FORMALIZED MATHEMATICS
Vol. 23, No. 2, Pages 127–160, 2015
DOI: 10.1515/forma-2015-0013

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Groups – Additive Notation

Roland Coghetto
Rue de la Brasserie 5
7100 La Louvière, Belgium

Summary. We translate the articles covering group theory already available in the Mizar Mathematical Library from multiplicative into additive notation. We adapt the works of Wojciech A. Trybulec [41, 42, 43] and Artur Korniłowicz [25].

In particular, these authors have defined the notions of group, abelian group, power of an element of a group, order of a group and order of an element, subgroup, coset of a subgroup, index of a subgroup, conjugation, normal subgroup, topological group, dense subset and basis of a topological group. Lagrange's theorem and some other theorems concerning these notions [9, 24, 22] are presented.

Note that "The term $\mathbb{Z}$-module is simply another name for an additive abelian group" [27]. We take an approach different than that used by Futa et al. [21] to use in a future article the results obtained by Artur Korniłowicz [25]. Indeed, Hölzl et al. showed that it was possible to build "a generic theory of limits based on filters" in Isabelle/HOL [23, 10]. Our goal is to define the convergence of a sequence and the convergence of a series in an abelian topological group [11] using the notion of filters.

MSC: 20A05 20K00 03B35

Keywords: additive group; subgroup; Lagrange theorem; conjugation; normal subgroup; index; additive topological group; basis; neighborhood; additive abelian group; Z-module

MML identifier: `GROUP_1A`, version: `8.1.04 5.32.1240`

The notation and terminology used in this paper have been introduced in the following articles: [12], [32], [31], [2], [18], [28], [33], [13], [19], [39], [14], [15], [1], [40], [26], [35], [36], [5], [6], [16], [30], [8], [46], [47], [44], [29], [37], [45], [25], [48], [20], [7], [38], and [17].

ISSN 1426–2630(Print), 1898-9934(Online)

1. Additive Notation for Groups – GROUP_1

From now on m, n denote natural numbers, i, j denote integers, S denotes a non empty additive magma, and r, r_1, r_2, s, s_1, s_2, t, t_1, t_2 denote elements of S.

The scheme *SeqEx2Dbis* deals with non empty sets $\mathcal{X}$, $\mathcal{Z}$ and a ternary predicate $\mathcal{P}$ and states that

(Sch. 1) There exists a function f from $\mathbb{N} \times \mathcal{X}$ into $\mathcal{Z}$ such that for every natural number x for every element y of $\mathcal{X}$, $\mathcal{P}[x, y, f(x, y)]$

provided

- for every natural number x and for every element y of $\mathcal{X}$, there exists an element z of $\mathcal{Z}$ such that $\mathcal{P}[x, y, z]$.

Let I_1 be an additive magma. We say that I_1 is add-unital if and only if

(Def. 1) there exists an element e of I_1 such that for every element h of I_1, $h + e = h$ and $e + h = h$.

We say that I_1 is additive group-like if and only if

(Def. 2) there exists an element e of I_1 such that for every element h of I_1, $h + e = h$ and $e + h = h$ and there exists an element g of I_1 such that $h + g = e$ and $g + h = e$.

Let us note that every additive magma which is additive group-like is also add-unital and there exists an additive magma which is strict, additive group-like, add-associative, and non empty.

An additive group is an additive group-like, add-associative, non empty additive magma. Now we state the propositions:

(1) Suppose for every r, s, and t, $(r + s) + t = r + (s + t)$ and there exists t such that for every s_1, $s_1 + t = s_1$ and $t + s_1 = s_1$ and there exists s_2 such that $s_1 + s_2 = t$ and $s_2 + s_1 = t$. Then S is an additive group.

(2) Suppose for every r, s, and t, $(r + s) + t = r + (s + t)$ and for every r and s, there exists t such that $r + t = s$ and there exists t such that $t + r = s$. Then S is add-associative and additive group-like.

(3) $\langle \mathbb{R}, +_{\mathbb{R}} \rangle$ is add-associative and additive group-like.

From now on G denotes an additive group-like, non empty additive magma and e, h denote elements of G.

Let G be an additive magma. Assume G is add-unital. The functor 0_G yielding an element of G is defined by

(Def. 3) for every element h of G, $h + it = h$ and $it + h = h$.

Now we state the proposition:

(4) If for every h, $h + e = h$ and $e + h = h$, then $e = 0_G$.

From now on G denotes an additive group and f, g, h denote elements of G.

Let us consider G and h. The functor $-h$ yielding an element of G is defined by

(Def. 4) $h + it = 0_G$ and $it + h = 0_G$.

Let us note that the functor is involutive.

Now we state the propositions:

(5) If $h + g = 0_G$ and $g + h = 0_G$, then $g = -h$.

(6) If $h + g = h + f$ or $g + h = f + h$, then $g = f$.

(7) If $h + g = h$ or $g + h = h$, then $g = 0_G$. The theorem is a consequence of (6).

(8) $-0_G = 0_G$.

(9) If $-h = -g$, then $h = g$. The theorem is a consequence of (6).

(10) If $-h = 0_G$, then $h = 0_G$. The theorem is a consequence of (8).

(11) If $h + g = 0_G$, then $h = -g$ and $g = -h$. The theorem is a consequence of (6).

(12) $h + f = g$ if and only if $f = -h + g$. The theorem is a consequence of (6).

(13) $f + h = g$ if and only if $f = g + -h$. The theorem is a consequence of (6).

(14) There exists f such that $g + f = h$. The theorem is a consequence of (12).

(15) There exists f such that $f + g = h$. The theorem is a consequence of (13).

(16) $-(h + g) = -g + -h$. The theorem is a consequence of (11).

(17) $g + h = h + g$ if and only if $-(g + h) = -g + -h$. The theorem is a consequence of (16) and (6).

(18) $g + h = h + g$ if and only if $-g + -h = -h + -g$. The theorem is a consequence of (16) and (17).

(19) $g+h = h+g$ if and only if $g+-h = -h+g$. The theorem is a consequence of (18), (11), and (16).

From now on u denotes a unary operation on G.

Let us consider G. The functor add inverse G yielding a unary operation on G is defined by

(Def. 5) $it(h) = -h$.

Let G be an add-associative, non empty additive magma. Let us note that the addition of G is associative.

Let us consider an add-unital, non empty additive magma G. Now we state the propositions:

(20) 0_G is a unity w.r.t. the addition of G.

(21) $\mathbf{1}_\alpha = 0_G$, where α is the addition of G. The theorem is a consequence of (20).

Let G be an add-unital, non empty additive magma. Let us note that the addition of G is unital.

Now we state the proposition:

(22) add inverse G is an inverse operation w.r.t. the addition of G. The theorem is a consequence of (21).

Let us consider G. One can verify that the addition of G has inverse operation.

Now we state the proposition:

(23) The inverse operation w.r.t. the addition of G = add inverse G. The theorem is a consequence of (22).

Let G be a non empty additive magma. The functor mult G yielding a function from $\mathbb{N} \times$ (the carrier of G) into the carrier of G is defined by

(Def. 6) for every element h of G, $it(0, h) = 0_G$ and for every natural number n, $it(n+1, h) = it(n, h) + h$.

Let us consider G, i, and h. The functor $i \cdot h$ yielding an element of G is defined by the term

(Def. 7) $\begin{cases} (\mathrm{mult}\, G)(|i|, h), & \textbf{if } 0 \leqslant i, \\ -(\mathrm{mult}\, G)(|i|, h), & \textbf{otherwise}. \end{cases}$

Let us consider n. One can check that the functor $n \cdot h$ is defined by the term

(Def. 8) $(\mathrm{mult}\, G)(n, h)$.

Now we state the propositions:

(24) $0 \cdot h = 0_G$.

(25) $1 \cdot h = h$.

(26) $2 \cdot h = h + h$. The theorem is a consequence of (25).

(27) $3 \cdot h = h + h + h$. The theorem is a consequence of (26).

(28) $2 \cdot h = 0_G$ if and only if $-h = h$. The theorem is a consequence of (26) and (11).

(29) If $i \leqslant 0$, then $i \cdot h = -|i| \cdot h$. The theorem is a consequence of (8).

(30) $i \cdot 0_G = 0_G$. The theorem is a consequence of (8).

(31) $(-1) \cdot h = -h$. The theorem is a consequence of (25).

(32) $(i + j) \cdot h = i \cdot h + j \cdot h$.

PROOF: Define $\mathcal{P}[\text{integer}] \equiv$ for every i, $(i + \$_1) \cdot h = i \cdot h + \$_1 \cdot h$. For every j such that $\mathcal{P}[j]$ holds $\mathcal{P}[j-1]$ and $\mathcal{P}[j+1]$. $\mathcal{P}[0]$. For every j, $\mathcal{P}[j]$ from [40, Sch. 4]. □

(33) (i) $(i+1) \cdot h = i \cdot h + h$, and

(ii) $(i+1) \cdot h = h + i \cdot h$.

The theorem is a consequence of (25) and (32).

(34) $(-i) \cdot h = -i \cdot h$.

Let us assume that $g + h = h + g$. Now we state the propositions:

(35) $i \cdot (g + h) = i \cdot g + i \cdot h$. The theorem is a consequence of (16).

(36) $i \cdot g + j \cdot h = j \cdot h + i \cdot g$. The theorem is a consequence of (19) and (16).

(37) $g + i \cdot h = i \cdot h + g$. The theorem is a consequence of (25) and (36).

Let us consider G and h. We say that h is of order 0 if and only if

(Def. 9) if $n \cdot h = 0_G$, then $n = 0$.

