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Definition and Properties of Direct Sum Decomposition of Groups[1]

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Summary. In this article, direct sum decomposition of group is mainly discussed. In the second section, support of element of direct product group is defined and its properties are formalized. It is formalized here that an element of direct product group belongs to its direct sum if and only if support of the element is finite. In the third section, product map and sum map are prepared. In the fourth section, internal and external direct sum are defined. In the last section, an equivalent form of internal direct sum is proved. We referred to [23], [22], [8] and [18] in the formalization.

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The notation and terminology used in this paper have been introduced in the following articles: [1], [2], [6], [16], [24], [10], [11], [12], [13], [7], [27], [20], [21], [28], [29], [30], [17], [33], [25], [3], [5], [14], [19], [32], [31], and [15].

1. Miscellanies

Let D be a non empty set and x be an element of D. Observe that the functor $\langle x \rangle$ yields a finite sequence of elements of D. Let I be a set.

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A family of groups of I is an associative, group-like multiplicative magma family of I. Let G be a group. Note that there exists a subgroup of G which is commutative.

Now we state the proposition:

(1) Let us consider a set I, a family F of groups of I, and an object i. If $i \in I$, then $F(i)$ is a group.

Let I be a set, F be a family of groups of I, and i be an object. Assume $i \in I$. One can verify that the functor $F(i)$ yields a group. One can verify that $\operatorname{sum} F$ is non empty and constituted functions.

Now we state the propositions:

(2) Let us consider a set I, and a function F. Suppose $I = \operatorname{dom} F$ and for every object i such that $i \in I$ holds $F(i)$ is a group. Then F is a family of groups of I.

(3) Let us consider a set I, a family F of groups of I, and an element a of $\prod F$. Then $\operatorname{dom} a = I$.

(4) Let us consider a non empty set I, a family F of groups of I, and an element x of I. Then (the support of F)$(x) = \Omega_{F(x)}$.

(5) Let us consider a non empty set I, a family F of groups of I, a function x, and an element i of I. If $x \in \prod F$, then $x(i) \in F(i)$. The theorem is a consequence of (4).

(6) Let us consider groups G, H, and a subgroup I of H. Then every homomorphism from G to I is a homomorphism from G to H.

2. Support of Element of Direct Product Group

Let I be a set, F be a family of groups of I, and a be a function. The functor $\operatorname{support}(a, F)$ yielding a subset of I is defined by

(Def. 1) for every object i, $i \in it$ iff $a(i) \neq \mathbf{1}_{F(i)}$ and $i \in I$.

Now we state the proposition:

(7) Let us consider a set I, a family F of groups of I, and an element a of $\operatorname{sum} F$. Then there exists a finite subset J of I and there exists a many sorted set b indexed by J such that $J = \operatorname{support}(a, F)$ and $a = \mathbf{1}_{\prod F} {+\cdot} b$ and for every object j such that $j \in I \setminus J$ holds $a(j) = \mathbf{1}_{F(j)}$ and for every object j such that $j \in J$ holds $a(j) = b(j)$.

Proof: Consider g being an element of $\prod$(the support of F), J being a finite subset of I, b being a many sorted set indexed by J such that $g = \mathbf{1}_{\prod F}$ and $a = g {+\cdot} b$ and for every set j such that $j \in J$ there exists a group-like, non empty multiplicative magma G such that $G = F(j)$ and

$b(j) \in$ the carrier of G and $b(j) \neq \mathbf{1}_G$. $\operatorname{dom} \mathbf{1}_{\prod F} = I$. For every object j such that $j \in I \setminus J$ holds $a(j) = \mathbf{1}_{F(j)}$ by [13, (11)], [17, (6)]. For every object j, $j \in \operatorname{support}(a, F)$ iff $j \in J$ by [13, (13)]. $\square$

Let I be a set, F be a family of groups of I, and a be an element of sum F. One can verify that $\operatorname{support}(a, F)$ is finite.

Let G be a group and a be a function from I into G. The functor support a yielding a subset of I is defined by

(Def. 2) for every object i, $i \in it$ iff $a(i) \neq \mathbf{1}_G$ and $i \in I$.

We say that a is finite-support if and only if

(Def. 3) support a is finite.

Let us observe that there exists a function from I into G which is finite-support. Let a be a finite-support function from I into G. One can verify that support a is finite.

The functor $\prod a$ yielding an element of G is defined by the term

(Def. 4) $\prod (a \restriction \operatorname{support} a)$.

