

# $\sigma$ -ring and $\sigma$ -algebra of Sets<sup>1</sup>

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**Summary.** In this article, semiring and semialgebra of sets are formalized so as to construct a measure of a given set in the next step. Although a semiring of sets has already been formalized in [13], that is, strictly speaking, a definition of a quasi semiring of sets suggested in the last few decades [15]. We adopt a classical definition of a semiring of sets here to avoid such a confusion. Ring of sets and algebra of sets have been formalized as non empty preboolean set [23] and field of subsets [18], respectively. In the second section, definitions of a ring and a  $\sigma$ -ring of sets, which are based on a semiring and a ring of sets respectively, are formalized and their related theorems are proved. In the third section, definitions of an algebra and a  $\sigma$ -algebra of sets, which are based on a semialgebra and an algebra of sets respectively, are formalized and their related theorems are proved. In the last section, mutual relationships between  $\sigma$ -ring and  $\sigma$ -algebra of sets are formalized and some related examples are given. The formalization is based on [15], and also referred to [9] and [16].

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The notation and terminology used in this paper have been introduced in the following articles: [1], [2], [3], [17], [21], [6], [14], [23], [10], [11], [7], [8], [22], [4], [5], [18], [19], [26], [27], [20], [13], [25], and [12].

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## 1. PRELIMINARIES

Now we state the propositions:

- (1) Let us consider finite sequences  $f_1, f_2$ , and a natural number  $k$ . Suppose  $k \in \text{Seg}(\text{len } f_1 \cdot \text{len } f_2)$ . Then
  - (i)  $(k -' 1 \text{ mod } \text{len } f_2) + 1 \in \text{dom } f_2$ , and
  - (ii)  $(k -' 1 \text{ div } \text{len } f_2) + 1 \in \text{dom } f_1$ .
- (2) Let us consider a non empty, finite set  $S$ . Then  $\bigcup \text{CFS}(S) = \bigcup S$ .
- (3) Let us consider an object  $x$ . Then  $\langle x \rangle$  is a disjoint valued finite sequence.
- (4) Let us consider sets  $x, y$ , and a finite sequence  $F$ . If  $F = \langle x, y \rangle$  and  $x$  misses  $y$ , then  $F$  is disjoint valued.
- (5) Let us consider finite sequences  $f_1, f_2$ . Then there exists a finite sequence  $f$  such that
  - (i)  $\bigcup f_1 \cap \bigcup f_2 = \bigcup f$ , and
  - (ii)  $\text{dom } f = \text{Seg}(\text{len } f_1 \cdot \text{len } f_2)$ , and
  - (iii) for every natural number  $i$  such that  $i \in \text{dom } f$  holds  $f(i) = f_1((i -' 1 \text{ div } \text{len } f_2) + 1) \cap f_2((i -' 1 \text{ mod } \text{len } f_2) + 1)$ .

PROOF: For every natural number  $k$  such that  $k \in \text{Seg}(\text{len } f_1 \cdot \text{len } f_2)$  holds  $(k -' 1 \text{ mod } \text{len } f_2) + 1 \in \text{dom } f_2$  and  $(k -' 1 \text{ div } \text{len } f_2) + 1 \in \text{dom } f_1$ . Define  $\mathcal{P}[\text{natural number, object}] \equiv \$_2 = f_1((\$_1 -' 1 \text{ div } \text{len } f_2) + 1) \cap f_2((\$_1 -' 1 \text{ mod } \text{len } f_2) + 1)$ . Consider  $f$  being a finite sequence such that  $\text{dom } f = \text{Seg}(\text{len } f_1 \cdot \text{len } f_2)$  and for every natural number  $k$  such that  $k \in \text{Seg}(\text{len } f_1 \cdot \text{len } f_2)$  holds  $\mathcal{P}[k, f(k)]$  from [6, Sch. 1].  $\square$

- (6) Let us consider disjoint valued finite sequences  $f_1, f_2$ . Then there exists a disjoint valued finite sequence  $f$  such that
  - (i)  $\bigcup f_1 \cap \bigcup f_2 = \bigcup f$ , and
  - (ii)  $\text{dom } f = \text{Seg}(\text{len } f_1 \cdot \text{len } f_2)$ , and
  - (iii) for every natural number  $i$  such that  $i \in \text{dom } f$  holds  $f(i) = f_1((i -' 1 \text{ div } \text{len } f_2) + 1) \cap f_2((i -' 1 \text{ mod } \text{len } f_2) + 1)$ .

