

Flexary Operations¹

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Summary. In this article we introduce necessary notation and definitions to prove the Euler's Partition Theorem according to H.S. Wilf's lecture notes [31]. Our aim is to create an environment which allows to formalize the theorem in a way that is as similar as possible to the original informal proof.

Euler's Partition Theorem is listed as item #45 from the "Formalizing 100 Theorems" list maintained by Freek Wiedijk at <http://www.cs.ru.nl/F.Wiedijk/100/> [30].

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The notation and terminology used in this paper have been introduced in the following articles: [1], [2], [6], [8], [15], [27], [13], [14], [23], [9], [10], [7], [25], [24], [3], [4], [19], [5], [22], [32], [33], [11], [21], [28], [18], and [12].

1. AUXILIARY FACTS ABOUT FINITE SEQUENCES CONCATENATION

From now on x, y denote objects, D, D_1, D_2 denote non empty sets, i, j, k, m, n denote natural numbers, f, g denote finite sequences of elements of D^* , f_1 denotes a finite sequence of elements of D_1^* , and f_2 denotes a finite sequence of elements of D_2^* .

Now we state the propositions:

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(1) Let us consider a function yielding function F , and an object a . Then $a \in \text{Values } F$ if and only if there exists x and there exists y such that $x \in \text{dom } F$ and $y \in \text{dom}(F(x))$ and $a = F(x)(y)$.

(2) Let us consider a set D , and finite sequences f, g of elements of D^* . Then $\text{Values } f \wedge g = \text{Values } f \cup \text{Values } g$.

PROOF: Set $F = f \wedge g$. $\text{Values } f \subseteq \text{Values } F$ by (1), [6, (26)]. $\text{Values } g \subseteq \text{Values } F$ by (1), [6, (28)]. $\text{Values } F \subseteq \text{Values } f \cup \text{Values } g$ by (1), [6, (25)].

□

(3) The concatenation of $D \odot f \wedge g = (\text{the concatenation of } D \odot f) \wedge (\text{the concatenation of } D \odot g)$.

(4) $\text{rng}(\text{the concatenation of } D \odot f) = \text{Values } f$.

PROOF: Set $D_3 = \text{the concatenation of } D$. Define $\mathcal{P}[\text{natural number}] \equiv$ for every finite sequence f of elements of D^* such that $\text{len } f = \$_1$ holds $\text{rng}(D_3 \odot f) = \text{Values } f$. $\mathcal{P}[0]$. If $\mathcal{P}[i]$, then $\mathcal{P}[i+1]$ by [8, (19), (16)], (3), [27, (11)]. $\mathcal{P}[i]$ from [4, Sch. 2]. □

(5) If $f_1 = f_2$, then the concatenation of $D_1 \odot f_1 = \text{the concatenation of } D_2 \odot f_2$.

PROOF: Set $C = \text{the concatenation of } D_2$. Set $N = \text{the concatenation of } D_1$. Define $\mathcal{P}[\text{natural number}] \equiv$ for every finite sequence f_4 of elements of D_1^* for every finite sequence f_3 of elements of D_2^* such that $\$_1 = \text{len } f_4$ and $f_4 = f_3$ holds $N \odot f_4 = C \odot f_3$. $\mathcal{P}[0]$. If $\mathcal{P}[i]$, then $\mathcal{P}[i+1]$ by [8, (19), (16)], (3), [27, (11)]. $\mathcal{P}[i]$ from [4, Sch. 2]. □

(6) $i \in \text{dom}(\text{the concatenation of } D \odot f)$ if and only if there exists n and there exists k such that $n+1 \in \text{dom } f$ and $k \in \text{dom}(f(n+1))$ and $i = k + \text{len}(\text{the concatenation of } D \odot f \upharpoonright n)$.