One can check that 0_G is non of order 0.

Let us consider h. The functor $\text{ord}(h)$ yielding an element of $\mathbb{N}$ is defined by

(Def. 10) (i) $it = 0$, **if** h is of order 0,

(ii) $it \cdot h = 0_G$ and $it \neq 0$ and for every m such that $m \cdot h = 0_G$ and $m \neq 0$ holds $it \leqslant m$, **otherwise**.

Now we state the propositions:

(38) $\text{ord}(h) \cdot h = 0_G$.

(39) $\text{ord}(0_G) = 1$.

(40) If $\text{ord}(h) = 1$, then $h = 0_G$. The theorem is a consequence of (25).

Observe that there exists an additive group which is strict and Abelian.

Now we state the proposition:

(41) $\langle \mathbb{R}, +_{\mathbb{R}} \rangle$ is an Abelian additive group. The theorem is a consequence of (3).

In the sequel A denotes an Abelian additive group and a, b denote elements of A.

Now we state the propositions:

(42) $-(a + b) = -a + -b$.

(43) $i \cdot (a + b) = i \cdot a + i \cdot b$.

(44) $\langle \text{the carrier of } A, \text{the addition of } A, 0_A \rangle$ is Abelian, add-associative, right zeroed, and right complementable.

Let us consider an add-unital, non empty additive magma L and an element x of L. Now we state the propositions:

(45) $(\text{mult}\, L)(1, x) = x$.

(46) $(\operatorname{mult} L)(2, x) = x + x$. The theorem is a consequence of (45).

Now we state the proposition:

(47) Let us consider an add-associative, Abelian, add-unital, non empty additive magma L, elements x, y of L, and a natural number n. Then $(\operatorname{mult} L)(n, x + y) = (\operatorname{mult} L)(n, x) + (\operatorname{mult} L)(n, y)$.

PROOF: Define $\mathcal{P}[\text{natural number}] \equiv (\operatorname{mult} L)(\$_1, x+y) = (\operatorname{mult} L)(\$_1, x)+ (\operatorname{mult} L)(\$_1, y)$. For every natural number n, $\mathcal{P}[n]$ from [5, Sch. 2]. $\square$

Let G, H be additive magmas and I_1 be a function from G into H. We say that I_1 preserves zero if and only if

(Def. 11) $I_1(0_G) = 0_H$.

2. Subgroups and Lagrange Theorem – GROUP_2

In the sequel x denotes an object, y, y_1, y_2, Y, Z denote sets, k denotes a natural number, G denotes an additive group, a, g, h denote elements of G, and A denotes a subset of G.

Let us consider G and A. The functor $-A$ yielding a subset of G is defined by the term

(Def. 12) $\{-g : g \in A\}$.

One can check that the functor is involutive.

Now we state the propositions:

(48) $x \in -A$ if and only if there exists g such that $x = -g$ and $g \in A$.

(49) $-\{g\} = \{-g\}$.

(50) $-\{g, h\} = \{-g, -h\}$.

(51) $-\emptyset_\alpha = \emptyset$, where α is the carrier of G.

(52) $-\Omega_\alpha =$ the carrier of G, where α is the carrier of G.

(53) $A \neq \emptyset$ if and only if $-A \neq \emptyset$. The theorem is a consequence of (48).

Let us consider G. Let A be an empty subset of G. Observe that $-A$ is empty.

Let A be a non empty subset of G. One can check that $-A$ is non empty.

In the sequel G denotes a non empty additive magma, A, B, C denote subsets of G, and a, b, g, g_1, g_2, h, h_1, h_2 denote elements of G.

Let G be an Abelian, non empty additive magma and A, B be subsets of G. One can check that the functor $A + B$ is commutative.

(54) $x \in A + B$ if and only if there exists g and there exists h such that $x = g + h$ and $g \in A$ and $h \in B$.

(55) $A \neq \emptyset$ and $B \neq \emptyset$ if and only if $A + B \neq \emptyset$. The theorem is a consequence of (54).

(56) If G is add-associative, then $(A+B)+C = A+(B+C)$.

(57) Let us consider an additive group G, and subsets A, B of G. Then $-(A+B) = -B + -A$. The theorem is a consequence of (16).

(58) $A+(B \cup C) = A+B \cup (A+C)$.

(59) $(A \cup B)+C = A+C \cup (B+C)$.

(60) $A+B \cap C \subseteq (A+B) \cap (A+C)$.

(61) $A \cap B+C \subseteq (A+C) \cap (B+C)$.

(62) (i) $\emptyset_\alpha + A = \emptyset$, and

(ii) $A + \emptyset_\alpha = \emptyset$,

where α is the carrier of G. The theorem is a consequence of (54).

(63) Let us consider an additive group G, and a subset A of G. Suppose $A \neq \emptyset$. Then

(i) $\Omega_\alpha + A =$ the carrier of G, and

(ii) $A + \Omega_\alpha =$ the carrier of G,

where α is the carrier of G.

(64) $\{g\} + \{h\} = \{g+h\}$.

(65) $\{g\} + \{g_1, g_2\} = \{g+g_1, g+g_2\}$.

(66) $\{g_1, g_2\} + \{g\} = \{g_1+g, g_2+g\}$.

(67) $\{g, h\} + \{g_1, g_2\} = \{g+g_1, g+g_2, h+g_1, h+g_2\}$.

Let us consider an additive group G and a subset A of G. Now we state the propositions:

(68) Suppose for every elements g_1, g_2 of G such that $g_1, g_2 \in A$ holds $g_1+g_2 \in A$ and for every element g of G such that $g \in A$ holds $-g \in A$. Then $A + A = A$.

(69) If for every element g of G such that $g \in A$ holds $-g \in A$, then $-A = A$.

(70) If for every a and b such that $a \in A$ and $b \in B$ holds $a+b = b+a$, then $A+B = B+A$.

(71) If G is an Abelian additive group, then $A+B = B+A$.

(72) Let us consider an Abelian additive group G, and subsets A, B of G. Then $-(A+B) = -A + -B$. The theorem is a consequence of (42).

Let us consider G, g, and A. The functors: $g+A$ and $A+g$ yielding subsets of G are defined by terms,

(Def. 13) $\{g\} + A$,

(Def. 14) $A + \{g\}$,

respectively. Now we state the propositions:

(73) $x \in g+A$ if and only if there exists h such that $x = g+h$ and $h \in A$.

(74) $x \in A + g$ if and only if there exists h such that $x = h + g$ and $h \in A$.

Let us assume that G is add-associative. Now we state the propositions:

(75) $(g + A) + B = g + (A + B)$.

(76) $(A + g) + B = A + (g + B)$.

(77) $(A + B) + g = A + (B + g)$.

(78) $(g + h) + A = g + (h + A)$. The theorem is a consequence of (64) and (56).

(79) $(g + A) + h = g + (A + h)$.

(80) $(A + g) + h = A + (g + h)$. The theorem is a consequence of (56) and (64).

(81) (i) $\emptyset_\alpha + a = \emptyset$, and

(ii) $a + \emptyset_\alpha = \emptyset$,

where α is the carrier of G.

From now on G denotes an additive group-like, non empty additive magma, h, g, g_1, g_2 denote elements of G, and A denotes a subset of G.

(82) Let us consider an additive group G, and an element a of G. Then

(i) $\Omega_\alpha + a =$ the carrier of G, and

(ii) $a + \Omega_\alpha =$ the carrier of G,

where α is the carrier of G.

(83) (i) $0_G + A = A$, and

(ii) $A + 0_G = A$.

The theorem is a consequence of (73) and (74).

(84) If G is an Abelian additive group, then $g + A = A + g$.

Let G be an additive group-like, non empty additive magma.

A subgroup of G is an additive group-like, non empty additive magma and is defined by

(Def. 15) the carrier of *it* $\subseteq$ the carrier of G and the addition of *it* $=$ (the addition of G) $\restriction$ (the carrier of *it*).

In the sequel H denotes a subgroup of G and h, h_1, h_2 denote elements of H.

Now we state the propositions:

(85) If G is finite, then H is finite.

(86) If $x \in H$, then $x \in G$.

(87) $h \in G$.

(88) h is an element of G.

(89) If $h_1 = g_1$ and $h_2 = g_2$, then $h_1 + h_2 = g_1 + g_2$.

Let G be an additive group. Let us observe that every subgroup of G is add-associative.

In the sequel G, G_1, G_2, G_3 denote additive groups, a, a_1, a_2, b, b_1, b_2, g, g_1, g_2 denote elements of G, A, B denote subsets of G, H, H_1, H_2, H_3 denote subgroups of G, and h, h_1, h_2 denote elements of H.

(90) $0_H = 0_G$. The theorem is a consequence of (87), (89), and (7).

(91) $0_{H_1} = 0_{H_2}$. The theorem is a consequence of (90).

(92) $0_G \in H$. The theorem is a consequence of (90).

(93) $0_{H_1} \in H_2$. The theorem is a consequence of (90) and (92).

(94) If $h = g$, then $-h = -g$. The theorem is a consequence of (87), (89), (90), and (11).

(95) add inverse H = add inverse $G\restriction$(the carrier of H). The theorem is a consequence of (87) and (94).

(96) If $g_1, g_2 \in H$, then $g_1 + g_2 \in H$. The theorem is a consequence of (89).

(97) If $g \in H$, then $-g \in H$. The theorem is a consequence of (94).

Let us consider G. Observe that there exists a subgroup of G which is strict.