The theorem is a consequence of (5).

- (7) Let us consider a set  $X$ , and a non empty,  $\setminus$ -closed family  $S$  of subsets of  $X$ . Then  $\emptyset \in S$ .

Let  $X$  be a set. One can check that every family of subsets of  $X$  which is non empty and  $\setminus$ -closed has also the empty element.

2. CLASSICAL SEMIRING, RING AND  $\sigma$ -RING OF SETS

Let  $I_1$  be a set. We say that  $I_1$  is semi  $\setminus$ -closed if and only if

(Def. 1) for every sets  $X, Y$  such that  $X, Y \in I_1$  there exists a disjoint valued finite sequence  $F$  of elements of  $I_1$  such that  $X \setminus Y = \bigcup F$ .

Let  $X$  be a set. Let us note that  $2^X$  is semi  $\setminus$ -closed and there exists a family of subsets of  $X$  which is non empty, semi  $\setminus$ -closed, and  $\cap$ -closed and there exists a family of subsets of  $X$  which is semi  $\setminus$ -closed and  $\cap$ -closed and has the empty element.

A semiring of  $X$  is a semi  $\setminus$ -closed,  $\cap$ -closed family of subsets of  $X$  with the empty element. Now we state the propositions:

- (8) Let us consider a set  $X$ , a family  $S$  of subsets of  $X$ , and sets  $S_1, S_2$ . Suppose  $S_1, S_2 \in S$  and  $S$  is semi  $\setminus$ -closed. Then there exists a finite subset  $x$  of  $S$  such that  $x$  is a partition of  $S_1 \setminus S_2$ .
- (9) Let us consider a set  $X$ , and a non empty family  $S$  of subsets of  $X$ . Suppose  $S$  is semi  $\setminus$ -closed. Then  $S$  is  $\underset{fp}{\subseteq}$ -closed. The theorem is a consequence of (8).
- (10) Let us consider a set  $X$ , and a family  $S$  of subsets of  $X$ . Suppose  $S$  is  $\cap_{fp}$ -closed and  $\underset{fp}{\subseteq}$ -closed and has the empty element. Then  $S$  is semi  $\setminus$ -closed. The theorem is a consequence of (2).

Note that every set which is  $\setminus$ -closed is also semi  $\setminus$ -closed and  $\cap$ -closed.

Let  $X$  be a set. Observe that there exists a family of subsets of  $X$  which is non empty and preboolean and every set which is non empty and preboolean has also the empty element.

Let  $X$  be a set and  $S$  be a semi  $\setminus$ -closed,  $\cap$ -closed family of subsets of  $X$  with the empty element. The ring generated by  $S$  yielding a non empty, preboolean family of subsets of  $X$  is defined by the term

(Def. 2)  $\cap\{Z, \text{ where } Z \text{ is a non empty, preboolean family of subsets of } X : S \subseteq Z\}$ .

Now we state the proposition:

- (11) Let us consider a set  $X$ , and a semi  $\setminus$ -closed,  $\cap$ -closed family  $P$  of subsets of  $X$  with the empty element. Then  $P \subseteq$  the ring generated by  $P$ .

Let  $X$  be a set and  $S$  be a semi  $\setminus$ -closed,  $\cap$ -closed family of subsets of  $X$  with the empty element. The functor  $\text{DisUnion } S$  yielding a non empty family of subsets of  $X$  is defined by the term

(Def. 3)  $\{A, \text{ where } A \text{ is a subset of } X : \text{ there exists a disjoint valued finite sequence } F \text{ of elements of } S \text{ such that } A = \bigcup F\}$ .

Let us consider a set  $X$  and a semi  $\setminus$ -closed,  $\cap$ -closed family  $S$  of subsets of  $X$  with the empty element. Now we state the propositions:

- (12)  $S \subseteq \text{DisUnion } S$ .
- (13)  $\text{DisUnion } S$  is  $\cap$ -closed. The theorem is a consequence of (6) and (1).

Now we state the proposition:

- (14) Let us consider a set  $X$ , a semi  $\setminus$ -closed,  $\cap$ -closed family  $S$  of subsets of  $X$  with the empty element, and sets  $A, B, P$ . If  $P = \text{DisUnion } S$  and  $A, B \in P$  and  $A$  misses  $B$ , then  $A \cup B \in P$ .