Now we state the propositions:

(8) Let us consider a set I, a family F of groups of I, and an element a of $\prod F$. Then $a \in \operatorname{sum} F$ if and only if $\operatorname{support}(a, F)$ is finite.
Proof: Reconsider $J = \operatorname{support}(a, F)$ as a finite subset of I. Set $k = a \restriction J$. Set $x = \mathbf{1}_{\prod F} {+\cdot} k$. For every set j such that $j \in J$ there exists a group-like, non empty multiplicative magma G such that $G = F(j)$ and $k(j) \in$ the carrier of G and $k(j) \neq \mathbf{1}_G$ by [10, (49)], (5). $\operatorname{dom} x = I$. For every object i such that $i \in \operatorname{dom} x$ holds $x(i) = a(i)$ by [13, (11)], [17, (6)], [13, (13)], [10, (49)]. $x = a$. $\square$

(9) Let us consider a set I, a group G, a family H of groups of I, a function x from I into G, and an element y of $\prod H$. Suppose $x = y$ and for every object i such that $i \in I$ holds $H(i)$ is a subgroup of G. Then support $x = \operatorname{support}(y, H)$.
Proof: For every object i such that $i \in I$ holds $\mathbf{1}_{H(i)} = \mathbf{1}_G$ by [28, (44)]. For every object i, $i \in \operatorname{support}(y, H)$ iff $i \in$ support x. $\square$

(10) Let us consider a set I, a group G, a family F of groups of I, and an object a. Suppose $a \in \operatorname{sum} F$ and for every object i such that $i \in I$ holds $F(i)$ is a subgroup of G. Then a is a finite-support function from I into G.
Proof: Reconsider $b = a$ as an element of $\prod F$. For every object i such that $i \in I$ holds $\Omega_{F(i)} \subseteq \Omega_G$. $\operatorname{dom} b = I$. For every object z such that $z \in \operatorname{rng} b$ holds $z \in \Omega_G$ by (3), (5), [28, (40)]. $\operatorname{support}(b, F) = \operatorname{support} b$. $\square$

(11) Let us consider a non empty set I, and a family F of groups of I. Then $\mathrm{support}(\mathbf{1}_{\prod F}, F)$ is empty.

PROOF: For every object i, $i \notin \mathrm{support}(\mathbf{1}_{\prod F}, F)$ by [17, (6)]. □

(12) Let us consider a non empty set I, a group G, and a function a from I into G. If $a = I \longmapsto \mathbf{1}_G$, then $\mathrm{support}\, a$ is empty.

PROOF: For every object i, $i \notin \mathrm{support}\, a$ by [24, (7)]. □

(13) Let us consider a non empty set I, a group G, and a family F of groups of I. Suppose for every element i of I, $F(i)$ is a subgroup of G. Then $\mathbf{1}_{\prod F} = I \longmapsto \mathbf{1}_G$.

PROOF: $\mathrm{dom}\, \mathbf{1}_{\prod F} = I$. For every object j such that $j \in I$ holds $\mathbf{1}_{\prod F}(j) = (I \longmapsto \mathbf{1}_G)(j)$ by [17, (6)], [24, (7)], [28, (44)]. □

(14) Let us consider a non empty set I, a family F of groups of I, a group G, and a finite-support function x from I into G. Suppose $\mathrm{support}\, x = \emptyset$ and for every object i such that $i \in I$ holds $F(i)$ is a subgroup of G. Then $x = \mathbf{1}_{\prod F}$.

PROOF: For every set i such that $i \in I$ there exists a group-like, non empty multiplicative magma G such that $G = F(i)$ and $x(i) = \mathbf{1}_G$ by [28, (44)]. □

(15) Let us consider a set I, a group G, and a finite-support function x from I into G. If $\mathrm{support}\, x = \emptyset$, then $\prod x = \mathbf{1}_G$.

(16) Let us consider a non empty set I, a group G, and a finite-support function a from I into G. If $a = I \longmapsto \mathbf{1}_G$, then $\prod a = \mathbf{1}_G$. The theorem is a consequence of (12) and (15).

Let us consider a non empty set I, a family F of groups of I, an element x of $\prod F$, an element i of I, and an element g of $F(i)$. Now we state the propositions:

(17) If $x = \mathbf{1}_{\prod F} +\cdot (i, g)$, then $\mathrm{support}(x, F) \subseteq \{i\}$.

PROOF: For every object j such that $j \in \mathrm{support}(x, F)$ holds $j \in \{i\}$ by [20, (1)]. □

(18) If $x = \mathbf{1}_{\prod F} +\cdot (i, g)$ and $g \neq \mathbf{1}_{F(i)}$, then $\mathrm{support}(x, F) = \{i\}$. The theorem is a consequence of (17).

Let us consider a non empty set I, a group G, an element i of I, an element g of G, and a function a from I into G. Now we state the propositions:

(19) If $a = (I \longmapsto \mathbf{1}_G) +\cdot (i, g)$, then $\mathrm{support}\, a \subseteq \{i\}$.

