1 from a to b, a morphism f_2 from b to c, and a morphism f_3 from c to d. Suppose $\hom(a, b) \neq \emptyset$ and $\hom(b, c) \neq \emptyset$ and $\hom(c, d) \neq \emptyset$. Then $f_3 \cdot (f_2 \cdot f_1) = (f_3 \cdot f_2) \cdot f_1$. The theorem is a consequence of (22) and (17).

(24) Let us consider a composable category structure $\mathscr{C}$ with identities, objects a, b, c of $\mathscr{C}$, a morphism f_1 from a to b, and a morphism f_2 from b to c. Suppose $\hom(a, b) \neq \emptyset$ and $\hom(b, c) \neq \emptyset$. Then

(i) if f_1 is identity, then $f_2 \cdot f_1 = f_2$, and

(ii) if f_2 is identity, then $f_2 \cdot f_1 = f_1$.

PROOF: $f_2 \triangleright f_1$. If f_1 is identity, then $f_2 \cdot f_1 = f_2$ by [17, (22), (23)]. $\square$

3. Monomorphisms, Epimorphisms and Isomorphisms

Let $\mathscr{C}$ be a composable category structure with identities, a, b be objects of $\mathscr{C}$, and f be a morphism from a to b. We say that f is monomorphic if and only if

(Def. 5) $\hom(a,b) \neq \emptyset$ and for every object c of $\mathscr{C}$ such that $\hom(c,a) \neq \emptyset$ for every morphisms g_1, g_2 from c to a such that $f \cdot g_1 = f \cdot g_2$ holds $g_1 = g_2$.

We say that f is epimorphic if and only if

(Def. 6) $\hom(a,b) \neq \emptyset$ and for every object c of $\mathscr{C}$ such that $\hom(b,c) \neq \emptyset$ for every morphisms g_1, g_2 from b to c such that $g_1 \cdot f = g_2 \cdot f$ holds $g_1 = g_2$.

Now we state the proposition:

(25) Let us consider a composable category structure $\mathscr{C}$ with identities, objects a, b of $\mathscr{C}$, and a morphism f_1 from a to b. Suppose $\hom(a,b) \neq \emptyset$ and f_1 is identity. Then f_1 is monomorphic. The theorem is a consequence of (24).

Let us consider a category $\mathscr{C}$, objects a, b, c of $\mathscr{C}$, a morphism f_1 from a to b, and a morphism f_2 from b to c. Now we state the propositions:

(26) If f_1 is monomorphic and f_2 is monomorphic, then $f_2 \cdot f_1$ is monomorphic. The theorem is a consequence of (22) and (23).

(27) If $f_2 \cdot f_1$ is monomorphic and $\hom(a,b) \neq \emptyset$ and $\hom(b,c) \neq \emptyset$, then f_1 is monomorphic. The theorem is a consequence of (23).

Let $\mathscr{C}$ be a composable category structure with identities, a, b be objects of $\mathscr{C}$, and f be a morphism from a to b. We say that f is a section if and only if

(Def. 7) $\hom(a,b) \neq \emptyset$ and $\hom(b,a) \neq \emptyset$ and there exists a morphism g from b to a such that $g \cdot f = \mathrm{id}\text{-}a$.

We say that f is a retraction if and only if

(Def. 8) $\hom(a,b) \neq \emptyset$ and $\hom(b,a) \neq \emptyset$ and there exists a morphism g from b to a such that $f \cdot g = \mathrm{id}\text{-}b$.

Now we state the propositions:

(28) Let us consider a category $\mathscr{C}$, objects a, b of $\mathscr{C}$, and a morphism f from a to b. If f is a section, then f is monomorphic. The theorem is a consequence of (23) and (18).

(29) Let us consider a composable category structure $\mathscr{C}$ with identities, objects a, b of $\mathscr{C}$, and a morphism f_1 from a to b. Suppose $\hom(a,b) \neq \emptyset$ and f_1 is identity. Then f_1 is epimorphic. The theorem is a consequence of (24).

Let us consider a category $\mathscr{C}$, objects a, b, c of $\mathscr{C}$, a morphism f_1 from a to b, and a morphism f_2 from b to c. Now we state the propositions:

(30) If f_1 is epimorphic and f_2 is epimorphic, then $f_2 \cdot f_1$ is epimorphic. The theorem is a consequence of (22) and (23).

(31) If $f_2 \cdot f_1$ is epimorphic and $\hom(a, b) \neq \emptyset$ and $\hom(b, c) \neq \emptyset$, then f_2 is epimorphic. The theorem is a consequence of (23).

(32) Let us consider a category $\mathscr{C}$, objects a, b of $\mathscr{C}$, and a morphism f from a to b. If f is a retraction, then f is epimorphic. The theorem is a consequence of (23) and (18).

Let $\mathscr{C}$ be a composable category structure with identities, a, b be objects of $\mathscr{C}$, and f be a morphism from a to b. We say that f is isomorphism if and only if

(Def. 9) $\hom(a, b) \neq \emptyset$ and $\hom(b, a) \neq \emptyset$ and there exists a morphism g from b to a such that $g \cdot f = \text{id-}a$ and $f \cdot g = \text{id-}b$.

We say that a and b are isomorphic if and only if

(Def. 10) there exists a morphism f from a to b such that f is isomorphism.

Note that a and b are isomorphic if and only if the condition (Def. 11) is satisfied.

(Def. 11) $\hom(a, b) \neq \emptyset$ and $\hom(b, a) \neq \emptyset$ and there exists a morphism f from a to b and there exists a morphism g from b to a such that $g \cdot f = \text{id-}a$ and $f \cdot g = \text{id-}b$.

Now we state the proposition:

(33) Let us consider a category $\mathscr{C}$, objects a, b of $\mathscr{C}$, and a morphism f from a to b. If f is isomorphism, then f is monomorphic and epimorphic. The theorem is a consequence of (28) and (32).

4. Ordinal Numbers as Categories

Let $\mathscr{C}$ be a category structure. We say that $\mathscr{C}$ is a preorder if and only if

(Def. 12) for every objects a, b of $\mathscr{C}$ and for every morphisms f_1, f_2 of $\mathscr{C}$ such that f_1, $f_2 \in \hom(a, b)$ holds $f_1 = f_2$.

Observe that every category structure which is empty is also a preorder and there exists a category structure which is strict and preorder and every composable category structure with identities which is a preorder is also associative.

