

Finite Product of Semiring of Sets

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Summary. We formalize that the image of a semiring of sets [17] by an injective function is a semiring of sets. We offer a non-trivial example of a semiring of sets in a topological space [21]. Finally, we show that the finite product of a semiring of sets is also a semiring of sets [21] and that the finite product of a classical semiring of sets [8] is a classical semiring of sets. In this case, we use here the notation from the book of Aliprantis and Border [1].

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The notation and terminology used in this paper have been introduced in the following articles: [9], [2], [3], [4], [22], [7], [15], [23], [10], [11], [6], [12], [20], [26], [27], [19], [14], [16], [25], [18], and [13].

1. PRELIMINARIES

From now on X_1, X_2, X_3, X_4 denote sets.

Now we state the propositions:

- (1) (i) $X_1 \cap X_4 \setminus (X_2 \cup X_3)$ misses $X_1 \setminus ((X_2 \cup X_3) \cup X_4)$, and
(ii) $X_1 \cap X_4 \setminus (X_2 \cup X_3)$ misses $(X_1 \cap X_3) \cap X_4 \setminus X_2$, and
(iii) $X_1 \setminus ((X_2 \cup X_3) \cup X_4)$ misses $(X_1 \cap X_3) \cap X_4 \setminus X_2$.
- (2) $(X_1 \setminus X_2) \setminus (X_3 \setminus X_4) = (X_1 \setminus (X_2 \cup X_3)) \cup (X_1 \cap X_4 \setminus X_2)$.
- (3) $(X_1 \setminus (X_2 \cup X_3)) \cup (X_1 \cap X_4 \setminus X_2) = ((X_1 \cap X_4 \setminus (X_2 \cup X_3)) \cup (X_1 \setminus ((X_2 \cup X_3) \cup X_4))) \cup ((X_1 \cap X_3) \cap X_4 \setminus X_2)$.

(4) $(X_1 \setminus X_2) \setminus (X_3 \setminus X_4) = ((X_1 \cap X_4 \setminus (X_2 \cup X_3)) \cup (X_1 \setminus ((X_2 \cup X_3) \cup X_4))) \cup ((X_1 \cap X_3) \cap X_4 \setminus X_2)$. The theorem is a consequence of (2) and (3).

(5) $\cup\{X_1, X_2, X_3\} = (X_1 \cup X_2) \cup X_3$.

2. THE DIRECT IMAGE OF A SEMIRING OF SETS BY AN INJECTIVE FUNCTION

Now we state the proposition:

(6) Let us consider sets T, S , a function f from T into S , and a family G of subsets of T . Then $f^\circ G = \{f^\circ A, \text{ where } A \text{ is a subset of } T : A \in G\}$.

Let T, S be sets, f be a function from T into S , and G be a finite family of subsets of T . Let us note that $f^\circ G$ is finite.

Let f be a function and A be a countable set. Let us note that $f^\circ A$ is countable.

The scheme *FraenkelCountable* deals with a set \mathcal{A} and a set \mathcal{X} and a unary functor \mathcal{F} yielding a set and states that

(Sch. 1) $\{\mathcal{F}(w), \text{ where } w \text{ is an element of } \mathcal{A} : w \in \mathcal{X}\}$ is countable provided

- \mathcal{X} is countable.

Let T, S be sets, f be a function from T into S , and G be a countable family of subsets of T . Let us note that $f^\circ G$ is countable.

Let X, Y be sets, S be a family of subsets of X with the empty element, and f be a function from X into Y . One can verify that $f^\circ S$ has the empty element.

Now we state the propositions:

(7) Let us consider sets X, Y , a function f from X into Y , and families S_1, S_2 of subsets of X . If $S_1 \subseteq S_2$, then $f^\circ S_1 \subseteq f^\circ S_2$. The theorem is a consequence of (6).

(8) Let us consider sets X, Y , a \cap -closed family S of subsets of X , and a function f from X into Y . Suppose f is one-to-one. Then $f^\circ S$ is a \cap -closed family of subsets of Y .

(9) Let us consider non empty sets X, Y , a \cap_{fp} -closed family S of subsets of X , and a function f from X into Y . Suppose f is one-to-one. Then $f^\circ S$ is a \cap_{fp} -closed family of subsets of Y .