Let us consider a set  $X$ , a semi  $\setminus$ -closed,  $\cap$ -closed family  $S$  of subsets of  $X$  with the empty element, and sets  $A, B$ . Now we state the propositions:

- (15) If  $A, B \in S$ , then  $B \setminus A \in \text{DisUnion } S$ .
- (16) If  $A \in S$  and  $B \in \text{DisUnion } S$ , then  $B \setminus A \in \text{DisUnion } S$ .

PROOF: Reconsider  $A_1 = A$  as a subset of  $X$ . Consider  $B_1$  being a subset of  $X$  such that  $B = B_1$  and there exists a disjoint valued finite sequence  $F$  of elements of  $S$  such that  $B_1 = \bigcup F$ . Consider  $g_1$  being a disjoint valued finite sequence of elements of  $S$  such that  $B_1 = \bigcup g_1$ . Reconsider  $R_1 = \text{DisUnion } S$  as a non empty set. Define  $\mathcal{P}[\text{natural number, object}] \equiv \$_2 = g_1(\$_1) \setminus A_1$ . For every natural number  $k$  such that  $k \in \text{Seg len } g_1$  there exists an element  $x$  of  $R_1$  such that  $\mathcal{P}[k, x]$  by [10, (3)], (15). Consider  $g_2$  being a finite sequence of elements of  $R_1$  such that  $\text{dom } g_2 = \text{Seg len } g_1$  and for every natural number  $k$  such that  $k \in \text{Seg len } g_1$  holds  $\mathcal{P}[k, g_2(k)]$  from [6, Sch. 5]. For every natural numbers  $n, m$  such that  $n, m \in \text{dom } g_2$  and  $n \neq m$  holds  $g_2(n)$  misses  $g_2(m)$ . Set  $R = \text{DisUnion } S$ . Define  $\mathcal{H}[\text{natural number}] \equiv \bigcup \text{rng}(g_2 \upharpoonright \$_1) \in R$ . For every natural number  $k$  such that  $\mathcal{H}[k]$  holds  $\mathcal{H}[k + 1]$  by [4, (13)], [6, (59), (82)], [24, (55)]. For every natural number  $k$ ,  $\mathcal{H}[k]$  from [4, Sch. 2].  $\square$

Now we state the propositions:

- (17) Let us consider a set  $X$ , a semi  $\setminus$ -closed,  $\cap$ -closed family  $S$  of subsets of  $X$  with the empty element, and sets  $A, B, R$ . Suppose  $R = \text{DisUnion } S$  and  $A, B \in R$  and  $A \neq \emptyset$ . Then  $B \setminus A \in R$ .

PROOF: Consider  $A_1$  being a subset of  $X$  such that  $A = A_1$  and there exists a disjoint valued finite sequence  $F$  of elements of  $S$  such that  $A_1 = \bigcup F$ . Consider  $f_1$  being a disjoint valued finite sequence of elements of  $S$  such that  $A_1 = \bigcup f_1$ . Consider  $B_1$  being a subset of  $X$  such that  $B = B_1$  and there exists a disjoint valued finite sequence  $F$  of elements of  $S$  such that  $B_1 = \bigcup F$ . Reconsider  $R_1 = R$  as a non empty set. Define  $\mathcal{P}[\text{natural number, object}] \equiv \$_2 = B_1 \setminus f_1(\$_1)$ . For every natural number  $k$  such that  $k \in \text{Seg len } f_1$  there exists an element  $x$  of  $R_1$  such that  $\mathcal{P}[k, x]$  by [10, (3)], (16). Consider  $F$  being a finite sequence of elements of  $R_1$  such

that  $\text{dom } F = \text{Seg len } f_1$  and for every natural number  $k$  such that  $k \in \text{Seg len } f_1$  holds  $\mathcal{P}[k, F(k)]$  from [6, Sch. 5]. Define  $\mathcal{P}[\text{natural number}] \equiv \bigcap \text{rng}(F \setminus \$_1) \in R$ . For every natural number  $k$  such that  $\mathcal{P}[k]$  holds  $\mathcal{P}[k+1]$  by [6, (82)], [4, (11)], [6, (59)], [24, (55)]. For every natural number  $k$ ,  $\mathcal{P}[k]$  from [4, Sch. 2].  $\square$

- (18) Let us consider a set  $X$ , and a semi  $\setminus$ -closed,  $\cap$ -closed family  $S$  of subsets of  $X$  with the empty element. Then the ring generated by  $S = \text{DisUnion } S$ . The theorem is a consequence of (13), (17), and (14).