PROOF: Set $D_3 = \text{the concatenation of } D$. Define $\mathcal{P}[\text{natural number}] \equiv$ for every i for every finite sequence f of elements of D^* such that $\text{len } f = \$_1$ holds $i \in \text{dom}(D_3 \odot f)$ iff there exists n and there exists k such that $n+1 \in \text{dom } f$ and $k \in \text{dom}(f(n+1))$ and $i = k + \text{len}(D_3 \odot f \upharpoonright n)$. $\mathcal{P}[0]$. If $\mathcal{P}[j]$, then $\mathcal{P}[j+1]$ by [8, (19), (16)], (3), [27, (11)]. $\mathcal{P}[j]$ from [4, Sch. 2]. □

(7) Suppose $i \in \text{dom}(\text{the concatenation of } D \odot f)$. Then

(i) $(\text{the concatenation of } D \odot f)(i) = (\text{the concatenation of } D \odot f \wedge g)(i)$,
and

(ii) $(\text{the concatenation of } D \odot f)(i) = (\text{the concatenation of } D \odot g \wedge f)(i + \text{len}(\text{the concatenation of } D \odot g))$.

The theorem is a consequence of (3).

(8) Suppose $k \in \text{dom}(f(n+1))$. Then $f(n+1)(k) = (\text{the concatenation of}$

$D \odot f)(k + \text{len}(\text{the concatenation of } D \odot f|n))$. The theorem is a consequence of (3).

2. FLEXARY PLUS

From now on f denotes a complex-valued function and g, h denote complex-valued finite sequences.

Let us consider k and n . Let f, g be complex-valued functions. The functor $(f, k) + \dots + (g, n)$ yielding a complex number is defined by

- (Def. 1) (i) $h(0 + 1) = f(0 + k)$ and ... and $h(n -' k + 1) = f(n -' k + k)$, then
 $it = \sum(h|(n -' k + 1))$, **if** $f = g$ and $k \leq n$,
 (ii) $it = 0$, **otherwise**.

Now we state the propositions:

- (9) Suppose $k \leq n$. Then there exists h such that
 (i) $(f, k) + \dots + (f, n) = \sum h$, and
 (ii) $\text{len } h = n -' k + 1$, and
 (iii) $h(0 + 1) = f(0 + k)$ and ... and $h(n -' k + 1) = f(n -' k + k)$.

PROOF: Define $\mathcal{P}(\text{natural number}) = f(k + \$_1 - 1)$. Set $n_3 = n -' k + 1$. Consider p being a finite sequence such that $\text{len } p = n_3$ and for every i such that $i \in \text{dom } p$ holds $p(i) = \mathcal{P}(i)$ from [6, Sch. 2]. $\text{rng } p \subseteq \mathbb{C}$. $p(1 + 0) = f(k + 0)$ and ... and $p(1 + (n -' k)) = f(k + (n -' k))$ by [4, (11)], [26, (25)]. \square

- (10) If $(f, k) + \dots + (f, n) \neq 0$, then there exists i such that $k \leq i \leq n$ and $i \in \text{dom } f$.

PROOF: Consider h such that $(f, k) + \dots + (f, n) = \sum h$ and $\text{len } h = n -' k + 1$ and $h(0 + 1) = f(0 + k)$ and ... and $h(n -' k + 1) = f(n -' k + k)$. $\text{rng } h \subseteq \{0\}$ by [26, (25)], [4, (11)]. \square

- (11) $(f, k) + \dots + (f, k) = f(k)$. The theorem is a consequence of (9).
 (12) If $k \leq n + 1$, then $(f, k) + \dots + (f, (n + 1)) = ((f, k) + \dots + (f, n)) + f(n + 1)$. The theorem is a consequence of (11) and (9).
 (13) If $k \leq n$, then $(f, k) + \dots + (f, n) = f(k) + ((f, (k + 1)) + \dots + (f, n))$. The theorem is a consequence of (11) and (9).
 (14) If $k \leq m \leq n$, then $((f, k) + \dots + (f, m)) + ((f, (m + 1)) + \dots + (f, n)) = (f, k) + \dots + (f, n)$.