PROOF: For every object j such that $j \in \mathrm{support}\, a$ holds $j \in \{i\}$ by [7, (32)], [24, (7)]. □

(20) If $a = (I \longmapsto \mathbf{1}_G) +\cdot (i, g)$ and $g \neq \mathbf{1}_G$, then $\mathrm{support}\, a = \{i\}$. The theorem is a consequence of (19).

Now we state the propositions:

(21) Let us consider a non empty set I, a group G, a finite-support function a from I into G, an element i of I, and an element g of G. If $a = (I \longmapsto \mathbf{1}_G) +\cdot (i, g)$, then $\prod a = g$. The theorem is a consequence of (20) and (16).

(22) Let us consider a non empty set I, a family F of groups of I, a function x, an element i of I, and an element g of $F(i)$. Suppose $\mathrm{support}(x, F)$ is finite. Then $\mathrm{support}(x +\cdot (i, g), F)$ is finite.
Proof: Reconsider $y = x +\cdot (i, g)$ as a function. For every object j such that $j \in \mathrm{support}(y, F)$ holds $j \in \mathrm{support}(x, F) \cup \{i\}$ by [7, (32)]. □

(23) Let us consider a non empty set I, a group G, a function a from I into G, an element i of I, and an element g of G. Suppose $\mathrm{support}\, a$ is finite. Then $\mathrm{support}(a +\cdot (i, g))$ is finite.
Proof: Reconsider $b = a +\cdot (i, g)$ as a function from I into G. For every object j such that $j \in \mathrm{support}\, b$ holds $j \in \mathrm{support}\, a \cup \{i\}$ by [7, (32)]. □

Let us consider a non empty set I, a family F of groups of I, a function x, an element i of I, and an element g of $F(i)$. Now we state the propositions:

(24) If $x \in \prod F$, then $x +\cdot (i, g) \in \prod F$.
Proof: $\mathrm{dom}\, x = I$. Set $y = x +\cdot (i, g)$. For every object j such that $j \in \mathrm{dom}(\text{the support of } F)$ holds $y(j) \in (\text{the support of } F)(j)$ by [7, (31)], (4), [7, (32)], [2, (9)]. □

(25) If $x \in \mathrm{sum}\, F$, then $x +\cdot (i, g) \in \mathrm{sum}\, F$.
Proof: Set $y = x +\cdot (i, g)$. $y \in \prod F$. For every object j such that $j \in \mathrm{support}(y, F)$ holds $j \in \mathrm{support}(x, F) \cup \{i\}$ by [7, (32)]. □

Now we state the propositions:

(26) Let us consider a non empty set I, a group G, a finite-support function a from I into G, an element i of I, and an element g of G. Then $a +\cdot (i, g)$ is a finite-support function from I into G. The theorem is a consequence of (23).

(27) Let us consider a non empty set I, a family F of groups of I, an object i, and functions a, b. Suppose $i \in I$ and $\mathrm{dom}\, a = I$ and $b = a +\cdot (i, \mathbf{1}_{F(i)})$. Then $\mathrm{support}(b, F) = \mathrm{support}(a, F) \setminus \{i\}$.
Proof: For every object j, $j \in \mathrm{support}(b, F)$ iff $j \in \mathrm{support}(a, F) \setminus \{i\}$ by [15, (11), (48)], [7, (32)], [15, (50)]. □

(28) Let us consider a non empty set I, a group G, an object i, and functions a, b from I into G. Suppose $i \in \mathrm{support}\, a$ and $b = a +\cdot (i, \mathbf{1}_G)$. Then $\mathrm{support}\, b = \mathrm{support}\, a \setminus \{i\}$.
Proof: For every object j, $j \in \mathrm{support}\, b$ iff $j \in \mathrm{support}\, a \setminus \{i\}$ by [15, (11), (48)], [7, (32)], [15, (50)]. □

(29) Let us consider a non empty set I, a family F of groups of I, an object i, an element a of $\mathrm{sum}\, F$, and a function b. Suppose $i \in \mathrm{support}(a, F)$ and

$b = a +\cdot (i, \mathbf{1}_{F(i)})$. Then $\overline{\overline{\mathrm{support}(b, F)}} = \overline{\overline{\mathrm{support}(a, F)}} - 1$. The theorem is a consequence of (3) and (27).

(30) Let us consider a non empty set I, a group G, an object i, a finite-support function a from I into G, and a function b from I into G. Suppose $i \in \mathrm{support}\, a$ and $b = a +\cdot (i, \mathbf{1}_G)$. Then $\overline{\overline{\mathrm{support}\, b}} = \overline{\overline{\mathrm{support}\, a}} - 1$. The theorem is a consequence of (28).

Let us consider a non empty set I, a family F of groups of I, and elements a, b of $\prod F$.