(10) Let us consider non empty sets X, Y , a \bigcap_{fp}^{\subseteq} -closed family S of subsets of X , and a function f from X into Y . Suppose f is one-to-one and $f^\circ S$ is not empty. Then $f^\circ S$ is a \bigcap_{fp}^{\subseteq} -closed family of subsets of Y .

PROOF: Reconsider $f_1 = f \circ S$ as a family of subsets of Y . f_1 is $\setminus_{fp}^{\subseteq}$ -closed by [10, (64), (87)], [11, (103)], [26, (123)]. \square

- (11) Let us consider non empty sets X, Y , a \setminus_{fp} -closed family S of subsets of X , and a function f from X into Y . Suppose f is one-to-one. Then $f \circ S$ is a \setminus_{fp} -closed family of subsets of Y .
- (12) Let us consider non empty sets X, Y , a semiring S of sets of X , and a function f from X into Y . If f is one-to-one, then $f \circ S$ is a semiring of sets of Y .

3. THE SET OF SET DIFFERENCES OF ALL ELEMENTS OF A SEMIRING OF SETS

Now we state the proposition:

- (13) Let us consider a 1-element finite sequence X . Suppose $X(1)$ is not empty. Then there exists a function I from $X(1)$ into $\prod X$ such that
- (i) I is one-to-one and onto, and
 - (ii) for every object x such that $x \in X(1)$ holds $I(x) = \langle x \rangle$.

Let X be a set. Observe that 2_*^X is \cap -closed and there exists a \cap -closed family of subsets of X which has the empty element and there exists a \cap -closed family of subsets of X with the empty element which is \cup -closed.

Let X, Y be non empty sets. Let us observe that $X \parallel Y$ is non empty.

Now we state the proposition:

- (14) Let us consider a set X , and a family S of subsets of X with the empty element. Then $S \parallel S =$ the set of all $A \setminus B$ where A, B are elements of S .

Let X be a set and S be a family of subsets of X with the empty element. The functor semidiff S yielding a family of subsets of X is defined by the term (Def. 1) $S \parallel S$.

Now we state the proposition:

- (15) Let us consider a set X , a family S of subsets of X with the empty element, and an object x . Suppose $x \in$ semidiff S . Then there exist elements A, B of S such that $x = A \setminus B$. The theorem is a consequence of (14).

Let X be a set and S be a family of subsets of X with the empty element. Observe that semidiff S has the empty element.

Let S be a \cap -closed, \cup -closed family of subsets of X with the empty element. Note that semidiff S is \cap -closed and \setminus_{fp} -closed.

Now we state the proposition:

- (16) Let us consider a set X , and a \cap -closed, \cup -closed family S of subsets of X with the empty element. Then semidiff S is a semiring of sets of X .

4. THE COLLECTION OF ALL LOCALLY CLOSED SETS $LC(X, \tau)$ OF A TOPOLOGICAL SPACE (X, τ)

Let T be a non empty topological space. The functor $LC(T)$ yielding a family of subsets of Ω_T is defined by the term

(Def. 2) $\{A \cap B, \text{ where } A, B \text{ are subsets of } T : A \text{ is open and } B \text{ is closed}\}.$

Let us note that $LC(T)$ is \cap -closed and \setminus_{fp} -closed and has the empty element.

(17) Let us consider a non empty topological space T . Then $LC(T)$ is a semiring of sets of Ω_T .

5. THE FINITE PRODUCT OF SEMIRINGS OF SETS

Let n be a natural number. Note that there exists an n -element finite sequence which is non-empty.

Let n be a non zero natural number and X be a non-empty, n -element finite sequence.

A semiring family of X is an n -element finite sequence and is defined by

(Def. 3) for every natural number i such that $i \in \text{Seg } n$ holds $it(i)$ is a semiring of sets of $X(i)$.

In the sequel n denotes a non zero natural number and X denotes a non-empty, n -element finite sequence. Now we state the propositions:

(18) Let us consider a semiring family S of X . Then $\text{dom } S = \text{dom } X$.

(19) Let us consider a semiring family S of X , and a natural number i . If $i \in \text{Seg } n$, then $\cup(S(i)) \subseteq X(i)$.

(20) Let us consider a function f , and an n -element finite sequence X . If $f \in \prod X$, then f is an n -element finite sequence.