(98) Suppose $A \neq \emptyset$ and for every g_1 and g_2 such that $g_1, g_2 \in A$ holds $g_1 + g_2 \in A$ and for every g such that $g \in A$ holds $-g \in A$. Then there exists a strict subgroup H of G such that the carrier of $H = A$.
PROOF: Reconsider $D = A$ as a non empty set. Set $o =$ (the addition of $G) \restriction A$. $\operatorname{rng} o \subseteq A$ by [17, (87)], [14, (47)]. Set $H = \langle D, o \rangle$. H is additive group-like. $\square$

(99) If G is an Abelian additive group, then H is Abelian. The theorem is a consequence of (87) and (89).

Let G be an Abelian additive group. One can check that every subgroup of G is Abelian.

(100) G is a subgroup of G.

(101) Suppose G_1 is a subgroup of G_2 and G_2 is a subgroup of G_1. Then the additive magma of G_1 = the additive magma of G_2.

(102) If G_1 is a subgroup of G_2 and G_2 is a subgroup of G_3, then G_1 is a subgroup of G_3.

(103) If the carrier of $H_1 \subseteq$ the carrier of H_2, then H_1 is a subgroup of H_2.

(104) If for every g such that $g \in H_1$ holds $g \in H_2$, then H_1 is a subgroup of H_2. The theorem is a consequence of (87) and (103).

(105) Suppose the carrier of H_1 = the carrier of H_2. Then the additive magma of H_1 = the additive magma of H_2. The theorem is a consequence of (103) and (101).

(106) Suppose for every g, $g \in H_1$ iff $g \in H_2$. Then the additive magma of $H_1 =$ the additive magma of H_2. The theorem is a consequence of (104) and (101).

Let us consider G. Let H_1, H_2 be strict subgroups of G. One can check that $H_1 = H_2$ if and only if the condition (Def. 16) is satisfied.

(Def. 16) for every g, $g \in H_1$ iff $g \in H_2$.

Now we state the propositions:

(107) Let us consider an additive group G, and a subgroup H of G. Suppose the carrier of $G \subseteq$ the carrier of H. Then the additive magma of $H =$ the additive magma of G. The theorem is a consequence of (100) and (105).

(108) Suppose for every element g of G, $g \in H$. Then the additive magma of $H =$ the additive magma of G. The theorem is a consequence of (100) and (106).

Let us consider G. The functor $\mathbf{0}_G$ yielding a strict subgroup of G is defined by

(Def. 17) the carrier of *it* $= \{0_G\}$.

The functor Ω_G yielding a strict subgroup of G is defined by the term

(Def. 18) the additive magma of G.

Note that the functor is projective.

Now we state the propositions:

(109) $\mathbf{0}_H = \mathbf{0}_G$. The theorem is a consequence of (90) and (102).

(110) $\mathbf{0}_{H_1} = \mathbf{0}_{H_2}$. The theorem is a consequence of (109).

(111) $\mathbf{0}_G$ is a subgroup of H. The theorem is a consequence of (109).

(112) Let us consider a strict additive group G. Then every subgroup of G is a subgroup of Ω_G.

(113) Every strict additive group is a subgroup of Ω_G.

(114) $\mathbf{0}_G$ is finite.

Let us consider G. Note that $\mathbf{0}_G$ is finite and there exists a subgroup of G which is strict and finite and there exists an additive group which is strict and finite.

Let G be a finite additive group. One can verify that every subgroup of G is finite.

Now we state the propositions:

(115) $\overline{\overline{\mathbf{0}_G}} = 1$.

(116) Let us consider a strict, finite subgroup H of G. If $\overline{\overline{H}} = 1$, then $H = \mathbf{0}_G$. The theorem is a consequence of (92).

(117) $\overline{\overline{H}} \subseteq \overline{\overline{G}}$.

Let us consider a finite additive group G and a subgroup H of G. Now we state the propositions:

(118) $\overline{\overline{H}} \leqslant \overline{\overline{G}}$.

(119) Suppose $\overline{\overline{G}} = \overline{\overline{H}}$. Then the additive magma of $H =$ the additive magma of G.

PROOF: The carrier of $H =$ the carrier of G by [3, (48)]. □

Let us consider G and H. The functor $\overline{H}$ yielding a subset of G is defined by the term

(Def. 19) the carrier of H.

Now we state the propositions:

(120) If $g_1, g_2 \in \overline{H}$, then $g_1 + g_2 \in \overline{H}$. The theorem is a consequence of (96).

(121) If $g \in \overline{H}$, then $-g \in \overline{H}$. The theorem is a consequence of (97).

(122) $\overline{H} + \overline{H} = \overline{H}$. The theorem is a consequence of (121), (120), and (68).

(123) $-\overline{H} = \overline{H}$. The theorem is a consequence of (121) and (69).

(124) (i) if $\overline{H_1} + \overline{H_2} = \overline{H_2} + \overline{H_1}$, then there exists a strict subgroup H of G such that the carrier of $H = \overline{H_1} + \overline{H_2}$, and

(ii) if there exists H such that the carrier of $H = \overline{H_1} + \overline{H_2}$, then $\overline{H_1} + \overline{H_2} = \overline{H_2} + \overline{H_1}$.

The theorem is a consequence of (121), (16), (120), (55), and (98).

(125) Suppose G is an Abelian additive group. Then there exists a strict subgroup H of G such that the carrier of $H = \overline{H_1} + \overline{H_2}$. The theorem is a consequence of (71) and (124).

Let us consider G, H_1, and H_2. The functor $H_1 \cap H_2$ yielding a strict subgroup of G is defined by

(Def. 20) the carrier of $it = \overline{H_1} \cap \overline{H_2}$.

Now we state the propositions:

(126) (i) for every subgroup H of G such that $H = H_1 \cap H_2$ holds the carrier of $H =$ (the carrier of H_1) $\cap$ (the carrier of H_2), and

(ii) for every strict subgroup H of G such that the carrier of $H =$ (the carrier of H_1) $\cap$ (the carrier of H_2) holds $H = H_1 \cap H_2$.

(127) $\overline{H_1 \cap H_2} = \overline{H_1} \cap \overline{H_2}$.

(128) $x \in H_1 \cap H_2$ if and only if $x \in H_1$ and $x \in H_2$.

(129) Let us consider a strict subgroup H of G. Then $H \cap H = H$. The theorem is a consequence of (105).

Let us consider G, H_1, and H_2. Note that the functor $H_1 \cap H_2$ is commutative.

(130) $(H_1 \cap H_2) \cap H_3 = H_1 \cap (H_2 \cap H_3)$. The theorem is a consequence of (105).

(131) (i) $\mathbf{0}_G \cap H = \mathbf{0}_G$, and

(ii) $H \cap \mathbf{0}_G = \mathbf{0}_G$.

The theorem is a consequence of (111).

(132) Let us consider a strict additive group G, and a strict subgroup H of G. Then

(i) $H \cap \Omega_G = H$, and

(ii) $\Omega_G \cap H = H$.

(133) Let us consider a strict additive group G. Then $\Omega_G \cap \Omega_G = G$.

(134) $H_1 \cap H_2$ is subgroup of H_1 and subgroup of H_2.

(135) Let us consider a subgroup H_1 of G. Then H_1 is a subgroup of H_2 if and only if the additive magma of $H_1 \cap H_2 =$ the additive magma of H_1.

(136) If H_1 is a subgroup of H_2, then $H_1 \cap H_3$ is a subgroup of H_2. The theorem is a consequence of (102).

(137) If H_1 is subgroup of H_2 and subgroup of H_3, then H_1 is a subgroup of $H_2 \cap H_3$. The theorem is a consequence of (86), (128), and (104).

(138) If H_1 is a subgroup of H_2, then $H_1 \cap H_3$ is a subgroup of $H_2 \cap H_3$. The theorem is a consequence of (126) and (103).

(139) If H_1 is finite or H_2 is finite, then $H_1 \cap H_2$ is finite.

Let us consider G, H, and A. The functors: $A + H$ and $H + A$ yielding subsets of G are defined by terms,

(Def. 21) $A + \overline{H}$,

(Def. 22) $\overline{H} + A$,

respectively. Now we state the propositions:

(140) $x \in A + H$ if and only if there exists g_1 and there exists g_2 such that $x = g_1 + g_2$ and $g_1 \in A$ and $g_2 \in H$.

(141) $x \in H + A$ if and only if there exists g_1 and there exists g_2 such that $x = g_1 + g_2$ and $g_1 \in H$ and $g_2 \in A$.

(142) $(A + B) + H = A + (B + H)$.

(143) $(A + H) + B = A + (H + B)$.

(144) $(H + A) + B = H + (A + B)$.

(145) $(A + H_1) + H_2 = A + (H_1 + \overline{H_2})$.

(146) $(H_1 + A) + H_2 = H_1 + (A + H_2)$.

(147) $(H_1 + \overline{H_2}) + A = H_1 + (H_2 + A)$.

(148) If G is an Abelian additive group, then $A + H = H + A$.

Let us consider G, H, and a. The functors: $a+H$ and $H+a$ yielding subsets of G are defined by terms,

(Def. 23) $a+\overline{H}$,

(Def. 24) $\overline{H}+a$,

respectively. Now we state the propositions:

(149) $x \in a+H$ if and only if there exists g such that $x=a+g$ and $g \in H$. The theorem is a consequence of (73).

(150) $x \in H+a$ if and only if there exists g such that $x=g+a$ and $g \in H$. The theorem is a consequence of (74).

(151) $(a+b)+H=a+(b+H)$.

(152) $(a+H)+b=a+(H+b)$.

(153) $(H+a)+b=H+(a+b)$.