Let $\mathscr{C}$ be category structure with identities. The functor $\mathrm{RelOb}\,\mathscr{C}$ yielding a binary relation on $\mathrm{Ob}\,\mathscr{C}$ is defined by the term

(Def. 13) $\{\langle a, b\rangle$, where a, b are objects of $\mathscr{C}$: there exists a morphism f of $\mathscr{C}$ such that $f \in \hom(a, b)\}$.

Let $\mathscr{C}$ be an empty category structure with identities. Let us note that $\mathrm{RelOb}\,\mathscr{C}$ is empty.

Now we state the propositions:

(34) Let us consider a composable category structure $\mathscr{C}$ with identities. Then

(i) $\operatorname{dom}\operatorname{RelOb}\mathscr{C} = \operatorname{Ob}\mathscr{C}$, and

(ii) $\operatorname{rng}\operatorname{RelOb}\mathscr{C} = \operatorname{Ob}\mathscr{C}$.

The theorem is a consequence of (6) and (19).

(35) Let us consider composable category structures $\mathscr{C}_1$, $\mathscr{C}_2$ with identities. Suppose $\mathscr{C}_1 \cong \mathscr{C}_2$. Then $\operatorname{RelOb}\mathscr{C}_1$ and $\operatorname{RelOb}\mathscr{C}_2$ are isomorphic. The theorem is a consequence of (15), (34), and (20).

Let $\mathscr{C}$ be a non empty, composable category structure with identities. One can verify that $\operatorname{RelOb}\mathscr{C}$ is non empty.

Now we state the propositions:

(36) Let us consider preorder, composable category structure $\mathscr{C}$ with identities. Suppose $\mathscr{C}$ is not empty. Then there exists a function $\mathcal{F}$ from $\mathscr{C}$ into $\operatorname{RelOb}\mathscr{C}$ such that

(i) $\mathcal{F}$ is bijective, and

(ii) for every morphism f of $\mathscr{C}$, $\mathcal{F}(f) = \langle \operatorname{dom} f, \operatorname{cod} f \rangle$.

Proof: Reconsider $\mathscr{C}_1 = \mathscr{C}$ as a non empty, composable category structure with identities. Define $\mathcal{P}[\text{object}, \text{object}] \equiv$ for every morphism f of $\mathscr{C}_1$ such that $\$_1 = f$ holds $\$_2 = \langle \operatorname{dom} f, \operatorname{cod} f \rangle$. For every element x of the carrier of $\mathscr{C}_1$, there exists an element y of $\operatorname{RelOb}\mathscr{C}_1$ such that $\mathcal{P}[x, y]$. Consider $\mathcal{F}$ being a function from the carrier of $\mathscr{C}_1$ into $\operatorname{RelOb}\mathscr{C}_1$ such that for every element x of the carrier of $\mathscr{C}_1$, $\mathcal{P}[x, \mathcal{F}(x)]$ from [10, Sch. 3]. For every object y such that $y \in \operatorname{RelOb}\mathscr{C}$ holds $y \in \operatorname{rng}\mathcal{F}$ by (20), [9, (3)]. For every objects x_1, x_2 such that x_1, $x_2 \in \operatorname{dom}\mathcal{F}$ and $\mathcal{F}(x_1) = \mathcal{F}(x_2)$ holds $x_1 = x_2$. $\square$

(37) Let us consider an ordinal number O. Then there exists a strict, a preorder category $\mathscr{C}$ such that

(i) $\operatorname{Ob}\mathscr{C} = O$, and

(ii) for every objects o_1, o_2 of $\mathscr{C}$ such that $o_1 \in o_2$ holds $\hom(o_1, o_2) = \{\langle o_1, o_2 \rangle\}$, and

(iii) $\operatorname{RelOb}\mathscr{C} = {\subseteq}_O$, and

(iv) $\operatorname{Mor}\mathscr{C} = O \cup \{\langle o_1, o_2 \rangle$, where o_1, o_2 are elements of $O : o_1 \in o_2\}$.

The theorem is a consequence of (6), (20), and (21).

Let O be an ordinal number and $\mathscr{C}$ be a composable category structure with identities. We say that $\mathscr{C}$ is O-ordered if and only if

(Def. 14) $\operatorname{RelOb}\mathscr{C}$ and ${\subseteq}_O$ are isomorphic.

Let O be a non empty, ordinal number. Let us observe that every composable category structure with identities which is O-ordered is also non empty.

Let O be an ordinal number. Note that there exists a composable category structure with identities which is strict, O-ordered, and preorder.

Let O be an empty, ordinal number. Let us observe that every composable category structure with identities which is O-ordered is also empty.

Now we state the proposition:

(38) Let us consider ordinal numbers O_1, O_2, a O_1-ordered, a preorder category $\mathscr{C}_1$, and a O_2-ordered, a preorder category $\mathscr{C}_2$. Then $O_1 = O_2$ if and only if $\mathscr{C}_1 \cong \mathscr{C}_2$.

PROOF: If $O_1 = O_2$, then $\mathscr{C}_1 \cong \mathscr{C}_2$ by (13), [4, (39), (41)], (36). If $\mathscr{C}_1 \cong \mathscr{C}_2$, then $O_1 = O_2$ by (35), [4, (42), (40)], [5, (10)]. □

Let O be an ordinal number. The functor $\mathbf{O}$ yielding a strict, O-ordered, a preorder category is defined by the term

(Def. 15) the strict, O-ordered, a preorder category.

Now we state the proposition:

(39) There exists a morphism f of $\mathbf{2}$ such that

(i) f is not identity, and

(ii) $\mathrm{Ob}\,\mathbf{2} = \{\mathrm{dom}\,f, \mathrm{cod}\,f\}$, and

(iii) $\mathrm{Mor}\,\mathbf{2} = \{\mathrm{dom}\,f, \mathrm{cod}\,f, f\}$, and

(iv) $\mathrm{dom}\,f$, $\mathrm{cod}\,f$, f are mutually different.