Let n be a non zero natural number and X be an n -element finite sequence. The functor $\text{SemiringProduct } X$ yielding a set is defined by

(Def. 4) for every object $f, f \in it$ iff there exists a function g such that $f = \prod g$ and $g \in \prod X$.

Now we state the propositions:

(21) Let us consider an n -element finite sequence X .

Then $\text{SemiringProduct } X \subseteq 2(\cup \cup X)^{\text{dom } X}$.

(22) Let us consider a semiring family S of X . Then $\text{SemiringProduct } S$ is a family of subsets of $\prod X$.

PROOF: Reconsider $S_1 = \text{SemiringProduct } S$ as a subset of $2(\cup \cup S)^{\text{dom } S}$. $S_1 \subseteq 2\prod X$ by [3, (9)], (18), [7, (89)], (19). \square

- (23) Let us consider a non-empty, 1-element finite sequence X . Then $\prod X =$ the set of all $\langle x \rangle$ where x is an element of $X(1)$. The theorem is a consequence of (13).

One can check that $\prod \langle \emptyset \rangle$ is empty. Now we state the propositions:

- (24) Let us consider a non empty set x . Then $\prod \langle x \rangle =$ the set of all $\langle y \rangle$ where y is an element of x . The theorem is a consequence of (23).
- (25) Let us consider a non-empty, 1-element finite sequence X , and a semiring family S of X . Then $\text{SemiringProduct } S =$ the set of all $\prod \langle s \rangle$ where s is an element of $S(1)$. PROOF: S is non-empty by (18), [7, (3)]. $\prod S =$ the set of all $\langle s \rangle$ where s is an element of $S(1)$. \square

Let us consider sets x, y . Now we state the propositions:

- (26) $\prod \langle x \rangle \cap \prod \langle y \rangle = \prod \langle x \cap y \rangle$. The theorem is a consequence of (24).
- (27) $\prod \langle x \rangle \setminus \prod \langle y \rangle = \prod \langle x \setminus y \rangle$. The theorem is a consequence of (24).

Let us consider a non-empty, 1-element finite sequence X and a semiring family S of X . Now we state the propositions:

- (28) the set of all $\prod \langle s \rangle$ where s is an element of $S(1)$ is a semiring of sets of the set of all $\langle x \rangle$ where x is an element of $X(1)$. The theorem is a consequence of (24), (26), and (27).
- (29) $\text{SemiringProduct } S$ is a semiring of sets of $\prod X$. The theorem is a consequence of (23), (25), and (28).
- (30) Let us consider sets X_1, X_2 , a semiring S_1 of sets of X_1 , and a semiring S_2 of sets of X_2 . Then the set of all $s_1 \times s_2$ where s_1 is an element of S_1 , s_2 is an element of S_2 is a semiring of sets of $X_1 \times X_2$.
- (31) Let us consider a non-empty, n -element finite sequence X_3 , a non-empty, 1-element finite sequence X_1 , a semiring family S_3 of X_3 , and a semiring family S_1 of X_1 . Suppose $\text{SemiringProduct } S_3$ is a semiring of sets of $\prod X_3$ and $\text{SemiringProduct } S_1$ is a semiring of sets of $\prod X_1$. Let us consider a family S_4 of subsets of $\prod X_3 \times \prod X_1$. Suppose $S_4 =$ the set of all $s_1 \times s_2$ where s_1 is an element of $\text{SemiringProduct } S_3$, s_2 is an element of $\text{SemiringProduct } S_1$. Then there exists a function I from $\prod X_3 \times \prod X_1$ into $\prod (X_3 \cap X_1)$ such that

- (i) I is one-to-one and onto, and
- (ii) for every finite sequences x, y such that $x \in \prod X_3$ and $y \in \prod X_1$ holds $I(x, y) = x \cap y$, and
- (iii) $I^\circ S_4 = \text{SemiringProduct}(S_3 \cap S_1)$.