(154) (i) $a \in a+H$, and

(ii) $a \in H+a$.

The theorem is a consequence of (92), (149), and (150).

(155) (i) $0_G+H=\overline{H}$, and

(ii) $H+0_G=\overline{H}$.

(156) (i) $\mathbf{0}_G+a=\{a\}$, and

(ii) $a+\mathbf{0}_G=\{a\}$.

The theorem is a consequence of (64).

(157) (i) $a+\Omega_G=$ the carrier of G, and

(ii) $\Omega_G+a=$ the carrier of G.

The theorem is a consequence of (63).

(158) If G is an Abelian additive group, then $a+H=H+a$.

(159) $a \in H$ if and only if $a+H=\overline{H}$. The theorem is a consequence of (149), (96), (97), and (92).

(160) $a+H=b+H$ if and only if $-b+a \in H$. The theorem is a consequence of (78), (83), and (159).

(161) $a+H=b+H$ if and only if $a+H$ meets $b+H$. The theorem is a consequence of (154), (149), (97), (13), (12), (96), and (160).

(162) $(a+b)+H \subseteq a+H+(b+H)$. The theorem is a consequence of (149) and (92).

(163) (i) $\overline{H} \subseteq a+H+(-a+H)$, and

(ii) $\overline{H} \subseteq -a+H+(a+H)$.

The theorem is a consequence of (83) and (162).

(164) $2 \cdot a + H \subseteq a + H + (a + H)$. The theorem is a consequence of (26) and (162).

(165) $a \in H$ if and only if $H + a = \overline{H}$. The theorem is a consequence of (150), (96), (97), and (92).

(166) $H + a = H + b$ if and only if $b + -a \in H$. The theorem is a consequence of (83), (80), and (165).

(167) $H + a = H + b$ if and only if $H + a$ meets $H + b$. The theorem is a consequence of (154), (150), (97), (12), (13), (96), and (166).

(168) $(H + a) + b \subseteq H + a + (H + b)$. The theorem is a consequence of (92), (150), and (80).

(169) (i) $\overline{H} \subseteq H + a + (H + -a)$, and

(ii) $\overline{H} \subseteq H + -a + (H + a)$.

The theorem is a consequence of (80), (83), and (168).

(170) $H + 2 \cdot a \subseteq H + a + (H + a)$. The theorem is a consequence of (80), (26), and (168).

(171) $a + H_1 \cap H_2 = (a + H_1) \cap (a + H_2)$. The theorem is a consequence of (149), (128), and (6).

(172) $H_1 \cap H_2 + a = (H_1 + a) \cap (H_2 + a)$. The theorem is a consequence of (150), (128), and (6).

(173) There exists a strict subgroup H_1 of G such that the carrier of $H_1 = a + H_2 + -a$. The theorem is a consequence of (154), (74), (149), (97), (150), (16), (73), (56), (96), and (98).

(174) $a + H \approx b + H$.

PROOF: Define $\mathcal{P}[\text{object}, \text{object}] \equiv$ there exists g_1 such that $\$_1 = g_1$ and $\$_2 = b + -a + g_1$. For every object x such that $x \in a + H$ there exists an object y such that $\mathcal{P}[x, y]$. Consider f being a function such that $\operatorname{dom} f = a + H$ and for every object x such that $x \in a + H$ holds $\mathcal{P}[x, f(x)]$ from [4, Sch. 1]. $\operatorname{rng} f = b + H$. f is one-to-one. □

(175) $a + H \approx H + b$.

PROOF: Define $\mathcal{P}[\text{object}, \text{object}] \equiv$ there exists g_1 such that $\$_1 = g_1$ and $\$_2 = -a + g_1 + b$. For every object x such that $x \in a + H$ there exists an object y such that $\mathcal{P}[x, y]$. Consider f being a function such that $\operatorname{dom} f = a + H$ and for every object x such that $x \in a + H$ holds $\mathcal{P}[x, f(x)]$ from [4, Sch. 1]. $\operatorname{rng} f = H + b$. f is one-to-one. □

(176) $H + a \approx H + b$. The theorem is a consequence of (175).

(177) (i) $\overline{H} \approx a + H$, and

(ii) $\overline{H} \approx H + a$.

The theorem is a consequence of (83), (174), and (176).

(178) (i) $\overline{\overline{H}} = \overline{\overline{a + H}}$, and

(ii) $\overline{\overline{H}} = \overline{\overline{H + a}}$.

(179) Let us consider a finite subgroup H of G. Then there exist finite sets B, C such that

(i) $B = a + H$, and

(ii) $C = H + a$, and

(iii) $\overline{\overline{H}} = \overline{\overline{B}}$, and

(iv) $\overline{\overline{H}} = \overline{\overline{C}}$.

The theorem is a consequence of (177).

Let us consider G and H. The functors: the left cosets of H and the right cosets of H yielding families of subsets of G are defined by conditions,

(Def. 25) $A \in$ the left cosets of H iff there exists a such that $A = a + H$,

(Def. 26) $A \in$ the right cosets of H iff there exists a such that $A = H + a$,

respectively. Now we state the propositions:

(180) If G is finite, then the right cosets of H is finite and the left cosets of H is finite.

(181) (i) $\overline{H} \in$ the left cosets of H, and

(ii) $\overline{H} \in$ the right cosets of H.

The theorem is a consequence of (83).

(182) The left cosets of $H \approx$ the right cosets of H.

PROOF: Define $\mathcal{P}[\text{object}, \text{object}] \equiv$ there exists g such that $\$_1 = g + H$ and $\$_2 = H + -g$. For every object x such that $x \in$ the left cosets of H there exists an object y such that $\mathcal{P}[x, y]$. Consider f being a function such that $\operatorname{dom} f =$ the left cosets of H and for every object x such that $x \in$ the left cosets of H holds $\mathcal{P}[x, f(x)]$ from [4, Sch. 1]. $\operatorname{rng} f =$ the right cosets of H. f is one-to-one. □

(183) (i) $\bigcup$(the left cosets of H) = the carrier of G, and

(ii) $\bigcup$(the right cosets of H) = the carrier of G.

The theorem is a consequence of (87), (149), and (150).

(184) The left cosets of $\mathbf{0}_G =$ the set of all $\{a\}$. The theorem is a consequence of (156).

(185) The right cosets of $\mathbf{0}_G =$ the set of all $\{a\}$. The theorem is a consequence of (156).

Let us consider a strict subgroup H of G. Now we state the propositions:

(186) If the left cosets of $H =$ the set of all $\{a\}$, then $H = \mathbf{0}_G$. The theorem is a consequence of (87), (149), (92), and (6).

(187) If the right cosets of H = the set of all $\{a\}$, then $H = \mathbf{0}_G$. The theorem is a consequence of (87), (150), (92), and (6).

(188) (i) the left cosets of Ω_G = {the carrier of G}, and

(ii) the right cosets of Ω_G = {the carrier of G}.

The theorem is a consequence of (157).

Let us consider a strict additive group G and a strict subgroup H of G. Now we state the propositions:

(189) If the left cosets of H = {the carrier of G}, then $H = G$. The theorem is a consequence of (149), (6), and (108).

(190) If the right cosets of H = {the carrier of G}, then $H = G$. The theorem is a consequence of (150), (6), and (108).

Let us consider G and H. The functor $|\bullet : H|$ yielding a cardinal number is defined by the term

(Def. 27) $\overline{\overline{\alpha}}$, where α is the left cosets of H.

Now we state the proposition:

(191) (i) $|\bullet : H| = \overline{\overline{\alpha}}$, and

(ii) $|\bullet : H| = \overline{\overline{\beta}}$,

where α is the left cosets of H and β is the right cosets of H.

Let us consider G and H. Assume the left cosets of H is finite. The functor $|\bullet : H|_{\mathbb{N}}$ yielding an element of $\mathbb{N}$ is defined by

(Def. 28) there exists a finite set B such that B = the left cosets of H and $it = \overline{\overline{B}}$.

Now we state the proposition:

(192) Suppose the left cosets of H is finite. Then

(i) there exists a finite set B such that B = the left cosets of H and $|\bullet : H|_{\mathbb{N}} = \overline{\overline{B}}$, and

(ii) there exists a finite set C such that C = the right cosets of H and $|\bullet : H|_{\mathbb{N}} = \overline{\overline{C}}$.

The theorem is a consequence of (182).

Let us consider a finite additive group G and a subgroup H of G. Now we state the propositions:

(193) Lagrange theorem for additive groups: $\overline{\overline{G}} = \overline{\overline{H}} \cdot |\bullet : H|_{\mathbb{N}}$. The theorem is a consequence of (179), (174), (161), and (183).

(194) $\overline{\overline{H}} \mid \overline{\overline{G}}$. The theorem is a consequence of (193).

(195) Let us consider a finite additive group G, subgroups I, H of G, and a subgroup J of H. Suppose $I = J$. Then $|\bullet : I|_{\mathbb{N}} = |\bullet : J|_{\mathbb{N}} \cdot |\bullet : H|_{\mathbb{N}}$. The theorem is a consequence of (193).

(196) $|\bullet : \Omega_G|_{\mathbb{N}} = 1$. The theorem is a consequence of (188).

(197) Let us consider a strict additive group G, and a strict subgroup H of G. Suppose the left cosets of H is finite and $|\bullet : H|_{\mathbb{N}} = 1$. Then $H = G$. The theorem is a consequence of (183) and (189).

(198) $|\bullet : \mathbf{0}_G| = \overline{\overline{G}}$.