PROOF: Define $\mathcal{P}[\text{natural number}] \equiv ((f, k) + \dots + (f, m)) + ((f, (m + 1)) + \dots + (f, (m + \$_1))) = (f, k) + \dots + (f, (m + \$_1))$. $\mathcal{P}[0]$ by [4, (13)]. If $\mathcal{P}[i]$, then $\mathcal{P}[i + 1]$ by [4, (11)], (12). $\mathcal{P}[i]$ from [4, Sch. 2]. \square

- (15) If $k > \text{len } h$, then $(h, k) + \dots + (h, n) = 0$. The theorem is a consequence of (9).
- (16) If $n \geq \text{len } h$, then $(h, k) + \dots + (h, n) = (h, k) + \dots + (h, \text{len } h)$. The theorem is a consequence of (15) and (12).
- (17) $(h, 0) + \dots + (h, k) = (h, 1) + \dots + (h, k)$. The theorem is a consequence of (13).
- (18) $(h, 1) + \dots + (h, \text{len } h) = \sum h$. The theorem is a consequence of (9).
- (19) $(g \frown h, k) + \dots + (g \frown h, n) = ((g, k) + \dots + (g, n)) + ((h, (k - ' \text{len } g)) + \dots + (h, (n - ' \text{len } g)))$. The theorem is a consequence of (11), (15), (16), (17), and (14).

Let us consider n and k . Let f be a real-valued finite sequence. One can check that $(f, k) + \dots + (f, n)$ is real.

Let f be a natural-valued finite sequence. Note that $(f, k) + \dots + (f, n)$ is natural.

Let f be a complex-valued function. Assume $\text{dom } f \cap \mathbb{N}$ is finite. The functor $(f, n) + \dots$ yielding a complex number is defined by

- (Def. 2) for every k such that for every i such that $i \in \text{dom } f$ holds $i \leq k$ holds $it = (f, n) + \dots + (f, k)$.

Let us consider h . One can check that the functor $(h, n) + \dots$ yields a complex number and is defined by the term

- (Def. 3) $(h, n) + \dots + (h, \text{len } h)$.

Let n be a natural number and h be a natural-valued finite sequence. Let us note that $(h, n) + \dots$ is natural.

Now we state the propositions:

- (20) Let us consider a finite, complex-valued function f . Then $f(n) + (f, (n + 1)) + \dots = (f, n) + \dots$. The theorem is a consequence of (13).
- (21) $\sum h = (h, 1) + \dots$
- (22) $\sum h = h(1) + (h, 2) + \dots$. The theorem is a consequence of (18) and (20).

The scheme TT deals with complex-valued finite sequences f, g and natural numbers a, b and non zero natural numbers n, k and states that

- (Sch. 1) $(f, a) + \dots = (g, b) + \dots$

provided

- for every j , $(f, (a + j \cdot n)) + \dots + (f, (a + j \cdot n + (n - ' 1))) = (g, (b + j \cdot k) + \dots + (g, (b + j \cdot k + (k - ' 1)))$.

3. POWER FUNCTION

Let r be a real number and f be a real-valued function. The functor r^f yielding a real-valued function is defined by

(Def. 4) $\text{dom } it = \text{dom } f$ and for every x such that $x \in \text{dom } f$ holds $it(x) = r^{f(x)}$.

Let n be a natural number and f be a natural-valued function. One can verify that n^f is natural-valued.

Let r be a real number and f be a real-valued finite sequence. One can check that r^f is finite sequence-like and r^f is $(\text{len } f)$ -element.

Let f be a one-to-one, natural-valued function. Observe that $(2 + n)^f$ is one-to-one.

(23) Let us consider real numbers r, s . Then $r^{(s)} = \langle r^s \rangle$.

(24) Let us consider a real number r , and real-valued finite sequences f, g . Then $r^{f \wedge g} = r^f \wedge r^g$.