PROOF: $\cup(S_1(1)) \subseteq X_1(1)$. Consider I being a function from $\prod X_3 \times \prod X_1$ into $\prod (X_3 \cap X_1)$ such that I is one-to-one and I is onto and for every finite

sequences x, y such that $x \in \prod X_3$ and $y \in \prod X_1$ holds $I(x, y) = x \wedge y$. $I^\circ S_4 = \text{SemiringProduct}(S_3 \wedge S_1)$ by (25), (20), [7, (89)], [24, (153)]. \square

(32) Let us consider a non-empty, n -element finite sequence X_3 , a non-empty, 1-element finite sequence X_1 , a semiring family S_3 of X_3 , and a semiring family S_1 of X_1 . Suppose $\text{SemiringProduct } S_3$ is a semiring of sets of $\prod X_3$ and $\text{SemiringProduct } S_1$ is a semiring of sets of $\prod X_1$. Then $\text{SemiringProduct}(S_3 \wedge S_1)$ is a semiring of sets of $\prod(X_3 \wedge X_1)$. The theorem is a consequence of (30), (31), (9), and (10).

(33) Let us consider a semiring family S of X . Then $\text{SemiringProduct } S$ is a semiring of sets of $\prod X$. PROOF: Define $\mathcal{P}[\text{non zero natural number}] \equiv$ for every non-empty, $\$1$ -element finite sequence X for every semiring family S of X , $\text{SemiringProduct } S$ is a semiring of sets of $\prod X$. $\mathcal{P}[1]$. For every non zero natural number n , $\mathcal{P}[n]$ from [5, Sch. 10]. \square

Let n be a non zero natural number, X be a non-empty, n -element finite sequence, and S be a semiring family of X . We say that S is \cap -closed yielding if and only if

(Def. 5) for every natural number i such that $i \in \text{Seg } n$ holds $S(i)$ is \cap -closed.

Note that there exists a semiring family of X which is \cap -closed yielding.

6. THE FINITE PRODUCT OF CLASSICAL SEMIRINGS OF SETS

Let X be a set. Note that there exists a semiring of sets of X which is \cap -closed.

Let us consider a non-empty, 1-element finite sequence X and a \cap -closed yielding semiring family S of X . Now we state the propositions:

(34) the set of all $\prod \langle s \rangle$ where s is an element of $S(1)$ is a \cap -closed semiring of sets of the set of all $\langle x \rangle$ where x is an element of $X(1)$. The theorem is a consequence of (26) and (28).

(35) $\text{SemiringProduct } S$ is a \cap -closed semiring of sets of $\prod X$. The theorem is a consequence of (23), (25), and (34).

Now we state the propositions:

(36) Let us consider sets X_1, X_2 , a \cap -closed semiring S_1 of sets of X_1 , and a \cap -closed semiring S_2 of sets of X_2 . Then the set of all $s_1 \times s_2$ where s_1 is an element of S_1 , s_2 is an element of S_2 is a \cap -closed semiring of sets of $X_1 \times X_2$.

(37) Let us consider a non-empty, n -element finite sequence X_3 , a non-empty, 1-element finite sequence X_1 , a \cap -closed yielding semiring family S_3 of X_3 , and a \cap -closed yielding semiring family S_1 of X_1 . Suppose SemiringProduct

S_3 is a \cap -closed semiring of sets of $\prod X_3$ and SemiringProduct S_1 is a \cap -closed semiring of sets of $\prod X_1$. Then SemiringProduct($S_3 \cap S_1$) is a \cap -closed semiring of sets of $\prod(X_3 \cap X_1)$. The theorem is a consequence of (30), (31), (36), (8), and (10).

Let us consider n and X . Let S be a \cap -closed yielding semiring family of X . One can check that SemiringProduct S is \cap -closed.

(38) Let us consider a \cap -closed yielding semiring family S of X .

Then SemiringProduct S is a \cap -closed semiring of sets of $\prod X$.

7. MEASURABLE RECTANGLE

Let n be a non zero natural number and X be a non-empty, n -element finite sequence.

A classical semiring family of X is an n -element finite sequence and is defined by

(Def. 6) for every natural number i such that $i \in \text{Seg } n$ holds $it(i)$ is a semi-diff-closed, \cap -closed family of subsets of $X(i)$ with the empty element.

Let X be an n -element finite sequence. We introduce MeasurableRectangle X as a synonym of SemiringProduct X . Now we state the propositions:

(39) Every classical semiring family of X is a \cap -closed yielding semiring family of X .

(40) Let us consider a classical semiring family S of X .

Then MeasurableRectangle S is a semi-diff-closed, \cap -closed family of subsets of $\prod X$ with the empty element. The theorem is a consequence of (39) and (33).

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