Let us assume that $\mathrm{support}(a, F)$ misses $\mathrm{support}(b, F)$. Now we state the propositions:

(31) $a +\cdot b \restriction \mathrm{support}(b, F) = a \cdot b$.
PROOF: Reconsider $c = a +\cdot b \restriction \mathrm{support}(b, F)$ as a function. Reconsider $d = a \cdot b$ as an element of $\prod F$. $\mathrm{dom}\, a = I$. $\mathrm{dom}\, b = I$. $\mathrm{dom}\, d = I$. For every object i such that $i \in I$ holds $c(i) = d(i)$ by (5), [13, (11)], [17, (1)], [13, (13)]. $\square$

(32) $a \cdot b = b \cdot a$.
PROOF: Reconsider $c = a \cdot b$ as an element of $\prod F$. Reconsider $d = b \cdot a$ as an element of $\prod F$. $\mathrm{dom}\, c = I$. $\mathrm{dom}\, d = I$. For every object i such that $i \in I$ holds $c(i) = d(i)$ by (5), [17, (1)]. $\square$

(33) Let us consider a non empty set I, a family F of groups of I, and an element i of I. Then $\mathrm{ProjGroup}(F, i)$ is a subgroup of $\mathrm{sum}\, F$.
PROOF: Set $S = \mathrm{ProjGroup}(F, i)$. Set $G = \mathrm{sum}\, F$. For every object x such that $x \in \Omega_S$ holds $x \in \Omega_G$ by [28, (40)], (17), (8). $\square$

(34) Let us consider a non empty set I, families F, G of groups of I, and functions x, y. Suppose for every object i such that $i \in I$ there exists a homomorphism h_1 from $F(i)$ to $G(i)$ such that $y(i) = h_1(x(i))$. Then $\mathrm{support}(y, G) \subseteq \mathrm{support}(x, F)$.
PROOF: For every object i such that $i \in \mathrm{support}(y, G)$ holds $i \in \mathrm{support}(x, F)$ by [30, (31)]. $\square$

3. Product Map and Sum Map

Let F, G be non-empty, non empty functions and h be a non empty function. Assume $\mathrm{dom}\, F = \mathrm{dom}\, G = \mathrm{dom}\, h$ and for every object i such that $i \in \mathrm{dom}\, h$ holds $h(i)$ is a function from $F(i)$ into $G(i)$. The functor $\mathrm{ProductMap}(F, G, h)$ yielding a function from $\prod F$ into $\prod G$ is defined by

(Def. 5) for every element x of $\prod F$ and for every object i such that $i \in \mathrm{dom}\, h$ there exists a function h_1 from $F(i)$ into $G(i)$ such that $h_1 = h(i)$ and $(it(x))(i) = h_1(x(i))$.

Let us consider non-empty, non empty functions F, G and a non empty function h.

Let us assume that $\operatorname{dom} F = \operatorname{dom} G = \operatorname{dom} h$ and for every object i such that $i \in \operatorname{dom} h$ there exists a function h_1 from $F(i)$ into $G(i)$ such that $h_1 = h(i)$ and h_1 is onto. Now we state the propositions:

(35) ProductMap(F, G, h) is onto.
PROOF: Set $p =$ ProductMap(F, G, h). For every object i such that $i \in \operatorname{dom} h$ holds $h(i)$ is a function from $F(i)$ into $G(i)$. For every object y such that $y \in \prod G$ there exists an object x such that $x \in \prod F$ and $y = p(x)$ by [2, (9)], [11, (11)], [10, (2)]. $\square$

(36) ProductMap(F, G, h) is one-to-one.
PROOF: Set $p =$ ProductMap(F, G, h). For every object i such that $i \in \operatorname{dom} h$ holds $h(i)$ is a function from $F(i)$ into $G(i)$. For every objects x_1, x_2 such that x_1, $x_2 \in \prod F$ and $p(x_1) = p(x_2)$ holds $x_1 = x_2$ by [2, (9)], [11, (19)], [10, (2)]. $\square$

(37) ProductMap(F, G, h) is bijective. The theorem is a consequence of (35) and (36).

Now we state the proposition:

(38) Let us consider a non empty set I, families F, G of groups of I, a non empty function h, an element x of $\prod F$, and an element y of $\prod G$. Suppose $I = \operatorname{dom} h$ and $y =$ (ProductMap(the support of F, the support of G, h))(x) and for every object i such that $i \in I$ holds $h(i)$ is a homomorphism from $F(i)$ to $G(i)$. Let us consider an object i. Suppose $i \in I$. Then there exists a homomorphism h_1 from $F(i)$ to $G(i)$ such that

(i) $h_1 = h(i)$, and

(ii) $y(i) = h_1(x(i))$.