PROOF: Set $f_5 = f \wedge g$. Set $r_2 = r^f$. Set $r_3 = r^g$. For every i such that $1 \leq i \leq \text{len } f_5$ holds $r^{f_5}(i) = (r_2 \wedge r_3)(i)$ by [26, (25)], [6, (25)]. \square

(25) Let us consider a real-valued function f , and a function g . Then $2^f \cdot g = 2^{f \cdot g}$. PROOF: Set $h = 2^f$. Set $f_5 = f \cdot g$. $\text{dom}(h \cdot g) \subseteq \text{dom } 2^{f_5}$ by [9, (11)]. $\text{dom } 2^{f_5} \subseteq \text{dom}(h \cdot g)$ by [9, (11)]. For every x such that $x \in \text{dom } 2^{f_5}$ holds $(h \cdot g)(x) = 2^{f_5}(x)$ by [9, (11), (13)]. \square

(26) Let us consider an increasing, natural-valued finite sequence f . If $n > 1$, then $n^f(1) + (n^f, 2) + \dots < 2 \cdot n^{f(\text{len } f)}$.

PROOF: Define $\mathcal{P}[\text{natural number}] \equiv$ for every increasing, natural-valued finite sequence f such that $n > 1$ and $f(\text{len } f) \leq \$_1$ and $f \neq \emptyset$ holds $\sum n^f < 2 \cdot n^{f(\text{len } f)}$. For every natural-valued finite sequence f such that $n > 1$ and $\text{len } f = 1$ holds $\sum n^f < 2 \cdot n^{f(\text{len } f)}$ by [26, (25)], [19, (83)], [6, (40)], [11, (73)]. $\mathcal{P}[0]$ by [26, (25)], [4, (25)]. If $\mathcal{P}[i]$, then $\mathcal{P}[i + 1]$ by [4, (8), (25), (13)], [26, (25)]. $\mathcal{P}[i]$ from [4, Sch. 2]. $\sum n^f = n^f(1) + (n^f, 2) + \dots$ \square

(27) Let us consider increasing, natural-valued finite sequences f_1, f_2 . Suppose $n > 1$ and $n^{f_1}(1) + (n^{f_1}, 2) + \dots = n^{f_2}(1) + (n^{f_2}, 2) + \dots$. Then $f_1 = f_2$.

PROOF: For every natural-valued finite sequence f such that $n > 1$ and $\sum n^f \leq 0$ holds $f = \emptyset$ by [11, (85)], [19, (83)]. Define $\mathcal{P}[\text{natural number}] \equiv$ for every increasing, natural-valued finite sequences f_1, f_2 such that $n > 1$ and $\sum n^{f_1} \leq \$_1$ and $\sum n^{f_1} = \sum n^{f_2}$ holds $f_1 = f_2$. $\mathcal{P}[0]$. If $\mathcal{P}[i]$, then $\mathcal{P}[i + 1]$ by (21), (22), [4, (8)], [11, (72)]. $\mathcal{P}[i]$ from [4, Sch. 2]. $n^{f_1}(1) + (n^{f_1}, 2) + \dots = \sum n^{f_1}$. $n^{f_2}(1) + (n^{f_2}, 2) + \dots = \sum n^{f_2}$. \square

(28) Let us consider a natural-valued function f . If $n > 1$, then $\text{Coim}(n^f, n^k) = \text{Coim}(f, k)$. PROOF: $\text{Coim}(n^f, n^k) \subseteq \text{Coim}(f, k)$ by [17, (30)]. \square

(29) Let us consider natural-valued functions f_1, f_2 . Suppose $n > 1$. Then f_1 and f_2 are fiberwise equipotent if and only if n^{f_1} and n^{f_2} are fiberwise equipotent. PROOF: If f_1 and f_2 are fiberwise equipotent, then n^{f_1} and n^{f_2} are fiberwise equipotent by [9, (72)], [17, (30)], (28). For every object x , $\overline{\text{Coim}(f_1, x)} = \overline{\text{Coim}(f_2, x)}$ by [9, (72)], [17, (30)], (28). \square

(30) Let us consider one-to-one, natural-valued finite sequences f_1, f_2 . Suppose $n > 1$ and $n^{f_1}(1) + (n^{f_1}, 2) + \dots = n^{f_2}(1) + (n^{f_2}, 2) + \dots$. Then $\text{rng } f_1 = \text{rng } f_2$.