PROOF: Define $\mathcal{F}(\text{object}) = \{\$_1\}$. Consider f being a function such that $\operatorname{dom} f =$ the carrier of G and for every object x such that $x \in$ the carrier of G holds $f(x) = \mathcal{F}(x)$ from [14, Sch. 3]. $\operatorname{rng} f =$ the left cosets of $\mathbf{0}_G$. f is one-to-one by [17, (3)]. $\square$

(199) Let us consider a finite additive group G. Then $|\bullet : \mathbf{0}_G|_{\mathbb{N}} = \overline{\overline{G}}$. The theorem is a consequence of (193) and (115).

(200) Let us consider a finite additive group G, and a strict subgroup H of G. Suppose $|\bullet : H|_{\mathbb{N}} = \overline{\overline{G}}$. Then $H = \mathbf{0}_G$. The theorem is a consequence of (193) and (116).

(201) Let us consider a strict subgroup H of G. Suppose the left cosets of H is finite and $|\bullet : H| = \overline{\overline{G}}$. Then

(i) G is finite, and

(ii) $H = \mathbf{0}_G$.

The theorem is a consequence of (200).

3. CLASSES OF CONJUGATION AND NORMAL SUBGROUPS – GROUP_3

From now on x, y, y_1, y_2 denote sets, G denotes an additive group, a, b, c, d, g, h denote elements of G, A, B, C, D denote subsets of G, H, H_1, H_2, H_3 denote subgroups of G, n denotes a natural number, and i denotes an integer.

Now we state the propositions:

(202) (i) $a + b + -b = a$, and

(ii) $a + -b + b = a$, and

(iii) $-b + b + a = a$, and

(iv) $b + -b + a = a$, and

(v) $a + (b + -b) = a$, and

(vi) $a + (-b + b) = a$, and

(vii) $-b + (b + a) = a$, and

(viii) $b + (-b + a) = a$.

(203) G is an Abelian additive group if and only if the addition of G is commutative.

(204) $\mathbf{0}_G$ is Abelian.

(205) If $A \subseteq B$ and $C \subseteq D$, then $A + C \subseteq B + D$.

(206) If $A \subseteq B$, then $a + A \subseteq a + B$ and $A + a \subseteq B + a$.

(207) If H_1 is a subgroup of H_2, then $a + H_1 \subseteq a + H_2$ and $H_1 + a \subseteq H_2 + a$. The theorem is a consequence of (205).

(208) $a + H = \{a\} + H$.

(209) $H + a = H + \{a\}$.

(210) $(A + a) + H = A + (a + H)$. The theorem is a consequence of (142).

(211) $(a + H) + A = a + (H + A)$. The theorem is a consequence of (143).

(212) $(A + H) + a = A + (H + a)$. The theorem is a consequence of (143).

(213) $(H + a) + A = H + (a + A)$. The theorem is a consequence of (144).

(214) $(H_1 + a) + H_2 = H_1 + (a + H_2)$.

Let us consider G. The functor $\operatorname{SubGr} G$ yielding a set is defined by

(Def. 29) for every object x, $x \in it$ iff x is a strict subgroup of G.

Note that $\operatorname{SubGr} G$ is non empty.

Now we state the propositions:

(215) Let us consider a strict additive group G. Then $G \in \operatorname{SubGr} G$. The theorem is a consequence of (100).

(216) If G is finite, then $\operatorname{SubGr} G$ is finite.

PROOF: Define $\mathcal{P}[\text{object}, \text{object}] \equiv$ there exists a strict subgroup H of G such that $\$_1 = H$ and $\$_2 =$ the carrier of H. For every object x such that $x \in \operatorname{SubGr} G$ there exists an object y such that $\mathcal{P}[x, y]$. Consider f being a function such that $\operatorname{dom} f = \operatorname{SubGr} G$ and for every object x such that $x \in \operatorname{SubGr} G$ holds $\mathcal{P}[x, f(x)]$ from [4, Sch. 1]. $\operatorname{rng} f \subseteq 2^{\alpha}$, where α is the carrier of G. f is one-to-one. $\square$

Let us consider G, a, and b. The functor $a \cdot b$ yielding an element of G is defined by the term

(Def. 30) $-b + a + b$.

Now we state the propositions:

(217) If $a \cdot g = b \cdot g$, then $a = b$. The theorem is a consequence of (6).

(218) $0_G \cdot a = 0_G$.

(219) If $a \cdot b = 0_G$, then $a = 0_G$. The theorem is a consequence of (11) and (7).

(220) $a \cdot 0_G = a$. The theorem is a consequence of (8).

(221) $a \cdot a = a$.

(222) (i) $a \cdot (-a) = a$, and

(ii) $(-a) \cdot a = -a$.

(223) $a \cdot b = a$ if and only if $a + b = b + a$. The theorem is a consequence of (12).

(224) $(a + b) \cdot g = a \cdot g + b \cdot g$.

(225) $a \cdot g \cdot h = a \cdot (g + h)$. The theorem is a consequence of (16).

(226) (i) $a \cdot b \cdot (-b) = a$, and

(ii) $a \cdot (-b) \cdot b = a$.

The theorem is a consequence of (225) and (220).

(227) $(-a) \cdot b = -a \cdot b$. The theorem is a consequence of (16).

(228) $(n \cdot a) \cdot b = n \cdot (a \cdot b)$.

(229) $(i \cdot a) \cdot b = i \cdot (a \cdot b)$. The theorem is a consequence of (29) and (227).

(230) If G is an Abelian additive group, then $a \cdot b = a$. The theorem is a consequence of (202).

(231) If for every a and b, $a \cdot b = a$, then G is Abelian. The theorem is a consequence of (223).

Let us consider G, A, and B. The functor $A \cdot B$ yielding a subset of G is defined by the term

(Def. 31) $\{g \cdot h : g \in A \text{ and } h \in B\}$.

Now we state the propositions:

(232) $x \in A \cdot B$ if and only if there exists g and there exists h such that $x = g \cdot h$ and $g \in A$ and $h \in B$.

(233) $A \cdot B \neq \emptyset$ if and only if $A \neq \emptyset$ and $B \neq \emptyset$. The theorem is a consequence of (232).

(234) $A \cdot B \subseteq -B + A + B$.

(235) $(A + B) \cdot C \subseteq A \cdot C + B \cdot C$. The theorem is a consequence of (224).

(236) $A \cdot B \cdot C = A \cdot (B + C)$. The theorem is a consequence of (225).

(237) $(-A) \cdot B = -A \cdot B$. The theorem is a consequence of (227).

(238) $\{a\} \cdot \{b\} = \{a \cdot b\}$. The theorem is a consequence of (49), (64), (233), and (234).

(239) $\{a\} \cdot \{b, c\} = \{a \cdot b, a \cdot c\}$.

(240) $\{a, b\} \cdot \{c\} = \{a \cdot c, b \cdot c\}$.

(241) $\{a, b\} \cdot \{c, d\} = \{a \cdot c, a \cdot d, b \cdot c, b \cdot d\}$.

Let us consider G, A, and g. The functors: $A \cdot g$ and $g \cdot A$ yielding subsets of G are defined by terms, (Def. 32) $A \cdot \{g\}$,

(Def. 33) $\{g\} \cdot A$,

respectively. Now we state the propositions:

(242) $x \in A \cdot g$ if and only if there exists h such that $x = h \cdot g$ and $h \in A$.

(243) $x \in g \cdot A$ if and only if there exists h such that $x = g \cdot h$ and $h \in A$.

(244) $g \cdot A \subseteq -A + g + A$. The theorem is a consequence of (243) and (74).

(245) $A \cdot B \cdot g = A \cdot (B + g)$.

(246) $A \cdot g \cdot B = A \cdot (g + B)$.

(247) $g \cdot A \cdot B = g \cdot (A + B)$.

(248) $A \cdot a \cdot b = A \cdot (a + b)$. The theorem is a consequence of (236) and (64).

(249) $a \cdot A \cdot b = a \cdot (A + b)$.

(250) $a \cdot b \cdot A = a \cdot (b + A)$. The theorem is a consequence of (238) and (236).

(251) $A \cdot g = -g + A + g$. The theorem is a consequence of (234), (49), (74), (73), and (242).

(252) $(A + B) \cdot a \subseteq A \cdot a + B \cdot a$.

(253) $A \cdot 0_G = A$. The theorem is a consequence of (251), (83), and (8).

(254) If $A \neq \emptyset$, then $0_G \cdot A = \{0_G\}$. The theorem is a consequence of (243) and (218).

(255) (i) $A \cdot a \cdot (-a) = A$, and

(ii) $A \cdot (-a) \cdot a = A$.

The theorem is a consequence of (248) and (253).

(256) G is an Abelian additive group if and only if for every A and B such that $B \neq \emptyset$ holds $A \cdot B = A$. The theorem is a consequence of (230), (238), and (231).

(257) G is an Abelian additive group if and only if for every A and g, $A \cdot g = A$. The theorem is a consequence of (256), (238), and (231).

(258) G is an Abelian additive group if and only if for every A and g such that $A \neq \emptyset$ holds $g \cdot A = \{g\}$. The theorem is a consequence of (256), (238), and (231).

Let us consider G, H, and a. The functor $H \cdot a$ yielding a strict subgroup of G is defined by

(Def. 34) the carrier of $it = \overline{H} \cdot a$.

Now we state the propositions:

(259) $x \in H \cdot a$ if and only if there exists g such that $x = g \cdot a$ and $g \in H$. The theorem is a consequence of (242).

(260) The carrier of $H \cdot a = -a + H + a$. The theorem is a consequence of (251).

(261) $H \cdot a \cdot b = H \cdot (a + b)$. The theorem is a consequence of (248) and (105).