The theorem is a consequence of (4).

Let I be a non empty set, F, G be families of groups of I, and h be a non empty function. Assume $I = \operatorname{dom} h$ and for every object i such that $i \in I$ holds $h(i)$ is a homomorphism from $F(i)$ to $G(i)$. The functor ProductMap(F, G, h) yielding a homomorphism from $\prod F$ to $\prod G$ is defined by the term

(Def. 6) ProductMap(the support of F, the support of G, h).

Now we state the propositions:

(39) Let us consider a non empty set I, families F, G of groups of I, a non empty function h, an element x of $\prod F$, and an element y of $\prod G$. Suppose $I = \operatorname{dom} h$ and $y =$ (ProductMap(F, G, h))(x) and for every object i such that $i \in I$ holds $h(i)$ is a homomorphism from $F(i)$ to $G(i)$. Let us consider

an object i. Suppose $i \in I$. Then there exists a homomorphism h_1 from $F(i)$ to $G(i)$ such that

(i) $h_1 = h(i)$, and

(ii) $y(i) = h_1(x(i))$.

The theorem is a consequence of (38).

(40) Let us consider a non empty set I, families F, G of groups of I, and a non empty function h. Suppose $I = \mathrm{dom}\, h$ and for every object i such that $i \in I$ there exists a homomorphism h_1 from $F(i)$ to $G(i)$ such that $h_1 = h(i)$ and h_1 is bijective. Then $\mathrm{ProductMap}(F, G, h)$ is bijective. The theorem is a consequence of (4) and (37).

Let I be a non empty set, F be a family of groups of I, and i be an element of I. Observe that the functor $\mathrm{ProjGroup}(F, i)$ yields a strict subgroup of $\mathrm{sum}\, F$. Let F, G be families of groups of I and h be a non empty function. Assume $I = \mathrm{dom}\, h$ and for every object i such that $i \in I$ holds $h(i)$ is a homomorphism from $F(i)$ to $G(i)$. The functor $\mathrm{SumMap}(F, G, h)$ yielding a homomorphism from $\mathrm{sum}\, F$ to $\mathrm{sum}\, G$ is defined by the term

(Def. 7) $\mathrm{ProductMap}(F, G, h) \restriction \mathrm{sum}\, F$.

Now we state the propositions:

(41) Let us consider a non empty set I, families F, G of groups of I, and a non empty function h. Suppose $I = \mathrm{dom}\, h$ and for every object i such that $i \in I$ there exists a homomorphism h_1 from $F(i)$ to $G(i)$ such that $h_1 = h(i)$ and h_1 is bijective. Then $\mathrm{SumMap}(F, G, h)$ is bijective.

PROOF: For every object i such that $i \in I$ holds $h(i)$ is a homomorphism from $F(i)$ to $G(i)$. Set $p = \mathrm{ProductMap}(F, G, h)$. Set $s = \mathrm{SumMap}(F, G, h)$. p is bijective. For every object y such that $y \in \Omega_{\mathrm{sum}\, G}$ holds $y \in \mathrm{rng}\, s$ by [28, (40)], [30, (61)], (39), [30, (62)]. □

(42) Let us consider a non empty set I, families F, G of groups of I, and a non empty function h. Suppose $I = \mathrm{dom}\, h$ and for every object i such that $i \in I$ holds $h(i)$ is a homomorphism from $F(i)$ to $G(i)$. Let us consider an element i of I, an element f of $F(i)$, and a homomorphism h_1 from $F(i)$ to $G(i)$. Suppose $h_1 = h(i)$. Then $(\mathrm{SumMap}(F, G, h))((\mathrm{1ProdHom}(F, i))(f)) = (\mathrm{1ProdHom}(G, i))(h_1(f))$.

PROOF: Set $x = (\mathrm{1ProdHom}(F, i))(f)$. Set $y = (\mathrm{SumMap}(F, G, h))(x)$. $\mathrm{dom}\, y = I$. Consider h_2 being a homomorphism from $F(i)$ to $G(i)$ such that $h_2 = h(i)$ and $y(i) = h_2(x(i))$. For every element j of I such that $j \neq i$ holds $y(j) = \mathbf{1}_{G(j)}$ by [20, (1)], (39), [30, (31)]. □

4. Definition of Internal and External Direct Sum Decomposition

Now we state the proposition:

(43) Let us consider a non empty set I, a group G, and an object i. Suppose $i \in I$. Then there exists a family F of groups of I and there exists a homomorphism h from $\operatorname{sum} F$ to G such that h is bijective and $F = (I \longmapsto \{\mathbf{1}\}_G) {+}\cdot (\{i\} \longmapsto G)$ and for every object j such that $j \in I$ holds $\mathbf{1}_{F(j)} = \mathbf{1}_G$ and for every element x of $\operatorname{sum} F$, $h(x) = x(i)$ and for every element x of $\operatorname{sum} F$, there exists a finite subset J of I and there exists a many sorted set a indexed by J such that $J \subseteq \{i\}$ and $J = \operatorname{support}(x, F)$ and ($\operatorname{support}(x, F) = \emptyset$ or $\operatorname{support}(x, F) = \{i\}$) and $x = \mathbf{1}_{\prod F} {+}\cdot a$ and for every object j such that $j \in I \setminus J$ holds $x(j) = \mathbf{1}_{F(j)}$ and for every object j such that $j \in J$ holds $x(j) = a(j)$.