PROOF: Reconsider $F_1 = f_1, F_2 = f_2$ as a finite sequence of elements of \mathbb{R} . Set $s_1 = \text{sort}_a F_1$. Set $s_2 = \text{sort}_a F_2$. n^{F_1} and n^{s_1} are fiberwise equipotent. n^{F_2} and n^{s_2} are fiberwise equipotent. For every extended reals e_1, e_2 such that $e_1, e_2 \in \text{dom } s_1$ and $e_1 < e_2$ holds $s_1(e_1) < s_1(e_2)$ by [16, (2)], [2, (77)]. For every extended reals e_1, e_2 such that $e_1, e_2 \in \text{dom } s_2$ and $e_1 < e_2$ holds $s_2(e_1) < s_2(e_2)$ by [16, (2)], [2, (77)]. $\sum n^{s_1} = n^{s_1}(1) + (n^{s_1}, 2) + \dots$. $\sum n^{f_1} = n^{f_1}(1) + (n^{f_1}, 2) + \dots$. $\sum n^{s_1} = \sum n^{s_2}$. $n^{s_1}(1) + (n^{s_1}, 2) + \dots = n^{s_2}(1) + (n^{s_2}, 2) + \dots$ and s_1 is increasing and natural-valued. \square

(31) There exists an increasing, natural-valued finite sequence f such that $n = 2^f(1) + (2^f, 2) + \dots$

PROOF: Set $D = \text{digits}(n, 2)$. Consider d being a finite 0-sequence of \mathbb{N} such that $\text{dom } d = \text{dom } D$ and for every natural number i such that $i \in \text{dom } d$ holds $d(i) = D(i) \cdot 2^i$ and $\text{value}(D, 2) = \sum d$. Define $\mathcal{P}[\text{natural number}] \equiv$ if $\$1 \leq \text{len } d$, then there exists an increasing, natural-valued finite sequence f such that $(\text{len } f = 0 \text{ or } f(\text{len } f) < \$1)$ and $\sum 2^f = \sum(d \upharpoonright \$1)$. $\mathcal{P}[(0 \text{ qua natural number})]$ by [11, (72)]. If $\mathcal{P}[i]$, then $\mathcal{P}[i+1]$ by [4, (13)], [29, (86)], [20, (65)], [4, (25), (23)]. $\mathcal{P}[i]$ from [4, Sch. 2]. Consider f being an increasing, natural-valued finite sequence such that $\text{len } f = 0$ or $f(\text{len } f) < \text{len } d$ and $\sum 2^f = \sum(d \upharpoonright \text{len } d)$. $\sum 2^f = 2^f(1) + (2^f, 2) + \dots$. \square

4. VALUE-BASED FUNCTION (RE)ORGANIZATION

Let o be a function yielding function and x, y be objects. The functor $o_{x,y}$ yielding a set is defined by the term

(Def. 5) $o(x)(y)$.

Let F be a function yielding function. We say that F is double one-to-one if and only if

(Def. 6) for every objects x_1, x_2, y_1, y_2 such that $x_1 \in \text{dom } F$ and $y_1 \in \text{dom}(F(x_1))$ and $x_2 \in \text{dom } F$ and $y_2 \in \text{dom}(F(x_2))$ and $F_{x_1, y_1} = F_{x_2, y_2}$ holds $x_1 = x_2$ and $y_1 = y_2$.

Let D be a set. Observe that every finite sequence of elements of D^* which is empty is also double one-to-one and there exists a function yielding function which is double one-to-one and there exists a finite sequence of elements of D^* which is double one-to-one.

Let F be a double one-to-one, function yielding function and x be an object. One can check that $F(x)$ is one-to-one.

Let F be a one-to-one function. One can check that $\langle F \rangle$ is double one-to-one.