Let us consider a strict subgroup H of G. Now we state the propositions:

(262) $H \cdot 0_G = H$. The theorem is a consequence of (253) and (105).

(263) (i) $H \cdot a \cdot (-a) = H$, and

(ii) $H \cdot (-a) \cdot a = H$.

The theorem is a consequence of (261) and (262).

Now we state the propositions:

(264) $(H_1 \cap H_2) \cdot a = H_1 \cdot a \cap (H_2 \cdot a)$. The theorem is a consequence of (259), (128), and (217).

(265) $\overline{\overline{H}} = \overline{\overline{H \cdot a}}$.

Proof: Define $\mathcal{F}$(element of G) $= \$_1 \cdot a$. Consider f being a function from the carrier of G into the carrier of G such that for every g, $f(g) = \mathcal{F}(g)$ from [15, Sch. 4]. Set $g = f{\restriction}$(the carrier of H). $\operatorname{rng} g =$ the carrier of $H \cdot a$ by [46, (62)], (88), (242), [14, (47)]. g is one-to-one by [46, (62)], (88), [14, (47)], (217). $\square$

(266) H is finite if and only if $H \cdot a$ is finite. The theorem is a consequence of (265).

Let us consider G and a. Let H be a finite subgroup of G. Observe that $H \cdot a$ is finite.

Now we state the propositions:

(267) Let us consider a finite subgroup H of G. Then $\overline{\overline{H}} = \overline{\overline{H \cdot a}}$.

(268) $\mathbf{0}_G \cdot a = \mathbf{0}_G$. The theorem is a consequence of (238) and (218).

(269) Let us consider a strict subgroup H of G. If $H \cdot a = \mathbf{0}_G$, then $H = \mathbf{0}_G$. The theorem is a consequence of (266), (115), (265), and (116).

(270) Let us consider an additive group G, and an element a of G. Then $\Omega_G \cdot a = \Omega_G$. The theorem is a consequence of (225), (220), and (259).

(271) Let us consider a strict subgroup H of G. If $H \cdot a = G$, then $H = G$. The theorem is a consequence of (259), (217), and (108).

(272) $|\bullet : H| = |\bullet : H \cdot a|$.

Proof: Define $\mathcal{P}$[object, object] $\equiv$ there exists b such that $\$_1 = b + H$ and $\$_2 = b \cdot a + H \cdot a$. For every object x such that $x \in$ the left cosets of H there exists an object y such that $\mathcal{P}[x, y]$. Consider f being a function such that $\operatorname{dom} f =$ the left cosets of H and for every object x such that $x \in$ the left cosets of H holds $\mathcal{P}[x, f(x)]$ from [4, Sch. 1]. For every x, y_1, and y_2 such that $x \in$ the left cosets of H and $\mathcal{P}[x, y_1]$ and $\mathcal{P}[x, y_2]$ holds $y_1 = y_2$. $\operatorname{rng} f =$ the left cosets of $H \cdot a$. f is one-to-one. $\square$

(273) If the left cosets of H is finite, then $|\bullet : H|_{\mathbb{N}} = |\bullet : H \cdot a|_{\mathbb{N}}$. The theorem is a consequence of (272).

(274) If G is an Abelian additive group, then for every strict subgroup H of G and for every a, $H \cdot a = H$. The theorem is a consequence of (260), (158), (153), (155), and (105).

Let us consider G, a, and b. We say that a and b are conjugated if and only if

(Def. 35) there exists g such that $a = b \cdot g$.

Now we state the proposition:

(275) a and b are conjugated if and only if there exists g such that $b = a \cdot g$. The theorem is a consequence of (226).

Let us consider G, a, and b. Observe that a and b are conjugated is reflexive and symmetric.

Now we state the propositions:

(276) If a and b are conjugated and b and c are conjugated, then a and c are conjugated. The theorem is a consequence of (225).

(277) If a and 0_G are conjugated or 0_G and a are conjugated, then $a = 0_G$. The theorem is a consequence of (275) and (219).

(278) $a \cdot \overline{\Omega_G} = \{b : a \text{ and } b \text{ are conjugated}\}$. The theorem is a consequence of (243).

Let us consider G and a. The functor $a^\bullet$ yielding a subset of G is defined by the term

(Def. 36) $a \cdot \overline{\Omega_G}$.

Now we state the propositions:

(279) $x \in a^\bullet$ if and only if there exists b such that $b = x$ and a and b are conjugated. The theorem is a consequence of (278).

(280) $a \in b^\bullet$ if and only if a and b are conjugated. The theorem is a consequence of (279).

(281) $a \cdot g \in a^\bullet$.

(282) $a \in a^\bullet$.

(283) If $a \in b^\bullet$, then $b \in a^\bullet$. The theorem is a consequence of (280).

(284) $a^\bullet = b^\bullet$ if and only if $a^\bullet$ meets $b^\bullet$. The theorem is a consequence of (280), (279), and (276).

(285) $a^\bullet = \{0_G\}$ if and only if $a = 0_G$. The theorem is a consequence of (280), (279), and (277).

(286) $a^\bullet + A = A + a^\bullet$. The theorem is a consequence of (280), (202), (226), (224), (221), (225), (279), and (275).

Let us consider G, A, and B. We say that A and B are conjugated if and only if

(Def. 37) there exists g such that $A = B \cdot g$.

Now we state the propositions:

(287) A and B are conjugated if and only if there exists g such that $B = A \cdot g$. The theorem is a consequence of (255).

(288) A and A are conjugated. The theorem is a consequence of (253).

(289) If A and B are conjugated, then B and A are conjugated. The theorem is a consequence of (255).

Let us consider G, A, and B. Let us observe that A and B are conjugated is reflexive and symmetric.

Now we state the propositions:

(290) If A and B are conjugated and B and C are conjugated, then A and C are conjugated. The theorem is a consequence of (248).

(291) $\{a\}$ and $\{b\}$ are conjugated if and only if a and b are conjugated.
PROOF: If $\{a\}$ and $\{b\}$ are conjugated, then a and b are conjugated by (287), (238), (275), [17, (3)]. Consider g such that $a \cdot g = b$. $\{b\} = \{a\} \cdot g$. $\square$

(292) If A and $\overline{H_1}$ are conjugated, then there exists a strict subgroup H_2 of G such that the carrier of $H_2 = A$.

Let us consider G and A. The functor $A^\bullet$ yielding a family of subsets of G is defined by the term

(Def. 38) $\{B : A$ and B are conjugated$\}$.

Now we state the propositions:

(293) $x \in A^\bullet$ if and only if there exists B such that $x = B$ and A and B are conjugated.

(294) $A \in B^\bullet$ if and only if A and B are conjugated.

(295) $A \cdot g \in A^\bullet$. The theorem is a consequence of (287).

(296) $A \in A^\bullet$.

(297) If $A \in B^\bullet$, then $B \in A^\bullet$. The theorem is a consequence of (294).

(298) $A^\bullet = B^\bullet$ if and only if $A^\bullet$ meets $B^\bullet$. The theorem is a consequence of (294) and (290).

(299) $\{a\}^\bullet = \{\{b\} : b \in a^\bullet\}$. The theorem is a consequence of (287), (275), (280), (238), and (291).

(300) If G is finite, then $A^\bullet$ is finite.

Let us consider G, H_1, and H_2. We say that H_1 and H_2 are conjugated if and only if

(Def. 39) there exists g such that the additive magma of $H_1 = H_2 \cdot g$.

Now we state the propositions:

(301) Let us consider strict subgroups H_1, H_2 of G. Then H_1 and H_2 are conjugated if and only if there exists g such that $H_2 = H_1 \cdot g$. The theorem is a consequence of (263).

(302) Let us consider a strict subgroup H_1 of G. Then H_1 and H_1 are conjugated. The theorem is a consequence of (262).

(303) Let us consider strict subgroups H_1, H_2 of G. If H_1 and H_2 are conjugated, then H_2 and H_1 are conjugated. The theorem is a consequence of (263).

Let us consider G. Let H_1, H_2 be strict subgroups of G. Observe that H_1 and H_2 are conjugated is reflexive and symmetric.

Now we state the proposition:

(304) Let us consider strict subgroups H_1, H_2 of G. Suppose H_1 and H_2 are conjugated and H_2 and H_3 are conjugated. Then H_1 and H_3 are conjugated. The theorem is a consequence of (261).

In the sequel L denotes a subset of $\mathrm{SubGr}\, G$.

Let us consider G and H. The functor $H^\bullet$ yielding a subset of $\mathrm{SubGr}\, G$ is defined by

(Def. 40) for every object x, $x \in it$ iff there exists a strict subgroup H_1 of G such that $x = H_1$ and H and H_1 are conjugated.

Now we state the propositions:

(305) If $x \in H^\bullet$, then x is a strict subgroup of G.

(306) Let us consider strict subgroups H_1, H_2 of G. Then $H_1 \in {H_2}^\bullet$ if and only if H_1 and H_2 are conjugated.

Let us consider a strict subgroup H of G. Now we state the propositions:

(307) $H \cdot g \in H^\bullet$. The theorem is a consequence of (301).

(308) $H \in H^\bullet$.

Let us consider strict subgroups H_1, H_2 of G. Now we state the propositions:

(309) If $H_1 \in {H_2}^\bullet$, then $H_2 \in {H_1}^\bullet$. The theorem is a consequence of (306).

(310) ${H_1}^\bullet = {H_2}^\bullet$ if and only if ${H_1}^\bullet$ meets ${H_2}^\bullet$. The theorem is a consequence of (308), (305), (306), and (304).