PROOF: Set $v = I \longmapsto \{\mathbf{1}\}_G$. Set $w = \{i\} \longmapsto G$. Set $F = v {+}\cdot w$. For every object j such that $j \in I \setminus \{i\}$ holds $F(j) = \{\mathbf{1}\}_G$ by [24, (7)]. For every object j such that $j \in I$ holds $F(j)$ is a group. For every object j such that $j \in I$ holds $\mathbf{1}_{F(j)} = \mathbf{1}_G$ by [28, (44)]. Define $\mathcal{P}[\text{element of } \operatorname{sum} F, \text{element of } G] \equiv \$_2 = \$_1(i)$. For every element x of $\operatorname{sum} F$, there exists an element y of G such that $\mathcal{P}[x, y]$ by [28, (40)], (5), [24, (13), (7)]. Consider h being a function from $\operatorname{sum} F$ into G such that for every element x of $\operatorname{sum} F$, $\mathcal{P}[x, h(x)]$ from [11, Sch. 3]. For every object y such that $y \in \Omega_G$ there exists an object x such that $x \in \Omega_{\operatorname{sum} F}$ and $y = h(x)$ by [24, (7)], (24), (17), (8). For every element x of $\operatorname{sum} F$, $\operatorname{support}(x, F) \subseteq \{i\}$ by [28, (40)], (5), [28, (44)]. For every objects x_1, x_2 such that x_1, $x_2 \in \Omega_{\operatorname{sum} F}$ and $h(x_1) = h(x_2)$ holds $x_1 = x_2$ by [28, (40)], (3), (7), [10, (2)]. For every elements x, y of $\operatorname{sum} F$, $h(x \cdot y) = h(x) \cdot h(y)$ by [28, (40)], (5), [28, (43)], [17, (1)]. For every element x of $\operatorname{sum} F$, there exists a finite subset J of I and there exists a many sorted set a indexed by J such that $J \subseteq \{i\}$ and $J = \operatorname{support}(x, F)$ and ($\operatorname{support}(x, F) = \emptyset$ or $\operatorname{support}(x, F) = \{i\}$) and $x = \mathbf{1}_{\prod F} {+}\cdot a$ and for every object j such that $j \in I \setminus J$ holds $x(j) = \mathbf{1}_{F(j)}$ and for every object j such that $j \in J$ holds $x(j) = a(j)$ by [15, (31)], (7). $\square$

Let I be a non empty set and G be a group. A direct sum components of G and I is a family of groups of I and is defined by

(Def. 8) there exists a homomorphism h from sum *it* to G such that h is bijective.

Let F be a direct sum components of G and I. We say that F is internal if and only if

(Def. 9) for every element i of I, $F(i)$ is a subgroup of G and there exists a homomorphism h from $\operatorname{sum} F$ to G such that h is bijective and for every finite-support function x from I into G such that $x \in \operatorname{sum} F$ holds $h(x) = \prod x$.

One can verify that there exists a direct sum components of G and I which is internal.

5. Equivalent Expression of Internal Direct Sum Decomposition

Now we state the propositions:

(44) Let us consider a group G, and a non empty subset A of G. Suppose for every elements x, y of G such that $x, y \in A$ holds $x \cdot y = y \cdot x$. Then $\operatorname{gr}(A)$ is commutative.
Proof: For every elements x, y of G and for every elements i, j of $\mathbb{Z}$ such that $x, y \in A$ holds $x^i \cdot y^j = y^j \cdot x^i$ by [27, (39)]. For every element y of G and for every element j of $\mathbb{Z}$ such that $y \in A$ for every finite sequence F of elements of G for every finite sequence I of elements of $\mathbb{Z}$ such that $\operatorname{len} F = \operatorname{len} I$ and $\operatorname{rng} F \subseteq A$ holds $\prod F^I \cdot y^j = y^j \cdot \prod F^I$ by [29, (21), (8)], [32, (70)], [6, (4)]. For every elements x, g of G and for every element i of $\mathbb{Z}$ such that $x \in \operatorname{gr}(A)$ and $g \in A$ holds $x \cdot g^i = g^i \cdot x$ by [29, (28)]. For every element x of G such that $x \in \operatorname{gr}(A)$ for every finite sequence F of elements of G for every finite sequence I of elements of $\mathbb{Z}$ such that $\operatorname{len} F = \operatorname{len} I$ and $\operatorname{rng} F \subseteq A$ holds $\prod F^I \cdot x = x \cdot \prod F^I$ by [29, (21), (8)], [32, (70)], [6, (4)]. For every elements x, y of $\operatorname{gr}(A)$, $x \cdot y = y \cdot x$ by [28, (41)], [29, (28)], [28, (43)]. □