PROOF: Consider $\mathscr{C}$ being a strict, a preorder category such that $\mathrm{Ob}\,\mathscr{C} = 2$ and for every objects o_1, o_2 of $\mathscr{C}$ such that $o_1 \in o_2$ holds $\hom(o_1, o_2) = \{\langle o_1, o_2\rangle\}$ and $\mathrm{RelOb}\,\mathscr{C} = {}^{\subseteq}{}_2$ and $\mathrm{Mor}\,\mathscr{C} = 2 \cup \{\langle o_1, o_2\rangle$, where o_1, o_2 are elements of $2 : o_1 \in o_2\}$. $\mathscr{C} \cong \mathbf{2}$. Consider $\mathcal{F}$ being a functor from $\mathscr{C}$ to $\mathbf{2}$, $\mathcal{G}$ being a functor from $\mathbf{2}$ to $\mathscr{C}$ such that $\mathcal{F}$ is covariant and $\mathcal{G}$ is covariant and $\mathcal{G} \circ \mathcal{F} = \mathrm{id}_{\mathscr{C}}$ and $\mathcal{F} \circ \mathcal{G} = \mathrm{id}_{\mathbf{2}}$. Reconsider $g = \langle 0, 1\rangle$ as a morphism of $\mathscr{C}$. g is not identity by [17, (22)]. Set $f = \mathcal{F}(g)$. f is not identity by [9, (18)], [17, (34)]. $\overline{\overline{\mathrm{Ob}\,\mathbf{2}}} = \overline{\overline{2}}$. Consider x, y being objects such that $x \neq y$ and $\mathrm{Ob}\,\mathbf{2} = \{x, y\}$. $\mathrm{dom}\,f \neq \mathrm{cod}\,f$. For every object x, $x \in \mathrm{Mor}\,\mathbf{2}$ iff $x \in \{\mathrm{dom}\,f, \mathrm{cod}\,f, f\}$ by [17, (22)], [9, (18)], [17, (34)], [2, (50), (49)]. □

Let $\mathscr{C}$ be a non empty category and f be a morphism of $\mathscr{C}$. The functor $\mathcal{M}_{\mathrm{f}}$ yielding a covariant functor from $\mathbf{2}$ to $\mathscr{C}$ is defined by

(Def. 16) for every morphism g of $\mathbf{2}$ such that g is not identity holds $it(g) = f$.

Now we state the proposition:

(40) Let us consider a non empty category $\mathscr{C}$, and a morphism f of $\mathscr{C}$. Suppose f is identity. Let us consider a morphism g of $\mathbf{2}$. Then $(\mathcal{M}_{\mathrm{f}})(g) = f$. The theorem is a consequence of (39) and (6).

5. PULLBACKS

Let $\mathscr{C}$ be a category, c, c_1, c_2, d be objects of $\mathscr{C}$, and f_1 be a morphism from c_1 to c. Assume $\hom(c_1, c) \neq \emptyset$. Let f_2 be a morphism from c_2 to c. Assume $\hom(c_2, c) \neq \emptyset$. Let p_1 be a morphism from d to c_1. Assume $\hom(d, c_1) \neq \emptyset$. Let p_2 be a morphism from d to c_2. Assume $\hom(d, c_2) \neq \emptyset$. We say that $\langle d, p_1, p_2 \rangle$ is a pullback of f_1, f_2 if and only if

(Def. 17) $f_1 \cdot p_1 = f_2 \cdot p_2$ and for every object d_1 of $\mathscr{C}$ and for every morphism g_1 from d_1 to c_1 and for every morphism g_2 from d_1 to c_2 such that $\hom(d_1, c_1) \neq \emptyset$ and $\hom(d_1, c_2) \neq \emptyset$ and $f_1 \cdot g_1 = f_2 \cdot g_2$ holds $\hom(d_1, d) \neq \emptyset$ and there exists a morphism h from d_1 to d such that $p_1 \cdot h = g_1$ and $p_2 \cdot h = g_2$ and for every morphism h_1 from d_1 to d such that $p_1 \cdot h_1 = g_1$ and $p_2 \cdot h_1 = g_2$ holds $h = h_1$.

Now we state the proposition:

(41) Let us consider a category $\mathscr{C}$, objects c, c_1, c_2, d, e of $\mathscr{C}$, a morphism f_1 from c_1 to c, a morphism f_2 from c_2 to c, a morphism p_1 from d to c_1, a morphism p_2 from d to c_2, a morphism q_1 from e to c_1, and a morphism q_2 from e to c_2. Suppose $\hom(c_1, c) \neq \emptyset$ and $\hom(c_2, c) \neq \emptyset$ and $\hom(d, c_1) \neq \emptyset$ and $\hom(d, c_2) \neq \emptyset$ and $\hom(e, c_1) \neq \emptyset$ and $\hom(e, c_2) \neq \emptyset$ and $\langle d, p_1, p_2 \rangle$ is a pullback of f_1, f_2 and $\langle e, q_1, q_2 \rangle$ is a pullback of f_1, f_2. Then d and e are isomorphic. The theorem is a consequence of (23) and (18).

Let us consider a category $\mathscr{C}$, objects c, c_1, c_2, d of $\mathscr{C}$, a morphism f_1 from c_1 to c, a morphism f_2 from c_2 to c, a morphism p_1 from d to c_1, and a morphism p_2 from d to c_2. Now we state the propositions:

(42) Suppose $\hom(c_1, c) \neq \emptyset$ and $\hom(c_2, c) \neq \emptyset$ and $\hom(d, c_1) \neq \emptyset$ and $\hom(d, c_2) \neq \emptyset$ and $\langle d, p_1, p_2 \rangle$ is a pullback of f_1, f_2.
Then $\langle d, p_2, p_1 \rangle$ is a pullback of f_2, f_1.

(43) Suppose $\hom(c_1, c) \neq \emptyset$ and $\hom(c_2, c) \neq \emptyset$ and $\hom(d, c_1) \neq \emptyset$ and $\hom(d, c_2) \neq \emptyset$ and $\langle d, p_1, p_2 \rangle$ is a pullback of f_1, f_2 and f_1 is monomorphic. Then p_2 is monomorphic. The theorem is a consequence of (22) and (23).

(44) Suppose $\hom(c_1, c) \neq \emptyset$ and $\hom(c_2, c) \neq \emptyset$ and $\hom(d, c_1) \neq \emptyset$ and $\hom(d, c_2) \neq \emptyset$ and $\langle d, p_1, p_2 \rangle$ is a pullback of f_1, f_2 and f_1 is isomorphism. Then p_2 is isomorphism. The theorem is a consequence of (22), (23), and (18).

