

Morley's Trisector Theorem

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Summary. Morley's trisector theorem states that "The points of intersection of the adjacent trisectors of the angles of any triangle are the vertices of an equilateral triangle" [10].

There are many proofs of Morley's trisector theorem [12, 16, 9, 13, 8, 20, 3, 18]. We follow the proof given by A. Letac in [15].

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The notation and terminology used in this paper have been introduced in the following articles: [1], [11], [7], [14], [19], [2], [4], [23], [5], [24], [21], [22], and [6].

1. PRELIMINARIES

From now on A, B, C, D, E, F, G denote points of \mathcal{E}_T^2 .

Now we state the propositions:

- (1) $\sphericalangle(A, B, A) = 0$.
- (2) $0 \leq \sphericalangle(A, B, C) < 2 \cdot \pi$.
- (3) (i) $0 \leq \sphericalangle(A, B, C) < \pi$, or
(ii) $\sphericalangle(A, B, C) = \pi$, or
(iii) $\pi < \sphericalangle(A, B, C) < 2 \cdot \pi$.

The theorem is a consequence of (2).

- (4) $|F - E|^2 = |A - E|^2 + |A - F|^2 - 2 \cdot |A - E| \cdot |A - F| \cdot \cos \sphericalangle(E, A, F)$.
- (5) If A, B, C are mutually different and $0 < \sphericalangle(A, B, C) < \pi$, then $0 < \sphericalangle(B, C, A) < \pi$ and $0 < \sphericalangle(C, A, B) < \pi$.

- (6) Suppose A, B, C are mutually different and $\sphericalangle(A, B, C) = 0$. Then
- (i) $\sphericalangle(B, C, A) = 0$ and $\sphericalangle(C, A, B) = \pi$, or
 - (ii) $\sphericalangle(B, C, A) = \pi$ and $\sphericalangle(C, A, B) = 0$ and $\sphericalangle(A, B, C) + \sphericalangle(B, C, A) + \sphericalangle(C, A, B) = \pi$.
- (7) Suppose A, B, C are mutually different and $\sphericalangle(A, B, C) = \pi$. Then
- (i) $\sphericalangle(B, C, A) = 0$, and
 - (ii) $\sphericalangle(C, A, B) = 0$, and
 - (iii) $\sphericalangle(A, B, C) + \sphericalangle(B, C, A) + \sphericalangle(C, A, B) = \pi$.
- (8) If A, B, C are mutually different and $\sphericalangle(A, B, C) > \pi$, then $\sphericalangle(A, B, C) + \sphericalangle(B, C, A) + \sphericalangle(C, A, B) = 5 \cdot \pi$.

Let us assume that $\sphericalangle(C, B, A) < \pi$. Now we state the propositions:

- (9) $0 \leq \text{area of } \triangle(A, B, C)$. The theorem is a consequence of (2).
- (10) $0 \leq \varnothing_{\triangle}(A, B, C)$. The theorem is a consequence of (9).

2. MORLEY'S THEOREM

Now we state the propositions:

- (11) Suppose A, F, C form a triangle and $\sphericalangle(C, F, A) < \pi$ and $\sphericalangle(A, C, F) = \sphericalangle(A, C, B)/3$ and $\sphericalangle(F, A, C) = \sphericalangle(B, A, C)/3$ and $(\sphericalangle(A, C, B)/3) + (\sphericalangle(B, A, C)/3) + (\sphericalangle(C, B, A)/3) = \pi/3$.
Then $|A - F| \cdot \sin((\pi/3) - (\sphericalangle(C, B, A)/3)) = |A - C| \cdot \sin(\sphericalangle(A, C, B)/3)$.
- (12) Suppose A, B, C form a triangle and A, F, C form a triangle and $\sphericalangle(C, F, A) < \pi$ and $\sphericalangle(A, C, F) = \sphericalangle(A, C, B)/3$ and $\sphericalangle(F, A, C) = \sphericalangle(B, A, C)/3$ and $(\sphericalangle(A, C, B)/3) + (\sphericalangle(B, A, C)/3) + (\sphericalangle(C, B, A)/3) = \pi/3$ and $\sin((\pi/3) - (\sphericalangle(C, B, A)/3)) \neq 0$. Then $|A - F| = 4 \cdot \varnothing_{\triangle}(A, B, C) \cdot \sin(\sphericalangle(C, B, A)/3) \cdot \sin((\pi/3) + (\sphericalangle(C, B, A)/3)) \cdot \sin(\sphericalangle(A, C, B)/3)$. The theorem is a consequence of (11).
- (13) Suppose C, A, B form a triangle and A, F, C form a triangle and F, A, E form a triangle and E, A, B form a triangle and $\sphericalangle(B, A, E) = \sphericalangle(B, A, C)/3$ and $\sphericalangle(F, A, C) = \sphericalangle(B, A, C)/3$. Then $\sphericalangle(E, A, F) = \sphericalangle(B, A, C)/3$. PROOF: $\sphericalangle(E, A, F) \neq 4 \cdot \pi + (\sphericalangle(B, A, C)/3)$ by [17, (5)], (2), [7, (30)]. $\sphericalangle(E, A, F) \neq 2 \cdot \pi + (\sphericalangle(B, A, C)/3)$ by (2), [7, (30)]. \square
- (14) Suppose C, A, B form a triangle and $\sphericalangle(A, C, B) < \pi$ and A, F, C form a triangle and F, A, E form a triangle and E, A, B form a triangle and $\sphericalangle(B, A, E) = \sphericalangle(B, A, C)/3$ and $\sphericalangle(F, A, C) = \sphericalangle(B, A, C)/3$. Then $(\pi/3) + (\sphericalangle(A, C, B)/3) + ((\pi/3) + (\sphericalangle(C, B, A)/3)) + \sphericalangle(E, A, F) = \pi$. The theorem is a consequence of (13).