(45) Let us consider a group G, a subgroup H of G, a finite sequence a of elements of G, and a finite sequence b of elements of H. If $a = b$, then $\prod a = \prod b$.
Proof: Define $\mathcal{P}[\text{natural number}] \equiv$ for every finite sequence a of elements of G for every finite sequence b of elements of H such that $\operatorname{len} a = \$_1$ and $a = b$ holds $\prod a = \prod b$. $\mathcal{P}[0]$ by [29, (8)], [28, (44)]. For every natural number k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$ by [6, (4), (17)], [26, (55)], [29, (6)]. For every natural number k, $\mathcal{P}[k]$ from [4, Sch. 2]. □

(46) Let us consider a group G, an element h of G, and a finite sequence F of elements of G. Suppose for every natural number k such that $k \in \operatorname{dom} F$ holds $h \cdot F_k = F_k \cdot h$. Then $h \cdot \prod F = \prod F \cdot h$.
Proof: Define $\mathcal{P}[\text{natural number}] \equiv$ for every finite sequence F of elements of G such that $\operatorname{len} F = \$_1$ and for every natural number i such that $i \in \operatorname{dom} F$ holds $h \cdot F_i = F_i \cdot h$ holds $h \cdot \prod F = \prod F \cdot h$. $\mathcal{P}[0]$ by [29, (8)]. For every natural number k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$ by [6, (4), (17), (5)], [14, (80)]. For every natural number i, $\mathcal{P}[i]$ from [4, Sch. 2]. □

(47) Let us consider a group G, and finite sequences F, F_1, F_2 of elements of G. Suppose $\operatorname{len} F = \operatorname{len} F_1$ and $\operatorname{len} F = \operatorname{len} F_2$ and for every natural numbers i, j such that $i, j \in \operatorname{dom} F$ and $i \neq j$ holds $F_{1i} \cdot F_{2j} = F_{2j} \cdot F_{1i}$ and for every natural number k such that $k \in \operatorname{dom} F$ holds $F(k) = F_{1k} \cdot F_{2k}$. Then $\prod F = \prod F_1 \cdot \prod F_2$.
PROOF: Define $\mathcal{P}$[natural number] $\equiv$ for every finite sequences F, F_1, F_2 of elements of G such that $\operatorname{len} F = \$_1$ and $\operatorname{len} F = \operatorname{len} F_1$ and $\operatorname{len} F = \operatorname{len} F_2$ and for every natural numbers i, j such that $i, j \in \operatorname{dom} F$ and $i \neq j$ holds $F_{1i} \cdot F_{2j} = F_{2j} \cdot F_{1i}$ and for every natural number k such that $k \in \operatorname{dom} F$ holds $F(k) = F_{1k} \cdot F_{2k}$ holds $\prod F = \prod F_1 \cdot \prod F_2$. $\mathcal{P}[0]$ by [29, (8)]. For every natural number k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$ by [6, (4), (17), (5)], [14, (80)]. For every natural number i, $\mathcal{P}[i]$ from [4, Sch. 2]. $\square$

(48) Let us consider a group G, and a finite sequence a of elements of G. Suppose for every object i such that $i \in \operatorname{dom} a$ holds $a(i) = \mathbf{1}_G$. Then $\prod a = \mathbf{1}_G$.
PROOF: Set $n = \operatorname{len} a$. $a = n \mapsto \mathbf{1}_G$ by [24, (13)], [9, (57)], [10, (2)]. $\square$

(49) Let us consider a finite set I, a group G, and a (the carrier of G)-valued, total, I-defined function a. Suppose for every object i such that $i \in I$ holds $a(i) = \mathbf{1}_G$. Then $\prod a = \mathbf{1}_G$.
PROOF: Set $c_1 = \mathrm{CFS}(I)$. Set $a_2 = a \cdot c_1$. For every object i such that $i \in \operatorname{dom} a_2$ holds $a_2(i) = \mathbf{1}_G$ by [32, (27)], [10, (3), (12)]. $\square$

(50) Let us consider a finite set A, a non empty set B, and a B-valued, total, A-defined function f. Then $f \cdot \mathrm{CFS}(A)$ is a finite sequence of elements of B.