(45) Let us consider a category $\mathscr{C}$, objects c_1, c_1, c_2, c_3, c_4, c_5, c_6 of $\mathscr{C}$, a morphism f_1 from c_1 to c_2, a morphism f_2 from c_2 to c_3, a morphism f_3 from c_1 to c_4, a morphism f_4 from c_2 to c_5, a morphism f_5 from

c_3 to c_6, a morphism f_6 from c_4 to c_5, and a morphism f_7 from c_5 to c_6. Suppose $\hom(c_1, c_2) \neq \emptyset$ and $\hom(c_2, c_3) \neq \emptyset$ and $\hom(c_1, c_4) \neq \emptyset$ and $\hom(c_2, c_5) \neq \emptyset$ and $\hom(c_3, c_6) \neq \emptyset$ and $\hom(c_4, c_5) \neq \emptyset$ and $\hom(c_5, c_6) \neq \emptyset$ and $\langle c_2, f_2, f_4 \rangle$ is a pullback of f_5, f_7. Then $\langle c_1, f_1, f_3 \rangle$ is a pullback of f_4, f_6 if and only if $\langle c_1, f_2 \cdot f_1, f_3 \rangle$ is a pullback of $f_5, f_7 \cdot f_6$ and $f_4 \cdot f_1 = f_6 \cdot f_3$. The theorem is a consequence of (22) and (23).

6. Pullbacks of Functors

Let $\mathscr{C}$, $\mathscr{D}$ be categories and $\mathcal{F}$ be a functor from $\mathscr{C}$ to $\mathscr{D}$. We say that $\mathcal{F}$ is monomorphic if and only if

(Def. 18) $\mathcal{F}$ is covariant and for every category $\mathscr{B}$ and for every functors $\mathcal{G}_1$, $\mathcal{G}_2$ from $\mathscr{B}$ to $\mathscr{C}$ such that $\mathcal{G}_1$ is covariant and $\mathcal{G}_2$ is covariant and $\mathcal{F} \circ \mathcal{G}_1 = \mathcal{F} \circ \mathcal{G}_2$ holds $\mathcal{G}_1 = \mathcal{G}_2$.

We say that $\mathcal{F}$ is isomorphism if and only if

(Def. 19) $\mathcal{F}$ is covariant and there exists a functor $\mathcal{G}$ from $\mathscr{D}$ to $\mathscr{C}$ such that $\mathcal{G}$ is covariant and $\mathcal{G} \circ \mathcal{F} = \mathrm{id}_{\mathscr{C}}$ and $\mathcal{F} \circ \mathcal{G} = \mathrm{id}_{\mathscr{D}}$.

Let $\mathscr{C}$, $\mathscr{C}_1$, $\mathscr{C}_2$, $\mathscr{D}$ be categories and $\mathcal{F}_1$ be a functor from $\mathscr{C}_1$ to $\mathscr{C}$. Assume $\mathcal{F}_1$ is covariant. Let $\mathcal{F}_2$ be a functor from $\mathscr{C}_2$ to $\mathscr{C}$. Assume $\mathcal{F}_2$ is covariant. Let $\mathcal{P}_1$ be a functor from $\mathscr{D}$ to $\mathscr{C}_1$. Assume $\mathcal{P}_1$ is covariant. Let $\mathcal{P}_2$ be a functor from $\mathscr{D}$ to $\mathscr{C}_2$. Assume $\mathcal{P}_2$ is covariant. We say that $\langle \mathscr{D}, \mathcal{P}_1, \mathcal{P}_2 \rangle$ is a pullback of $\mathcal{F}_1, \mathcal{F}_2$ if and only if

(Def. 20) $\mathcal{F}_1 \circ \mathcal{P}_1 = \mathcal{F}_2 \circ \mathcal{P}_2$ and for every category $\mathscr{D}_1$ and for every functor $\mathcal{G}_1$ from $\mathscr{D}_1$ to $\mathscr{C}_1$ and for every functor $\mathcal{G}_2$ from $\mathscr{D}_1$ to $\mathscr{C}_2$ such that $\mathcal{G}_1$ is covariant and $\mathcal{G}_2$ is covariant and $\mathcal{F}_1 \circ \mathcal{G}_1 = \mathcal{F}_2 \circ \mathcal{G}_2$ there exists a functor $\mathcal{H}$ from $\mathscr{D}_1$ to $\mathscr{D}$ such that $\mathcal{H}$ is covariant and $\mathcal{P}_1 \circ \mathcal{H} = \mathcal{G}_1$ and $\mathcal{P}_2 \circ \mathcal{H} = \mathcal{G}_2$ and for every functor $\mathcal{H}_1$ from $\mathscr{D}_1$ to $\mathscr{D}$ such that $\mathcal{H}_1$ is covariant and $\mathcal{P}_1 \circ \mathcal{H}_1 = \mathcal{G}_1$ and $\mathcal{P}_2 \circ \mathcal{H}_1 = \mathcal{G}_2$ holds $\mathcal{H} = \mathcal{H}_1$.

Now we state the proposition:

(46) Let us consider categories $\mathscr{C}$, $\mathscr{C}_1$, $\mathscr{C}_2$, $\mathscr{D}$, $\mathscr{E}$, a functor $\mathcal{F}_1$ from $\mathscr{C}_1$ to $\mathscr{C}$, a functor $\mathcal{F}_2$ from $\mathscr{C}_2$ to $\mathscr{C}$, a functor $\mathcal{P}_1$ from $\mathscr{D}$ to $\mathscr{C}_1$, a functor $\mathcal{P}_2$ from $\mathscr{D}$ to $\mathscr{C}_2$, a functor $\mathcal{Q}_1$ from $\mathscr{E}$ to $\mathscr{C}_1$, and a functor $\mathcal{Q}_2$ from $\mathscr{E}$ to $\mathscr{C}_2$. Suppose $\mathcal{F}_1$ is covariant and $\mathcal{F}_2$ is covariant and $\mathcal{P}_1$ is covariant and $\mathcal{P}_2$ is covariant and $\mathcal{Q}_1$ is covariant and $\mathcal{Q}_2$ is covariant and $\langle \mathscr{D}, \mathcal{P}_1, \mathcal{P}_2 \rangle$ is a pullback of $\mathcal{F}_1, \mathcal{F}_2$ and $\langle \mathscr{E}, \mathcal{Q}_1, \mathcal{Q}_2 \rangle$ is a pullback of $\mathcal{F}_1, \mathcal{F}_2$. Then $\mathscr{D} \cong \mathscr{E}$.