Let  $X$  be a set.

A  $\sigma$ -ring of subsets of  $X$  is a non empty, preboolean family of subsets of  $X$  and is defined by

- (Def. 4) for every sequence  $F$  of subsets of  $X$  such that  $\text{rng } F \subseteq \text{it}$  holds  $\bigcup F \in \text{it}$ .

Let us observe that every  $\sigma$ -ring of subsets of  $X$  is  $\sigma$ -multiplicative.

Let  $S$  be a family of subsets of  $X$ . The functor  $\sigma\text{-ring}(S)$  yielding a  $\sigma$ -ring of subsets of  $X$  is defined by

- (Def. 5)  $S \subseteq \text{it}$  and for every set  $Z$  such that  $S \subseteq Z$  and  $Z$  is a  $\sigma$ -ring of subsets of  $X$  holds  $\text{it} \subseteq Z$ .

Now we state the proposition:

- (19) Let us consider a set  $X$ , and a semi  $\setminus$ -closed,  $\cap$ -closed family  $S$  of subsets of  $X$  with the empty element. Then  $\sigma\text{-ring}(\text{the ring generated by } S) = \sigma\text{-ring}(S)$ . The theorem is a consequence of (11).

### 3. SEMIALGEBRA, ALGEBRA AND $\sigma$ -ALGEBRA OF SETS

Let  $X$  be a set.

A semialgebra of sets of  $X$  is a semi  $\setminus$ -closed,  $\cap$ -closed family of subsets of  $X$  with the empty element and is defined by

- (Def. 6)  $X \in \text{it}$ .

Now we state the proposition:

- (20) Let us consider a set  $X$ . Then every field of subsets of  $X$  is a semialgebra of sets of  $X$ .

Let  $X$  be a set and  $S$  be a semialgebra of sets of  $X$ . The field generated by  $S$  yielding a non empty field of subsets of  $X$  is defined by the term

- (Def. 7)  $\bigcap \{Z, \text{ where } Z \text{ is a field of subsets of } X : S \subseteq Z\}$ .

Now we state the propositions:

- (21) Let us consider a set  $X$ , and a semialgebra  $P$  of sets of  $X$ . Then  $P \subseteq$  the field generated by  $P$ .

- (22) Let us consider a set  $X$ , and a semialgebra  $S$  of sets of  $X$ . Then the field generated by  $S = \text{DisUnion } S$ . The theorem is a consequence of (13), (17), and (14).
- (23) Let us consider a non empty set  $X$ , and a semialgebra  $S$  of sets of  $X$ . Then  $\sigma(\text{the field generated by } S) = \sigma(S)$ . The theorem is a consequence of (21).

#### 4. MUTUAL RELATIONSHIPS BETWEEN $\sigma$ -RING AND $\sigma$ -ALGEBRA OF SETS

Let us consider a set  $X$  and a set  $S$ . Now we state the propositions:

- (24) If  $S$  is a  $\sigma$ -field of subsets of  $X$ , then  $S$  is a  $\sigma$ -ring of subsets of  $X$ .
- (25) If  $S$  is a  $\sigma$ -ring of subsets of  $X$  and  $X \in S$ , then  $S$  is a  $\sigma$ -field of subsets of  $X$ .

Let us consider a family  $S$  of subsets of  $\mathbb{R}$ . Now we state the propositions:

- (26) Suppose  $S = \{I, \text{ where } I \text{ is a subset of } \mathbb{R} : I \text{ is left open interval}\}$ . Then  $S$  is semi  $\setminus$ -closed and  $\cap$ -closed and has the empty element. The theorem is a consequence of (10).
- (27) Suppose  $S = \{I, \text{ where } I \text{ is a subset of } \mathbb{R} : I \text{ is right open interval}\}$ . Then  $S$  is semi  $\setminus$ -closed and  $\cap$ -closed and has the empty element. The theorem is a consequence of (4) and (3).

Now we state the proposition:

- (28) the set of all  $I$  where  $I$  is an interval is a semialgebra of sets of  $\mathbb{R}$ . The theorem is a consequence of (3) and (4).

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