Let us consider a non empty set I, a group G, a finite-support function a from I into G, and a finite subset W of I. Now we state the propositions:

(51) If $\operatorname{support} a \subseteq W$ and for every elements i, j of I, $a(i) \cdot a(j) = a(j) \cdot a(i)$, then $\prod a = \prod(a \restriction W)$.
PROOF: Reconsider $r = \operatorname{rng} a$ as a non empty subset of G. For every elements x, y of G such that $x, y \in r$ holds $x \cdot y = y \cdot x$ by [11, (113)]. $\square$

(52) Suppose $\operatorname{support} a \subseteq W$. Then there exists a finite-support function a_1 from W into G such that

(i) $a_1 = a \restriction W$, and

(ii) $\operatorname{support} a = \operatorname{support} a_1$, and

(iii) $\prod a = \prod a_1$.

(53) Let us consider a non empty set I, a group G, a family F of groups of I, elements s_1, s_2 of $\operatorname{sum} F$, and finite-support functions x, y, x_3 from I into G. Suppose for every element i of I, $F(i)$ is a subgroup of G and for

every elements i, j of I and for every elements g_1, g_2 of G such that $i \neq j$ and $g_1 \in F(i)$ and $g_2 \in F(j)$ holds $g_1 \cdot g_2 = g_2 \cdot g_1$ and $s_1 = x$ and $s_2 = y$ and $s_1 \cdot s_2 = x_3$. Then $\prod x_3 = \prod x \cdot \prod y$.

PROOF: Reconsider $W = \operatorname{support} x \cup \operatorname{support} y$ as a finite subset of I. For every object i such that $i \in \operatorname{support} x_3$ holds $i \in W$ by (5), [28, (40), (43)], [17, (1)]. For every function a from I into G and for every elements i, j of I such that $a \in \prod F$ holds $a(i) \cdot a(j) = a(j) \cdot a(i)$. $\prod x = \prod(x \restriction W)$. $\prod y = \prod(y \restriction W)$. $\prod x_3 = \prod(x_3 \restriction W)$. Set $c_1 = \mathrm{CFS}(W)$. Reconsider $w_1 = (x \restriction W) \cdot c_1$ as a finite sequence of elements of G. Reconsider $w_3 = (y \restriction W) \cdot c_1$ as a finite sequence of elements of G. Reconsider $w_2 = (x_3 \restriction W) \cdot c_1$ as a finite sequence of elements of G. For every natural numbers i, j such that i, $j \in \operatorname{dom} w_2$ and $i \neq j$ holds $w_{1i} \cdot w_{3j} = w_{3j} \cdot w_{1i}$ by [10, (3), (12), (49)], (5). For every natural number i such that $i \in \operatorname{dom} w_2$ holds $w_2(i) = w_{1i} \cdot w_{3i}$ by [10, (3), (12), (49)], (5). $\prod w_2 = \prod w_1 \cdot \prod w_3$. □

(54) Let us consider a non empty set I, a group G, and a family F of groups of I. Then F is an internal direct sum components of G and I if and only if for every element i of I, $F(i)$ is a subgroup of G and for every elements i, j of I and for every elements g_1, g_2 of G such that $i \neq j$ and $g_1 \in F(i)$ and $g_2 \in F(j)$ holds $g_1 \cdot g_2 = g_2 \cdot g_1$ and for every element y of G, there exists a finite-support function x from I into G such that $x \in \operatorname{sum} F$ and $y = \prod x$ and for every finite-support functions x_1, x_2 from I into G such that x_1, $x_2 \in \operatorname{sum} F$ and $\prod x_1 = \prod x_2$ holds $x_1 = x_2$.

PROOF: Define $\mathcal{P}[\text{object}, \text{object}] \equiv$ there exists a finite-support function x from I into G such that $\$_1 = x$ and $\$_2 = \prod x$. For every element x of $\operatorname{sum} F$, there exists an element y of G such that $\mathcal{P}[x, y]$. Consider h being a function from $\operatorname{sum} F$ into G such that for every element x of $\operatorname{sum} F$, $\mathcal{P}[x, h(x)]$ from [11, Sch. 3]. For every object y such that $y \in \Omega_G$ there exists an object x such that $x \in \Omega_{\operatorname{sum} F}$ and $y = h(x)$. For every objects x_1, x_2 such that x_1, $x_2 \in \Omega_{\operatorname{sum} F}$ and $h(x_1) = h(x_2)$ holds $x_1 = x_2$. For every finite-support function a from I into G such that $a \in \operatorname{sum} F$ holds $h(a) = \prod a$. For every elements x, y of $\operatorname{sum} F$, $h(x \cdot y) = h(x) \cdot h(y)$. □

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