Now we state the propositions:

- (32) Let us consider a function yielding function f . Then f is double one-to-one if and only if for every x , $f(x)$ is one-to-one and for every x and y such that $x \neq y$ holds $\text{rng}(f(x))$ misses $\text{rng}(f(y))$.
- (33) Let us consider a set D , and double one-to-one finite sequences f_1, f_2 of elements of D^* . Suppose Values f_1 misses Values f_2 . Then $f_1 \wedge f_2$ is double one-to-one. The theorem is a consequence of (1).

Let D be a finite set.

A double reorganization of D is a double one-to-one finite sequence of elements of D^* and is defined by

(Def. 7) Values $it = D$.

Now we state the propositions:

- (34) (i) \emptyset is a double reorganization of \emptyset , and
(ii) $\langle \emptyset \rangle$ is a double reorganization of \emptyset .
- (35) Let us consider a finite set D , and a one-to-one, onto finite sequence F of elements of D . Then $\langle F \rangle$ is a double reorganization of D .
- (36) Let us consider finite sets D_1, D_2 . Suppose D_1 misses D_2 . Let us consider a double reorganization o_1 of D_1 , and a double reorganization o_2 of D_2 . Then $o_1 \wedge o_2$ is a double reorganization of $D_1 \cup D_2$. The theorem is a consequence of (33) and (2).
- (37) Let us consider a finite set D , a double reorganization o of D , and a one-to-one finite sequence F . Suppose $i \in \text{dom } o$ and $\text{rng } F \cap D \subseteq \text{rng}(o(i))$. Then $o + \cdot (i, F)$ is a double reorganization of $\text{rng } F \cup (D \setminus \text{rng}(o(i)))$.
PROOF: Set $r_1 = \text{rng } F$. Set $o_3 = o(i)$. Set $r_4 = \text{rng } o_3$. Set $o_4 = o + \cdot (i, F)$. $\text{rng } o_4 \subseteq (r_1 \cup (D \setminus r_4))^*$ by [7, (31), (32)]. o_4 is double one-to-one by [7, (32)], (1). Values $o_4 \subseteq r_1 \cup (D \setminus r_4)$ by (1), [7, (31), (32)]. $D \setminus r_4 \subseteq \text{Values } o_4$ by (1), [7, (32)]. $r_1 \subseteq \text{Values } o_4$. \square

Let D be a finite set and n be a non zero natural number. One can check that there exists a double reorganization of D which is n -element.

Let D be a finite, natural-membered set, o be a double reorganization of D , and x be an object. One can verify that $o(x)$ is natural-valued.

Now we state the propositions:

- (38) Let us consider a non empty finite sequence F , and a finite function G . Suppose $\text{rng } G \subseteq \text{rng } F$. Then there exists a $(\text{len } F)$ -element double reorganization o of $\text{dom } G$ such that for every n , $F(n) = G(o_{n,1})$ and ... and $F(n) = G(o_{n,\text{len}(o(n))})$.

PROOF: Set $D = \text{dom } G$. Set $d =$ the one-to-one , onto finite sequence of elements of D . Define $\mathcal{P}[\text{natural number}] \equiv$ if $\$1 \leq \overline{G}$, then there exists a $(\text{len } F)$ -element double reorganization o of $d^\circ(\text{Seg } \$1)$ such that for every k , $F(k) = G(o_{k,1})$ and ... and $F(k) = G(o_{k,\text{len}(o(k))})$. $\mathcal{P}[0]$. If $\mathcal{P}[i]$, then $\mathcal{P}[i + 1]$ by [4, (13)], [26, (29)], [4, (11)], [26, (25)]. $\mathcal{P}[i]$ from [4, Sch. 2]. \square

- (39) Let us consider a non empty finite sequence F , and a finite sequence G . Suppose $\text{rng } G \subseteq \text{rng } F$. Then there exists a $(\text{len } F)$ -element double reorganization o of $\text{dom } G$ such that for every n , $o(n)$ is increasing and $F(n) = G(o_{n,1})$ and ... and $F(n) = G(o_{n,\text{len}(o(n))})$.