Now we state the propositions:

(311) If G is finite, then $H^\bullet$ is finite.

(312) Let us consider a strict subgroup H_1 of G. Then H_1 and H_2 are conjugated if and only if $\overline{H_1}$ and $\overline{H_2}$ are conjugated.

Let us consider G. Let I_1 be a subgroup of G. We say that I_1 is normal if and only if

(Def. 41) for every a, $I_1 \cdot a =$ the additive magma of I_1.

Let us note that there exists a subgroup of G which is strict and normal.

From now on N_2 denotes a normal subgroup of G.

Now we state the propositions:

(313) (i) $\mathbf{0}_G$ is normal, and

(ii) Ω_G is normal.

(314) Let us consider strict, normal subgroups N_1, N_2 of G. Then $N_1 \cap N_2$ is normal. The theorem is a consequence of (264).

(315) Let us consider a strict subgroup H of G. If G is an Abelian additive group, then H is normal.

(316) H is a normal subgroup of G if and only if for every a, $a + H = H + a$. The theorem is a consequence of (260), (79), (151), (83), (153), (155), and (105).

Let us consider a subgroup H of G. Now we state the propositions:

(317) H is a normal subgroup of G if and only if for every a, $a + H \subseteq H + a$. The theorem is a consequence of (316), (205), (151), (155), (152), (80), and (83).

(318) H is a normal subgroup of G if and only if for every a, $H + a \subseteq a + H$. The theorem is a consequence of (316), (205), (151), (155), (152), (80), and (83).

(319) H is a normal subgroup of G if and only if for every A, $A + H = H + A$. The theorem is a consequence of (140), (149), (316), (150), and (141).

Let us consider a strict subgroup H of G. Now we state the propositions:

(320) H is a normal subgroup of G if and only if for every a, H is a subgroup of $H \cdot a$. The theorem is a consequence of (100), (260), (80), (83), (207), and (318).

(321) H is a normal subgroup of G if and only if for every a, $H \cdot a$ is a subgroup of H. The theorem is a consequence of (100), (260), (80), (83), (207), and (317).

(322) H is a normal subgroup of G if and only if $H^\bullet = \{H\}$.
PROOF: If H is a normal subgroup of G, then $H^\bullet = \{H\}$ by (301), (308), [17, (31)]. H is normal. $\square$

(323) H is a normal subgroup of G if and only if for every a such that $a \in H$ holds $a^\bullet \subseteq \overline{H}$. The theorem is a consequence of (279), (275), (259), and (226).

Let us consider strict, normal subgroups N_1, N_2 of G. Now we state the propositions:

(324) $\overline{N_1} + \overline{N_2} = \overline{N_2} + \overline{N_1}$.

(325) There exists a strict, normal subgroup N of G such that the carrier of $N = \overline{N_1} + \overline{N_2}$. The theorem is a consequence of (124), (75), (316), (76), and (77).

Now we state the propositions:

(326) Let us consider a normal subgroup N of G. Then the left cosets of $N =$ the right cosets of N. The theorem is a consequence of (316).

(327) Let us consider a subgroup H of G. Suppose the left cosets of H is finite and $|\bullet : H|_{\mathbb{N}} = 2$. Then H is a normal subgroup of G.
PROOF: There exists a finite set B such that $B =$ the left cosets of H and $|\bullet : H|_{\mathbb{N}} = \overline{\overline{B}}$. Consider x, y being objects such that $x \neq y$ and the left cosets of $H = \{x, y\}$. $\overline{H} \in$ the left cosets of H. Consider z_3 being an object such that $\{x, y\} = \{\overline{H}, z_3\}$. $\overline{H}$ misses z_3 by (155), (161), [34, (29)], [17, (4)]. $\bigcup$(the left cosets of H) = the carrier of G and $\bigcup$(the left cosets of H) $= \overline{H} \cup z_3$. $\bigcup$(the right cosets of H) = the carrier of G and $z_3 =$ (the carrier of G) $\setminus \overline{H}$. There exists a finite set C such that $C =$ the right cosets of H and $|\bullet : H|_{\mathbb{N}} = \overline{\overline{C}}$. Consider z_1, z_2 being objects such that $z_1 \neq z_2$ and the right cosets of $H = \{z_1, z_2\}$. $\overline{H} \in$ the right cosets of H. Consider z_4 being an object such that $\{z_1, z_2\} = \{\overline{H}, z_4\}$. $\overline{H}$ misses z_4 by (155), (167), [34, (29)], [17, (4)]. $\square$

Let us consider G and A. The functor $\mathrm{N}(A)$ yielding a strict subgroup of G is defined by

(Def. 42) the carrier of *it* $= \{h : A \cdot h = A\}$.

Now we state the propositions:

(328) $x \in \mathrm{N}(A)$ if and only if there exists h such that $x = h$ and $A \cdot h = A$.

(329) $\overline{\overline{A^\bullet}} = |\bullet : \mathrm{N}(A)|$.
PROOF: Define $\mathcal{P}[\text{object}, \text{object}] \equiv$ there exists a such that $\$_1 = A \cdot a$ and $\$_2 = \mathrm{N}(A) + a$. For every object x such that $x \in A^\bullet$ there exists an object y such that $\mathcal{P}[x, y]$. Consider f being a function such that $\operatorname{dom} f = A^\bullet$ and for every object x such that $x \in A^\bullet$ holds $\mathcal{P}[x, f(x)]$ from [4, Sch. 1]. For every x, y_1, and y_2 such that $x \in A^\bullet$ and $\mathcal{P}[x, y_1]$ and $\mathcal{P}[x, y_2]$ holds $y_1 = y_2$. $\operatorname{rng} f =$ the right cosets of $\mathrm{N}(A)$. f is one-to-one. $\square$

(330) Suppose $A^\bullet$ is finite or the left cosets of $\mathrm{N}(A)$ is finite. Then there exists a finite set C such that

(i) $C = A^\bullet$, and

(ii) $\overline{\overline{C}} = |\bullet : \mathrm{N}(A)|_{\mathbb{N}}$.

The theorem is a consequence of (329).

(331) $\overline{\overline{a^\bullet}} = |\bullet : \mathrm{N}(\{a\})|$.

PROOF: Define $\mathcal{F}(\text{object}) = \{\$_1\}$. Consider f being a function such that $\operatorname{dom} f = a^\bullet$ and for every object x such that $x \in a^\bullet$ holds $f(x) = \mathcal{F}(x)$ from [14, Sch. 3]. $\operatorname{rng} f = \{a\}^\bullet$. f is one-to-one by [17, (3)]. □

(332) Suppose $a^\bullet$ is finite or the left cosets of $\mathrm{N}(\{a\})$ is finite. Then there exists a finite set C such that

(i) $C = a^\bullet$, and

(ii) $\overline{\overline{C}} = |\bullet : \mathrm{N}(\{a\})|_{\mathbb{N}}$.

The theorem is a consequence of (331).

Let us consider G and H. The functor $\mathrm{N}(H)$ yielding a strict subgroup of G is defined by the term

(Def. 43) $\mathrm{N}(\overline{H})$.

Let us consider a strict subgroup H of G. Now we state the propositions:

(333) $x \in \mathrm{N}(H)$ if and only if there exists h such that $x = h$ and $H \cdot h = H$. The theorem is a consequence of (328).

(334) $\overline{\overline{H^\bullet}} = |\bullet : \mathrm{N}(H)|$.

PROOF: Define $\mathcal{P}[\text{object}, \text{object}] \equiv$ there exists a strict subgroup H_1 of G such that $\$_1 = H_1$ and $\$_2 = \overline{H_1}$. For every object x such that $x \in H^\bullet$ there exists an object y such that $\mathcal{P}[x, y]$. Consider f being a function such that $\operatorname{dom} f = H^\bullet$ and for every object x such that $x \in H^\bullet$ holds $\mathcal{P}[x, f(x)]$ from [4, Sch. 1]. $\operatorname{rng} f = \overline{H}^\bullet$. f is one-to-one. □

(335) Suppose $H^\bullet$ is finite or the left cosets of $\mathrm{N}(H)$ is finite. Then there exists a finite set C such that

(i) $C = H^\bullet$, and

(ii) $\overline{\overline{C}} = |\bullet : \mathrm{N}(H)|_{\mathbb{N}}$.

The theorem is a consequence of (334).

Now we state the proposition:

(336) Let us consider a strict additive group G, and a strict subgroup H of G. Then H is a normal subgroup of G if and only if $\mathrm{N}(H) = G$. The theorem is a consequence of (333) and (108).

Let us consider a strict additive group G. Now we state the propositions:

(337) $\mathrm{N}(\mathbf{0}_G) = G$. The theorem is a consequence of (313) and (336).

(338) $\mathrm{N}(\Omega_G) = G$. The theorem is a consequence of (313) and (336).

4. Topological Groups – TOPGRP_1

In the sequel S, R denote 1-sorted structures, X denotes a subset of R, T denotes a topological structure, x denotes a set, H denotes a non empty additive magma, P, Q, P_1, Q_1 denote subsets of H, and h denotes an element of H.

Now we state the proposition:

(339) If $P \subseteq P_1$ and $Q \subseteq Q_1$, then $P + Q \subseteq P_1 + Q_1$.

Let us assume that $P \subseteq Q$. Now we state the propositions:

(340) $P + h \subseteq Q + h$. The theorem is a consequence of (74).

(341) $h + P \subseteq h + Q$. The theorem is a consequence of (73).

From now on a denotes an element of G.

Now we state the propositions:

(342) $a \in -A$ if and only if $-a \in A$.