Proof: There exists a functor $\mathcal{F}_8$ from $\mathscr{D}$ to $\mathscr{E}$ and there exists a functor $\mathcal{G}_3$ from $\mathscr{E}$ to $\mathscr{D}$ such that $\mathcal{F}_8$ is covariant and $\mathcal{G}_3$ is covariant and $\mathcal{G}_3 \circ \mathcal{F}_8 = \mathrm{id}_{\mathscr{D}}$ and $\mathcal{F}_8 \circ \mathcal{G}_3 = \mathrm{id}_{\mathscr{E}}$ by (10), (11), [17, (35)]. $\square$

Let us consider categories $\mathscr{C}$, $\mathscr{C}_1$, $\mathscr{C}_2$, $\mathscr{D}$, a functor $\mathcal{F}_1$ from $\mathscr{C}_1$ to $\mathscr{C}$, a functor $\mathcal{F}_2$ from $\mathscr{C}_2$ to $\mathscr{C}$, a functor $\mathcal{P}_1$ from $\mathscr{D}$ to $\mathscr{C}_1$, and a functor $\mathcal{P}_2$ from $\mathscr{D}$ to $\mathscr{C}_2$. Now we state the propositions:

(47) Suppose $\mathcal{F}_1$ is covariant and $\mathcal{F}_2$ is covariant and $\mathcal{P}_1$ is covariant and $\mathcal{P}_2$ is covariant and $\langle \mathscr{D}, \mathcal{P}_1, \mathcal{P}_2 \rangle$ is a pullback of $\mathcal{F}_1, \mathcal{F}_2$.
Then $\langle \mathscr{D}, \mathcal{P}_2, \mathcal{P}_1 \rangle$ is a pullback of $\mathcal{F}_2, \mathcal{F}_1$.

(48) Suppose $\mathcal{F}_1$ is covariant and $\mathcal{F}_2$ is covariant and $\mathcal{P}_1$ is covariant and $\mathcal{P}_2$ is covariant and $\langle \mathscr{D}, \mathcal{P}_1, \mathcal{P}_2 \rangle$ is a pullback of $\mathcal{F}_1, \mathcal{F}_2$ and $\mathcal{F}_1$ is monomorphic. Then $\mathcal{P}_2$ is monomorphic.
Proof: For every category $\mathscr{D}_1$ and for every functors $\mathcal{Q}_1$, $\mathcal{Q}_2$ from $\mathscr{D}_1$ to $\mathscr{D}$ such that $\mathcal{Q}_1$ is covariant and $\mathcal{Q}_2$ is covariant and $\mathcal{P}_2 \circ \mathcal{Q}_1 = \mathcal{P}_2 \circ \mathcal{Q}_2$ holds $\mathcal{Q}_1 = \mathcal{Q}_2$ by [17, (35)], (10). □

(49) Suppose $\mathcal{F}_1$ is covariant and $\mathcal{F}_2$ is covariant and $\mathcal{P}_1$ is covariant and $\mathcal{P}_2$ is covariant and $\langle \mathscr{D}, \mathcal{P}_1, \mathcal{P}_2 \rangle$ is a pullback of $\mathcal{F}_1, \mathcal{F}_2$ and $\mathcal{F}_1$ is isomorphism. Then $\mathcal{P}_2$ is isomorphism. The theorem is a consequence of (10) and (11).

(50) Let us consider categories $\mathscr{C}_1$, $\mathscr{C}_2$, $\mathscr{C}_3$, $\mathscr{C}_4$, $\mathscr{C}_5$, $\mathscr{C}_6$, a functor $\mathcal{F}_1$ from $\mathscr{C}_1$ to $\mathscr{C}_2$, a functor $\mathcal{F}_2$ from $\mathscr{C}_2$ to $\mathscr{C}_3$, a functor $\mathcal{F}_3$ from $\mathscr{C}_1$ to $\mathscr{C}_4$, a functor $\mathcal{F}_4$ from $\mathscr{C}_2$ to $\mathscr{C}_5$, a functor $\mathcal{F}_5$ from $\mathscr{C}_3$ to $\mathscr{C}_6$, a functor $\mathcal{F}_6$ from $\mathscr{C}_4$ to $\mathscr{C}_5$, and a functor $\mathcal{F}_7$ from $\mathscr{C}_5$ to $\mathscr{C}_6$. Suppose $\mathcal{F}_1$ is covariant and $\mathcal{F}_2$ is covariant and $\mathcal{F}_3$ is covariant and $\mathcal{F}_4$ is covariant and $\mathcal{F}_5$ is covariant and $\mathcal{F}_6$ is covariant and $\mathcal{F}_7$ is covariant and $\langle \mathscr{C}_2, \mathcal{F}_2, \mathcal{F}_4 \rangle$ is a pullback of $\mathcal{F}_5, \mathcal{F}_7$. Then $\langle \mathscr{C}_1, \mathcal{F}_1, \mathcal{F}_3 \rangle$ is a pullback of $\mathcal{F}_4, \mathcal{F}_6$ if and only if $\langle \mathscr{C}_1, \mathcal{F}_2 \circ \mathcal{F}_1, \mathcal{F}_3 \rangle$ is a pullback of $\mathcal{F}_5, \mathcal{F}_7 \circ \mathcal{F}_6$ and $\mathcal{F}_4 \circ \mathcal{F}_1 = \mathcal{F}_6 \circ \mathcal{F}_3$.
Proof: For every category $\mathscr{D}_1$ and for every functor $\mathcal{G}_1$ from $\mathscr{D}_1$ to $\mathscr{C}_2$ and for every functor $\mathcal{G}_2$ from $\mathscr{D}_1$ to $\mathscr{C}_4$ such that $\mathcal{G}_1$ is covariant and $\mathcal{G}_2$ is covariant and $\mathcal{F}_4 \circ \mathcal{G}_1 = \mathcal{F}_6 \circ \mathcal{G}_2$ there exists a functor $\mathcal{H}$ from $\mathscr{D}_1$ to $\mathscr{C}_1$ such that $\mathcal{H}$ is covariant and $\mathcal{F}_1 \circ \mathcal{H} = \mathcal{G}_1$ and $\mathcal{F}_3 \circ \mathcal{H} = \mathcal{G}_2$ and for every functor $\mathcal{H}_1$ from $\mathscr{D}_1$ to $\mathscr{C}_1$ such that $\mathcal{H}_1$ is covariant and $\mathcal{F}_1 \circ \mathcal{H}_1 = \mathcal{G}_1$ and $\mathcal{F}_3 \circ \mathcal{H}_1 = \mathcal{G}_2$ holds $\mathcal{H} = \mathcal{H}_1$ by [17, (35)], (10). □