- (15) If A, C, B form a triangle, then $\sin((\pi/3) - (\angle(A, C, B)/3)) \neq 0$. The theorem is a consequence of (2).
- (16) Suppose A, B, C form a triangle and A, B, E form a triangle and $\angle(E, B, A) = \angle(C, B, A)/3$ and $\angle(B, A, E) = \angle(B, A, C)/3$ and A, F, C form a triangle and $\angle(A, C, F) = \angle(A, C, B)/3$ and $\angle(F, A, C) = \angle(B, A, C)/3$ and $\angle(A, C, B) < \pi$. Then $|F - E| = 4 \cdot \varnothing_{\square}(A, B, C) \cdot \sin(\angle(A, C, B)/3) \cdot \sin(\angle(C, B, A)/3) \cdot \sin(\angle(B, A, C)/3)$.
 PROOF: $\sin((\pi/3) - (\angle(A, C, B)/3)) \neq 0$. $\sin((\pi/3) - (\angle(C, B, A)/3)) \neq 0$. $0 < \angle(A, C, B)$. $\angle(C, B, A) < \pi$. $0 < \angle(A, C, B) < \pi$ and A, C, B are mutually different. $\angle(B, A, C) < \pi$. $0 < \angle(B, A, E) < \pi$. $\angle(A, E, B) < \pi$. $0 < \angle(F, A, C) < \pi$. $\angle(C, F, A) < \pi$. F, A, E form a triangle by [19, (4)], (5), [17, (5)], [7, (31)]. $|A - F| = \varnothing_{\square}(A, B, C) \cdot 4 \cdot \sin(\angle(C, B, A)/3) \cdot \sin((\pi/3) + (\angle(C, B, A)/3)) \cdot \sin(\angle(A, C, B)/3)$. $(\pi/3) + (\angle(A, C, B)/3) + ((\pi/3) + (\angle(C, B, A)/3)) + \angle(E, A, F) = \pi$. $|F - E|^2 = |A - E|^2 + |A - F|^2 - 2 \cdot |A - E| \cdot |A - F| \cdot \cos \angle(E, A, F)$. \square
- (17) Suppose A, B, C form a triangle and $\angle(E, B, A) = \angle(C, B, A)/3$ and $\angle(B, A, E) = \angle(B, A, C)/3$. Then A, B, E form a triangle. The theorem is a consequence of (1) and (2).
- (18) Suppose A, B, C form a triangle and $\angle(A, C, F) = \angle(A, C, B)/3$ and $\angle(F, A, C) = \angle(B, A, C)/3$. Then A, F, C form a triangle. The theorem is a consequence of (1) and (2).
- (19) Suppose A, B, C form a triangle and $\angle(C, B, G) = \angle(C, B, A)/3$ and $\angle(G, C, B) = \angle(A, C, B)/3$. Then C, G, B form a triangle. The theorem is a consequence of (1) and (2).

Let us assume that A, B, C form a triangle and $\angle(A, C, B) < \pi$ and $\angle(E, B, A) = \angle(C, B, A)/3$ and $\angle(B, A, E) = \angle(B, A, C)/3$ and $\angle(A, C, F) = \angle(A, C, B)/3$ and $\angle(F, A, C) = \angle(B, A, C)/3$ and $\angle(C, B, G) = \angle(C, B, A)/3$ and $\angle(G, C, B) = \angle(A, C, B)/3$. Now we state the propositions:

- (20) (i) $|F - E| = 4 \cdot \varnothing_{\square}(A, B, C) \cdot \sin(\angle(A, C, B)/3) \cdot \sin(\angle(C, B, A)/3) \cdot \sin(\angle(B, A, C)/3)$, and
 (ii) $|G - F| = 4 \cdot \varnothing_{\square}(C, A, B) \cdot \sin(\angle(C, B, A)/3) \cdot \sin(\angle(B, A, C)/3) \cdot \sin(\angle(A, C, B)/3)$, and
 (iii) $|E - G| = 4 \cdot \varnothing_{\square}(B, C, A) \cdot \sin(\angle(B, A, C)/3) \cdot \sin(\angle(A, C, B)/3) \cdot \sin(\angle(C, B, A)/3)$.

The theorem is a consequence of (17), (18), (19), (2), (5), and (16).

- (21) (i) $|F - E| = |G - F|$, and
 (ii) $|F - E| = |E - G|$, and
 (iii) $|G - F| = |E - G|$.

The theorem is a consequence of (20).

(22) MORLEY'S TRISECTOR THEOREM:

Suppose A, B, C form a triangle and $\angle(A, B, C) < \pi$ and $\angle(E, C, A) = \angle(B, C, A)/3$ and $\angle(C, A, E) = \angle(C, A, B)/3$ and $\angle(A, B, F) = \angle(A, B, C)/3$ and $\angle(F, A, B) = \angle(C, A, B)/3$ and $\angle(B, C, G) = \angle(B, C, A)/3$ and $\angle(G, B, C) = \angle(A, B, C)/3$. Then

- (i) $|F - E| = |G - F|$, and
- (ii) $|F - E| = |E - G|$, and
- (iii) $|G - F| = |E - G|$.

The theorem is a consequence of (21).

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