PROOF: Define $\mathcal{P}[\text{natural number}] \equiv$ if $\$1 \leq \text{len } G$, then there exists a $(\text{len } F)$ -element double reorganization o of $\text{Seg } \$1$ such that for every k , $o(k)$ is increasing and $F(k) = G(o_{k,1})$ and ... and $F(k) = G(o_{k,\text{len}(o(k))})$. $\mathcal{P}[0]$. If $\mathcal{P}[i]$, then $\mathcal{P}[i + 1]$ by [4, (13)], [26, (29)], [4, (11)], [26, (25)]. $\mathcal{P}[i]$ from [4, Sch. 2]. \square

Let f be a finite function, o be a double reorganization of $\text{dom } f$, and x be an object. One can check that $f \cdot o(x)$ is finite sequence-like and there exists a finite sequence which is complex-functions-valued and finite sequence-yielding.

Let f be a function yielding function and g be a function. We introduce $g \odot f$ as a synonym of $[g, f]$.

One can check that $g \odot f$ is function yielding.

Let f be a $((\text{dom } g)^*)$ -valued finite sequence. One can check that $g \odot f$ is finite sequence-yielding.

Let x be an object. Let us note that $(g \odot f)(x)$ is $(\text{len}(f(x)))$ -element.

Let f be a function yielding finite sequence. One can verify that $g \odot f$ is finite sequence-like and $g \odot f$ is $(\text{len } f)$ -element.

Let f be a function yielding function and g be a complex-valued function. One can check that $g \odot f$ is complex-functions-valued.

Let g be a natural-valued function. One can check that $g \odot f$ is natural-functions-valued.

Let us consider a function yielding function f and a function g . Now we state the propositions:

- (40) Values $g \odot f = g^\circ(\text{Values } f)$.

PROOF: Set $g_3 = g \odot f$. Values $g_3 \subseteq g^\circ(\text{Values } f)$ by (1), [9, (11), (12)]. Consider b being an object such that $b \in \text{dom } g$ and $b \in \text{Values } f$ and

$g(b) = a$. Consider x, y being objects such that $x \in \text{dom } f$ and $y \in \text{dom}(f(x))$ and $b = f(x)(y)$. \square

(41) $(g \odot f)(x) = g \cdot f(x)$.

Now we state the proposition:

(42) Let us consider a function yielding function f , a finite sequence g , and objects x, y . Then $(g \odot f)_{x,y} = g(f_{x,y})$. The theorem is a consequence of (41).

Let f be a complex-functions-valued, finite sequence-yielding function. The functor $\sum f$ yielding a complex-valued function is defined by

(Def. 8) $\text{dom } it = \text{dom } f$ and for every set $x, it(x) = \sum(f(x))$.

Let f be a complex-functions-valued, finite sequence-yielding finite sequence. One can verify that $\sum f$ is finite sequence-like and $\sum f$ is $(\text{len } f)$ -element.

Let f be a natural-functions-valued, finite sequence-yielding function. One can verify that $\sum f$ is natural-valued.

Let f, g be complex-functions-valued finite sequences. One can check that $f \wedge g$ is complex-functions-valued.

Let f, g be extended real-valued finite sequences. One can verify that $f \wedge g$ is extended real-valued.

Let f be a complex-functions-valued function and X be a set. One can check that $f|X$ is complex-functions-valued.

Let f be a finite sequence-yielding function. One can check that $f|X$ is finite sequence-yielding.

Let F be a complex-valued function. One can check that $\langle F \rangle$ is complex-functions-valued.

Let us consider finite sequences f, g . Now we state the propositions:

(43) If $f \wedge g$ is finite sequence-yielding, then f is finite sequence-yielding and g is finite sequence-yielding.

(44) If $f \wedge g$ is complex-functions-valued, then f is complex-functions-valued and g is complex-functions-valued.

Now we state the propositions:

(45) Let us consider a complex-valued finite sequence f . Then $\sum \langle f \rangle = \langle \sum f \rangle$.

(46) Let us consider complex-functions-valued, finite sequence-yielding finite sequences f, g . Then $\sum(f \wedge g) = \sum f \wedge \sum g$.