(51) Let us consider categories $\mathscr{C}$, $\mathscr{C}_1$, $\mathscr{C}_2$, a functor $\mathcal{F}_1$ from $\mathscr{C}_1$ to $\mathscr{C}$, and a functor $\mathcal{F}_2$ from $\mathscr{C}_2$ to $\mathscr{C}$. Suppose $\mathcal{F}_1$ is covariant and $\mathcal{F}_2$ is covariant. Then there exists a strict category $\mathscr{D}$ and there exists a functor $\mathcal{P}_1$ from $\mathscr{D}$ to $\mathscr{C}_1$ and there exists a functor $\mathcal{P}_2$ from $\mathscr{D}$ to $\mathscr{C}_2$ such that the carrier of $\mathscr{D} = \{\langle f_1, f_2 \rangle$, where f_1 is a morphism of $\mathscr{C}_1, f_2$ is a morphism of $\mathscr{C}_2 : f_1 \in$ the carrier of $\mathscr{C}_1$ and $f_2 \in$ the carrier of $\mathscr{C}_2$ and $\mathcal{F}_1(f_1) = \mathcal{F}_2(f_2)\}$ and the composition of $\mathscr{D} = \{\langle\langle f_1, f_2 \rangle, f_3 \rangle$, where f_1, f_2, f_3 are morphisms of $\mathscr{D} : f_1, f_2, f_3 \in$ the carrier of $\mathscr{D}$ and for every morphisms f_{11}, f_{12}, f_{13} of $\mathscr{C}_1$ and for every morphisms f_{21}, f_{22}, f_{23} of $\mathscr{C}_2$ such that $f_1 = \langle f_{11}, f_{21} \rangle$ and $f_2 = \langle f_{12}, f_{22} \rangle$ and $f_3 = \langle f_{13}, f_{23} \rangle$ holds

$f_{11} \triangleright f_{12}$ and $f_{21} \triangleright f_{22}$ and $f_{13} = f_{11} \circ f_{12}$ and $f_{23} = f_{21} \circ f_{22}\}$ and $\mathcal{P}_1$ is covariant and $\mathcal{P}_2$ is covariant and $\langle \mathscr{D}, \mathcal{P}_1, \mathcal{P}_2 \rangle$ is a pullback of $\mathcal{F}_1, \mathcal{F}_2$.

PROOF: Reconsider $c_7 = \{\langle f_1, f_2 \rangle$, where f_1 is a morphism of $\mathscr{C}_1, f_2$ is a morphism of $\mathscr{C}_2 : f_1 \in$ the carrier of $\mathscr{C}_1$ and $f_2 \in$ the carrier of $\mathscr{C}_2$ and $\mathcal{F}_1(f_1) = \mathcal{F}_2(f_2)\}$ as a set. Set $c_8 = \{\langle\langle x_1, x_2 \rangle, x_3 \rangle$, where x_1, x_2, x_3 are elements of $c_7 : x_1, x_2, x_3 \in c_7$ and for every morphisms f_{11}, f_{12}, f_{13} of $\mathscr{C}_1$ and for every morphisms f_{21}, f_{22}, f_{23} of $\mathscr{C}_2$ such that $x_1 = \langle f_{11}, f_{21} \rangle$ and $x_2 = \langle f_{12}, f_{22} \rangle$ and $x_3 = \langle f_{13}, f_{23} \rangle$ holds $f_{11} \triangleright f_{12}$ and $f_{21} \triangleright f_{22}$ and $f_{13} = f_{11} \circ f_{12}$ and $f_{23} = f_{21} \circ f_{22}\}$. For every object x such that $x \in c_8$ holds $x \in (c_7 \times c_7) \times c_7$. For every objects x, y_1, y_2 such that $\langle x, y_1 \rangle$, $\langle x, y_2 \rangle \in c_8$ holds $y_1 = y_2$. Set $\mathscr{D} = \langle c_7, c_8 \rangle$. For every morphisms g_1, g_2 of $\mathscr{D}$ such that $g_1 \triangleright g_2$ there exist morphisms f_{11}, f_{12}, f_{13} of $\mathscr{C}_1$ and there exist morphisms f_{21}, f_{22}, f_{23} of $\mathscr{C}_2$ such that $g_1 = \langle f_{11}, f_{21} \rangle$ and $g_2 = \langle f_{12}, f_{22} \rangle$ and $\mathcal{F}_1(f_{11}) = \mathcal{F}_2(f_{21})$ and $\mathcal{F}_1(f_{12}) = \mathcal{F}_2(f_{22})$ and $f_{11} \triangleright f_{12}$ and $f_{21} \triangleright f_{22}$ and $f_{13} = f_{11} \circ f_{12}$ and $f_{23} = f_{21} \circ f_{22}$ and $g_1 \circ g_2 = \langle f_{13}, f_{23} \rangle$ by (1), [17, (1)], [9, (1)]. For every morphisms g_1, g_2 of $\mathscr{D}$ such that there exist morphisms f_{11}, f_{12} of $\mathscr{C}_1$ and there exist morphisms f_{21}, f_{22} of $\mathscr{C}_2$ such that $g_1 = \langle f_{11}, f_{21} \rangle$ and $g_2 = \langle f_{12}, f_{22} \rangle$ and $\mathcal{F}_1(f_{11}) = \mathcal{F}_2(f_{21})$ and $\mathcal{F}_1(f_{12}) = \mathcal{F}_2(f_{22})$ and $f_{11} \triangleright f_{12}$ and $f_{21} \triangleright f_{22}$ holds $g_1 \triangleright g_2$ by (1), [17, (1)]. For every morphisms g, g_1, g_2 of $\mathscr{D}$ such that $g_1 \triangleright g_2$ holds $g_1 \circ g_2 \triangleright g$ iff $g_2 \triangleright g$. For every morphisms g, g_1, g_2 of $\mathscr{D}$ such that $g_1 \triangleright g_2$ holds $g \triangleright g_1 \circ g_2$ iff $g \triangleright g_1$. For every morphism g_1 of $\mathscr{D}$ such that $g_1 \in$ the carrier of $\mathscr{D}$ there exists a morphism g of $\mathscr{D}$ such that $g \triangleright g_1$ and g is left identity by (2), [17, (31), (32)]. For every morphism g_1 of $\mathscr{D}$ such that $g_1 \in$ the carrier of $\mathscr{D}$ there exists a morphism g of $\mathscr{D}$ such that $g_1 \triangleright g$ and g is right identity by (2), [17, (31), (32)]. For every morphisms g_1, g_2, g_3 of $\mathscr{D}$ such that $g_1 \triangleright g_2$ and $g_2 \triangleright g_3$ and $g_1 \circ g_2 \triangleright g_3$ and $g_1 \triangleright g_2 \circ g_3$ holds $g_1 \circ (g_2 \circ g_3) = (g_1 \circ g_2) \circ g_3$. For every object x, $x \in c_8$ iff $x \in \{\langle\langle f_1, f_2 \rangle, f_3 \rangle$, where f_1, f_2, f_3 are morphisms of $\mathscr{D} : f_1, f_2, f_3 \in$ the carrier of $\mathscr{D}$ and for every morphisms f_{11}, f_{12}, f_{13} of $\mathscr{C}_1$ and for every morphisms f_{21}, f_{22}, f_{23} of $\mathscr{C}_2$ such that $f_1 = \langle f_{11}, f_{21} \rangle$ and $f_2 = \langle f_{12}, f_{22} \rangle$ and $f_3 = \langle f_{13}, f_{23} \rangle$ holds $f_{11} \triangleright f_{12}$ and $f_{21} \triangleright f_{22}$ and $f_{13} = f_{11} \circ f_{12}$ and $f_{23} = f_{21} \circ f_{22}\}$. There exists a functor $\mathcal{P}_1$ from $\mathscr{D}$ to $\mathscr{C}_1$ and there exists a functor $\mathcal{P}_2$ from $\mathscr{D}$ to $\mathscr{C}_2$ such that $\mathcal{P}_1$ is covariant and $\mathcal{P}_2$ is covariant and $\mathcal{F}_1 \circ \mathcal{P}_1 = \mathcal{F}_2 \circ \mathcal{P}_2$ and for every category $\mathscr{D}_1$ and for every functor $\mathcal{G}_1$ from $\mathscr{D}_1$ to $\mathscr{C}_1$ and for every functor $\mathcal{G}_2$ from $\mathscr{D}_1$ to $\mathscr{C}_2$ such that $\mathcal{G}_1$ is covariant and $\mathcal{G}_2$ is covariant and $\mathcal{F}_1 \circ \mathcal{G}_1 = \mathcal{F}_2 \circ \mathcal{G}_2$ there exists a functor $\mathcal{H}$ from $\mathscr{D}_1$ to $\mathscr{D}$ such that $\mathcal{H}$ is covariant and $\mathcal{P}_1 \circ \mathcal{H} = \mathcal{G}_1$ and $\mathcal{P}_2 \circ \mathcal{H} = \mathcal{G}_2$ and for every functor $\mathcal{H}_1$ from $\mathscr{D}_1$ to $\mathscr{D}$ such that $\mathcal{H}_1$ is covariant and $\mathcal{P}_1 \circ \mathcal{H}_1 = \mathcal{G}_1$ and $\mathcal{P}_2 \circ \mathcal{H}_1 = \mathcal{G}_2$ holds $\mathcal{H} = \mathcal{H}_1$ by [17, (31)], [9, (13)], (1), [17, (32), (34)]. Consider $\mathcal{P}_1$