(343) $A \subseteq B$ if and only if $-A \subseteq -B$.

(344) $(\text{add inverse}\, G)^{\circ} A = -A$.

(345) $(\text{add inverse}\, G)^{-1}(A) = -A$.

(346) add inverse G is one-to-one. The theorem is a consequence of (9).

(347) $\operatorname{rng} \text{add inverse}\, G =$ the carrier of G.

Let G be an additive group. One can verify that add inverse G is one-to-one and onto.

Now we state the propositions:

(348) $(\text{add inverse}\, G)^{-1} = \text{add inverse}\, G$.

(349) $(\text{The addition of } H)^{\circ}(P \times Q) = P + Q$.

Let G be a non empty additive magma and a be an element of G. The functors: a^{+} and ${}^{+}a$ yielding functions from G into G are defined by conditions,

(Def. 44) for every element x of G, $a^{+}(x) = a + x$,

(Def. 45) for every element x of G, ${}^{+}a(x) = x + a$,

respectively. Let G be an additive group. One can verify that a^{+} is one-to-one and onto and ${}^{+}a$ is one-to-one and onto.

Now we state the propositions:

(350) $(h^{+})^{\circ} P = h + P$. The theorem is a consequence of (73).

(351) $({}^{+}h)^{\circ} P = P + h$. The theorem is a consequence of (74).

(352) $(a^{+})^{-1} = (-a)^{+}$.

(353) $({}^{+}a)^{-1} = {}^{+}(-a)$.

We consider topological additive group structures which extend additive magmas and topological structures and are systems

$$\langle \text{a carrier}, \text{an addition}, \text{a topology} \rangle$$

where the carrier is a set, the addition is a binary operation on the carrier, the topology is a family of subsets of the carrier.

Let A be a non empty set, R be a binary operation on A, and T be a family of subsets of A. Let us observe that $\langle A, R, T\rangle$ is non empty.

Let x be a set, R be a binary operation on $\{x\}$, and T be a family of subsets of $\{x\}$. Observe that $\langle \{x\}, R, T\rangle$ is trivial and every 1-element additive magma is additive group-like, add-associative, and Abelian and there exists a topological additive group structure which is strict and non empty and there exists a topological additive group structure which is strict, topological space-like, and 1-element.

Let G be an additive group-like, add-associative, non empty topological additive group structure. We say that G is inverse-continuous if and only if

(Def. 46) add inverse G is continuous.

Let G be a topological space-like topological additive group structure. We say that G is continuous if and only if

(Def. 47) for every function f from $G \times G$ into G such that $f =$ the addition of G holds f is continuous.

One can check that there exists a topological space-like, additive group-like, add-associative, 1-element topological additive group structure which is strict, Abelian, inverse-continuous, and continuous.

A semi additive topological group is a topological space-like, additive group-like, add-associative, non empty topological additive group structure.

A topological additive group is an inverse-continuous, continuous semi additive topological group. Now we state the propositions:

(354) Let us consider a continuous, non empty, topological space-like topological additive group structure T, elements a, b of T, and a neighbourhood W of $a + b$. Then there exists an open neighbourhood A of a and there exists an open neighbourhood B of b such that $A + B \subseteq W$.

(355) Let us consider a topological space-like, non empty topological additive group structure T. Suppose for every elements a, b of T for every neighbourhood W of $a + b$, there exists a neighbourhood A of a and there exists a neighbourhood B of b such that $A + B \subseteq W$. Then T is continuous.

Proof: For every point W of $T \times T$ and for every neighbourhood G of $f(W)$, there exists a neighbourhood H of W such that $f^\circ H \subseteq G$ by [32, (10)], (349). $\square$

(356) Let us consider an inverse-continuous semi additive topological group T, an element a of T, and a neighbourhood W of $-a$. Then there exists an open neighbourhood A of a such that $-A \subseteq W$.

(357) Let us consider a semi additive topological group T. Suppose for every

element a of T for every neighbourhood W of $-a$, there exists a neighbourhood A of a such that $-A \subseteq W$. Then T is inverse-continuous. The theorem is a consequence of (344).

(358) Let us consider a topological additive group T, elements a, b of T, and a neighbourhood W of $a+-b$. Then there exists an open neighbourhood A of a and there exists an open neighbourhood B of b such that $A+-B \subseteq W$. The theorem is a consequence of (354) and (356).

(359) Let us consider a semi additive topological group T. Suppose for every elements a, b of T for every neighbourhood W of $a+-b$, there exists a neighbourhood A of a and there exists a neighbourhood B of b such that $A+-B \subseteq W$. Then T is a topological additive group.

Proof: For every element a of T and for every neighbourhood W of $-a$, there exists a neighbourhood A of a such that $-A \subseteq W$ by [28, (4)]. For every elements a, b of T and for every neighbourhood W of $a+b$, there exists a neighbourhood A of a and there exists a neighbourhood B of b such that $A+B \subseteq W$. □

Let G be a continuous, non empty, topological space-like topological additive group structure and a be an element of G. One can check that a^+ is continuous and ${}^+a$ is continuous.

Let us consider a continuous semi additive topological group G and an element a of G. Now we state the propositions:

(360) a^+ is a homeomorphism of G. The theorem is a consequence of (352).

(361) ${}^+a$ is a homeomorphism of G. The theorem is a consequence of (353).

Let G be a continuous semi additive topological group and a be an element of G. The functors: a^+ and ${}^+a$ yield homeomorphisms of G. Now we state the proposition:

(362) Let us consider an inverse-continuous semi additive topological group G. Then add inverse G is a homeomorphism of G. The theorem is a consequence of (348).

Let G be an inverse-continuous semi additive topological group. Let us note that the functor add inverse G yields a homeomorphism of G. Let us note that every semi additive topological group which is continuous is also homogeneous.

Let us consider a continuous semi additive topological group G, a closed subset F of G, and an element a of G. Now we state the propositions:

(363) $F+a$ is closed. The theorem is a consequence of (351).

(364) $a+F$ is closed. The theorem is a consequence of (350).

Let G be a continuous semi additive topological group, F be a closed subset of G, and a be an element of G. Let us note that $F+a$ is closed and $a+F$ is closed.

Now we state the proposition:

(365) Let us consider an inverse-continuous semi additive topological group G, and a closed subset F of G. Then $-F$ is closed. The theorem is a consequence of (344).

Let G be an inverse-continuous semi additive topological group and F be a closed subset of G. One can verify that $-F$ is closed.

Let us consider a continuous semi additive topological group G, an open subset O of G, and an element a of G. Now we state the propositions:

(366) $O + a$ is open. The theorem is a consequence of (351).

(367) $a + O$ is open. The theorem is a consequence of (350).

Let G be a continuous semi additive topological group, A be an open subset of G, and a be an element of G. One can check that $A + a$ is open and $a + A$ is open.

Now we state the proposition:

(368) Let us consider an inverse-continuous semi additive topological group G, and an open subset O of G. Then $-O$ is open. The theorem is a consequence of (344).

Let G be an inverse-continuous semi additive topological group and A be an open subset of G. Observe that $-A$ is open.

Let us consider a continuous semi additive topological group G and subsets A, O of G.

Let us assume that O is open. Now we state the propositions:

(369) $O + A$ is open.

PROOF: $\text{Int}(O + A) = O + A$ by [48, (16)], (74), [48, (22)]. $\square$

(370) $A + O$ is open.

PROOF: $\text{Int}(A + O) = A + O$ by [48, (16)], (73), [48, (22)]. $\square$

Let G be a continuous semi additive topological group, A be an open subset of G, and B be a subset of G. Note that $A + B$ is open and $B + A$ is open.

Now we state the propositions:

(371) Let us consider an inverse-continuous semi additive topological group G, a point a of G, and a neighbourhood A of a. Then $-A$ is a neighbourhood of $-a$. The theorem is a consequence of (343).

(372) Let us consider a topological additive group G, a point a of G, and a neighbourhood A of $a + -a$. Then there exists an open neighbourhood B of a such that $B + -B \subseteq A$. The theorem is a consequence of (358) and (342).

(373) Let us consider an inverse-continuous semi additive topological group G, and a dense subset A of G. Then $-A$ is dense. The theorem is a consequence of (345).

Let G be an inverse-continuous semi additive topological group and A be a dense subset of G. Observe that $-A$ is dense.

Let us consider a continuous semi additive topological group G, a dense subset A of G, and a point a of G. Now we state the propositions:

(374) $a + A$ is dense. The theorem is a consequence of (350).

(375) $A + a$ is dense. The theorem is a consequence of (351).

Let G be a continuous semi additive topological group, A be a dense subset of G, and a be a point of G. Let us observe that $A + a$ is dense and $a + A$ is dense.

Now we state the proposition:

(376) Let us consider a topological additive group G, a basis B of 0_G, and a dense subset M of G. Then $\{V + x$, where V is a subset of G, x is a point of $G : V \in B$ and $x \in M\}$ is a basis of G.

Proof: Set $Z = \{V + x$, where V is a subset of G, x is a point of $G : V \in B$ and $x \in M\}$. $Z \subseteq$ the topology of G by [38, (12)]. For every subset W of G such that W is open for every point a of G such that $a \in W$ there exists a subset V of G such that $V \in Z$ and $a \in V$ and $V \subseteq W$ by (8), [28, (3)], (74), (372). $Z \subseteq 2^{\alpha}$, where α is the carrier of G. □

One can check that every topological additive group is regular.

Acknowledgement: The author wants to express his gratitude to the anonymous referee for his/her work on merging the three initial articles.

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Received April 30, 2015