PROOF: For every i such that $1 \leq i \leq \text{len } f + \text{len } g$ holds $(\sum(f \wedge g))(i) = (\sum f \wedge \sum g)(i)$ by [26, (25)], [6, (25)]. \square

(47) Let us consider a complex-valued finite sequence f , and a double reorganization o of $\text{dom } f$. Then $\sum f = \sum \sum(f \odot o)$.

PROOF: Define \mathcal{P} [natural number] \equiv for every complex-valued finite sequence f for every double reorganization o of $\text{dom } f$ such that $\text{len } f = \$_1$ holds $\sum f = \sum \sum (f \odot o)$. $\mathcal{P}[0]$ by [26, (29)], [11, (72)], [23, (11)], [11, (81)]. If $\mathcal{P}[i]$, then $\mathcal{P}[i + 1]$ by [4, (11)], [26, (25)], (1), [12, (116)]. $\mathcal{P}[i]$ from [4, Sch. 2]. \square

Let us note that \mathbb{N}^* is natural-functions-membered and \mathbb{C}^* is complex-functions-membered.

Now we state the proposition:

(48) Let us consider a finite sequence f of elements of \mathbb{C}^* .

Then $\sum(\text{the concatenation of } \mathbb{C} \odot f) = \sum \sum f$.

PROOF: Set C = the concatenation of \mathbb{C} . Define \mathcal{P} [natural number] \equiv for every finite sequence f of elements of \mathbb{C}^* such that $\text{len } f = \$_1$ holds $\sum(C \odot f) = \sum \sum f$. $\mathcal{P}[0]$. If $\mathcal{P}[i]$, then $\mathcal{P}[i + 1]$ by [8, (19), (16)], (46), (45). $\mathcal{P}[i]$ from [4, Sch. 2]. \square

Let f be a finite function.

A valued reorganization of f is a double reorganization of $\text{dom } f$ and is defined by

(Def. 9) for every n , there exists x such that $x = f(it_{n,1})$ and ... and $x = f(it_{n,\text{len}(it(n))})$ and for every natural numbers n_1, n_2, i_1, i_2 such that $i_1 \in \text{dom}(it(n_1))$ and $i_2 \in \text{dom}(it(n_2))$ and $f(it_{n_1,i_1}) = f(it_{n_2,i_2})$ holds $n_1 = n_2$.

Now we state the propositions:

(49) Let us consider a finite function f , and a valued reorganization o of f . Then

(i) $\text{rng}((f \odot o)(n)) = \emptyset$, or

(ii) $\text{rng}((f \odot o)(n)) = \{f(o_{n,1})\}$ and $1 \in \text{dom}(o(n))$.

PROOF: Consider y such that $y \in \text{rng}((f \odot o)(n))$. Consider x such that $x \in \text{dom}((f \odot o)(n))$ and $(f \odot o)(n)(x) = y$. $n \in \text{dom}(f \odot o)$. Consider w being an object such that $w = f(o_{n,1})$ and ... and $w = f(o_{n,\text{len}(o(n))})$. $\text{rng}((f \odot o)(n)) \subseteq \{f(o_{n,1})\}$ by [9, (11), (12)], [26, (25)]. \square

(50) Let us consider a finite sequence f , and valued reorganizations o_1, o_2 of f . Suppose $\text{rng}((f \odot o_1)(i)) = \text{rng}((f \odot o_2)(i))$. Then $\text{rng}(o_1(i)) = \text{rng}(o_2(i))$.

(51) Let us consider a finite sequence f , a complex-valued finite sequence g , and double reorganizations o_1, o_2 of $\text{dom } g$. Suppose o_1 is a valued reorganization of f and o_2 is a valued reorganization of f and $\text{rng}((f \odot o_1)(i)) = \text{rng}((f \odot o_2)(i))$. Then $(\sum(g \odot o_1))(i) = (\sum(g \odot o_2))(i)$. The theorem is a consequence of (41).

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