being a functor from $\mathscr{D}$ to $\mathscr{C}_1$, $\mathcal{P}_2$ being a functor from $\mathscr{D}$ to $\mathscr{C}_2$ such that $\mathcal{P}_1$ is covariant and $\mathcal{P}_2$ is covariant and $\mathcal{F}_1 \circ \mathcal{P}_1 = \mathcal{F}_2 \circ \mathcal{P}_2$ and for every category $\mathscr{D}_1$ and for every functor $\mathcal{G}_1$ from $\mathscr{D}_1$ to $\mathscr{C}_1$ and for every functor $\mathcal{G}_2$ from $\mathscr{D}_1$ to $\mathscr{C}_2$ such that $\mathcal{G}_1$ is covariant and $\mathcal{G}_2$ is covariant and $\mathcal{F}_1 \circ \mathcal{G}_1 = \mathcal{F}_2 \circ \mathcal{G}_2$ there exists a functor $\mathcal{H}$ from $\mathscr{D}_1$ to $\mathscr{D}$ such that $\mathcal{H}$ is covariant and $\mathcal{P}_1 \circ \mathcal{H} = \mathcal{G}_1$ and $\mathcal{P}_2 \circ \mathcal{H} = \mathcal{G}_2$ and for every functor $\mathcal{H}_1$ from $\mathscr{D}_1$ to $\mathscr{D}$ such that $\mathcal{H}_1$ is covariant and $\mathcal{P}_1 \circ \mathcal{H}_1 = \mathcal{G}_1$ and $\mathcal{P}_2 \circ \mathcal{H}_1 = \mathcal{G}_2$ holds $\mathcal{H} = \mathcal{H}_1$. $\square$

Let $\mathscr{C}$, $\mathscr{C}_1$, $\mathscr{C}_2$ be categories and $\mathcal{F}_1$ be a functor from $\mathscr{C}_1$ to $\mathscr{C}$. Assume $\mathcal{F}_1$ is covariant. Let $\mathcal{F}_2$ be a functor from $\mathscr{C}_2$ to $\mathscr{C}$. Assume $\mathcal{F}_2$ is covariant.

A pullback of $\mathcal{F}_1$, $\mathcal{F}_2$ is a triple object and is defined by

(Def. 21) there exists a strict category $\mathscr{D}$ and there exists a functor $\mathcal{P}_1$ from $\mathscr{D}$ to $\mathscr{C}_1$ and there exists a functor $\mathcal{P}_2$ from $\mathscr{D}$ to $\mathscr{C}_2$ such that *it* $= \langle\!\langle \mathscr{D}, \mathcal{P}_1, \mathcal{P}_2 \rangle\!\rangle$ and $\mathcal{P}_1$ is covariant and $\mathcal{P}_2$ is covariant and $\langle \mathscr{D}, \mathcal{P}_1, \mathcal{P}_2 \rangle$ is a pullback of $\mathcal{F}_1, \mathcal{F}_2$.

Assume $\mathcal{F}_1$ is covariant. Assume $\mathcal{F}_2$ is covariant. The functor $[\![\mathcal{F}_1, \mathcal{F}_2]\!]$ yielding a strict category is defined by the term

(Def. 22) the pullback of $\mathcal{F}_1$, $\mathcal{F}_{2\mathbf{1},3}$.

Assume $\mathcal{F}_1$ is covariant. Assume $\mathcal{F}_2$ is covariant. The functor $\pi_1(\mathcal{F}_1 \boxtimes \mathcal{F}_2)$ yielding a functor from $[\![\mathcal{F}_1, \mathcal{F}_2]\!]$ to $\mathscr{C}_1$ is defined by the term

(Def. 23) the pullback of $\mathcal{F}_1$, $\mathcal{F}_{2\mathbf{2},3}$.

The functor $\pi_2(\mathcal{F}_1 \boxtimes \mathcal{F}_2)$ yielding a functor from $[\![\mathcal{F}_1, \mathcal{F}_2]\!]$ to $\mathscr{C}_2$ is defined by the term

(Def. 24) the pullback of $\mathcal{F}_1$, $\mathcal{F}_{2\mathbf{3},3}$.

Let us consider categories $\mathscr{C}, \mathscr{C}_1, \mathscr{C}_2$, a functor $\mathcal{F}_1$ from $\mathscr{C}_1$ to $\mathscr{C}$, and a functor $\mathcal{F}_2$ from $\mathscr{C}_2$ to $\mathscr{C}$. Let us assume that $\mathcal{F}_1$ is covariant and $\mathcal{F}_2$ is covariant. Now we state the propositions:

(52) (i) $\pi_1(\mathcal{F}_1 \boxtimes \mathcal{F}_2)$ is covariant, and

(ii) $\pi_2(\mathcal{F}_1 \boxtimes \mathcal{F}_2)$ is covariant, and

(iii) $\langle [\![\mathcal{F}_1, \mathcal{F}_2]\!], \pi_1(\mathcal{F}_1 \boxtimes \mathcal{F}_2), \pi_2(\mathcal{F}_1 \boxtimes \mathcal{F}_2) \rangle$ is a pullback of $\mathcal{F}_1, \mathcal{F}_2$.

(53) $[\![\mathcal{F}_1, \mathcal{F}_2]\!] \cong [\![\mathcal{F}_2, \mathcal{F}_1]\!]$. The theorem is a consequence of (52), (47), and (46).

(54) There exist object-categories $\mathscr{C}$, $\mathscr{C}_1$, $\mathscr{C}_2$ and there exists a functor $\mathcal{F}_1$ from $\mathscr{C}_1$ to $\mathscr{C}$ and there exists a functor $\mathcal{F}_2$ from $\mathscr{C}_2$ to $\mathscr{C}$ such that there exists no object-category $\mathscr{D}$ and there exists a functor $\mathcal{P}_1$ from $\mathscr{D}$ to $\mathscr{C}_1$ and there exists a functor $\mathcal{P}_2$ from $\mathscr{D}$ to $\mathscr{C}_2$ such that $\mathcal{F}_1 \cdot \mathcal{P}_1 = \mathcal{F}_2 \cdot \mathcal{P}_2$ and for every object-category $\mathscr{D}_1$ and for every functor $\mathcal{G}_1$ from $\mathscr{D}_1$ to $\mathscr{C}_1$ and for every functor $\mathcal{G}_2$ from $\mathscr{D}_1$ to $\mathscr{C}_2$ such that $\mathcal{F}_1 \cdot \mathcal{G}_1 = \mathcal{F}_2 \cdot \mathcal{G}_2$ there exists a functor $\mathcal{H}$ from $\mathscr{D}_1$ to $\mathscr{D}$ such that $\mathcal{P}_1 \cdot \mathcal{H} = \mathcal{G}_1$ and $\mathcal{P}_2 \cdot \mathcal{H} = \mathcal{G}_2$ and for

every functor $\mathcal{H}_1$ from $\mathscr{D}_1$ to $\mathscr{D}$ such that $\mathcal{P}_1 \cdot \mathcal{H}_1 = \mathcal{G}_1$ and $\mathcal{P}_2 \cdot \mathcal{H}_1 = \mathcal{G}_2$ holds $\mathcal{H} = \mathcal{H}_1$. The theorem is a consequence of (39) and (40).

References

[1] Jiri Adamek, Horst Herrlich, and George E. Strecker. *Abstract and Concrete Categories: The Joy of Cats*. Dover Publication, New York, 2009.

[2] Grzegorz Bancerek. Cardinal numbers. *Formalized Mathematics*, 1(**2**):377–382, 1990.

[3] Grzegorz Bancerek. The ordinal numbers. *Formalized Mathematics*, 1(**1**):91–96, 1990.

[4] Grzegorz Bancerek. The well ordering relations. *Formalized Mathematics*, 1(**1**):123–129, 1990.

[5] Grzegorz Bancerek. Zermelo theorem and axiom of choice. *Formalized Mathematics*, 1 (**2**):265–267, 1990.

[6] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Formalized Mathematics*, 1(**1**):107–114, 1990.

[7] Francis Borceaux. *Handbook of Categorical Algebra I. Basic Category Theory*, volume 50 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1994.

[8] Czesław Byliński. Introduction to categories and functors. *Formalized Mathematics*, 1 (**2**):409–420, 1990.

[9] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**): 55–65, 1990.

[10] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.

[11] Czesław Byliński. Partial functions. *Formalized Mathematics*, 1(**2**):357–367, 1990.

[12] Czesław Byliński. Some basic properties of sets. *Formalized Mathematics*, 1(**1**):47–53, 1990.

[13] Agata Darmochwał. Finite sets. *Formalized Mathematics*, 1(**1**):165–167, 1990.

[14] F. William Lawvere. Functorial semantics of algebraic theories and some algebraic problems in the context of functorial semantics of algebraic theories. *Reprints in Theory and Applications of Categories*, 5:1–121, 2004.

[15] Saunders Mac Lane. *Categories for the Working Mathematician*, volume 5 of *Graduate Texts in Mathematics*. Springer Verlag, New York, Heidelberg, Berlin, 1971.

[16] Beata Padlewska. Families of sets. *Formalized Mathematics*, 1(**1**):147–152, 1990.

[17] Marco Riccardi. Object-free definition of categories. *Formalized Mathematics*, 21(**3**): 193–205, 2013. doi:10.2478/forma-2013-0021.

[18] Andrzej Trybulec. Enumerated sets. *Formalized Mathematics*, 1(**1**):25–34, 1990.

[19] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(**1**):67–71, 1990.

[20] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1 (**1**):73–83, 1990.

[21] Edmund Woronowicz. Relations defined on sets. *Formalized Mathematics*, 1(**1**):181–186, 1990.

Received December 31, 2014