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Polish Notation

Taneli Huuskonen[1]
Department of Mathematics and Statistics
University of Helsinki
Finland

Summary. This article is the first in a series formalizing some results in my joint work with Prof. Joanna Golińska-Pilarek ([12] and [13]) concerning a logic proposed by Prof. Andrzej Grzegorczyk ([14]).

We present some *mathematical folklore* about representing formulas in "Polish notation", that is, with operators of fixed arity prepended to their arguments. This notation, which was published by Jan Łukasiewicz in [15], eliminates the need for parentheses and is generally well suited for rigorous reasoning about syntactic properties of formulas.

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The notation and terminology used in this paper have been introduced in the following articles: [5], [1], [4], [11], [7], [8], [3], [9], [16], [19], [17], [18], and [10].

1. Preliminaries

From now on k, m, n denote natural numbers, a, b, c, c_1, c_2 denote objects, x, y, z, X, Y, Z denote sets, D denotes a non empty set, p, q, r, s, t, u, v denote finite sequences, P, Q, R, P_1, P_2, Q_1, Q_2, R_1, R_2 denote finite sequence-membered sets, and S, T denote non empty, finite sequence-membered sets.

Let D be a non empty set and P, Q be subsets of D^*. The functor $\frown(D, P, Q)$ yielding a subset of D^* is defined by the term

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(Def. 1) $\{p \frown q$, where p is a finite sequence of elements of D, q is a finite sequence of elements of $D : p \in P$ and $q \in Q\}$.

Let us consider P and Q. The functor $P \frown Q$ yielding a finite sequence-membered set is defined by

(Def. 2) for every a, $a \in it$ iff there exists p and there exists q such that $a = p \frown q$ and $p \in P$ and $q \in Q$.

Let β be an empty set. One can check that $\beta \frown P$ is empty and $P \frown \beta$ is empty.

Let us consider S and T. One can check that $S \frown T$ is non empty.

Now we state the propositions:

(1) If $p \frown q = r \frown s$, then there exists t such that $p \frown t = r$ or $p = r \frown t$.

(2) $(P \frown Q) \frown R = P \frown (Q \frown R)$.

PROOF: For every a, $a \in (P \frown Q) \frown R$ iff $a \in P \frown (Q \frown R)$ by [4, (32)]. $\square$

Note that $\{\emptyset\}$ is non empty and finite sequence-membered.

(3) (i) $P \frown \{\emptyset\} = P$, and

(ii) $\{\emptyset\} \frown P = P$.

PROOF: For every a, $a \in P \frown \{\emptyset\}$ iff $a \in P$ by [4, (34)]. For every a, $a \in \{\emptyset\} \frown P$ iff $a \in P$ by [4, (34)]. $\square$

Let us consider P. The functor $P^{\frown\frown}$ yielding a function is defined by

(Def. 3) $\operatorname{dom} it = \mathbb{N}$ and $it(0) = \{\emptyset\}$ and for every n, there exists Q such that $Q = it(n)$ and $it(n+1) = Q \frown P$.

Let us consider n. The functor $P \frown n$ yielding a finite sequence-membered set is defined by the term

(Def. 4) $(P^{\frown\frown})(n)$.

Now we state the proposition:

(4) $\emptyset \in P \frown 0$.

Let us consider P. Let n be a zero natural number. Note that $P \frown n$ is non empty.

Let β be an empty set and n be a non zero natural number. One can verify that $\beta \frown n$ is empty.

Let us consider P. The functor P^* yielding a non empty, finite sequence-membered set is defined by the term

(Def. 5) $\bigcup$ the set of all $P \frown n$ where n is a natural number.

(5) $a \in P^*$ if and only if there exists n such that $a \in P \frown n$.

Let us consider P.

(6) (i) $P \frown 0 = \{\emptyset\}$, and

(ii) for every n, $P \frown (n+1) = (P \frown n) \frown P$.

(7) $P \frown 1 = P$. The theorem is a consequence of (6) and (3).

(8) $P \frown n \subseteq P^*$.

(9) (i) $\emptyset \in P^*$, and

(ii) $P \subseteq P^*$.

The theorem is a consequence of (4), (5), and (7).

(10) $P \frown (m+n) = (P \frown m) \frown (P \frown n)$.

PROOF: Define $\mathcal{X}$[natural number] $\equiv P \frown (m + \$_1) = (P \frown m) \frown (P \frown \$_1)$. $\mathcal{X}[0]$. For every k such that $\mathcal{X}[k]$ holds $\mathcal{X}[k+1]$. For every k, $\mathcal{X}[k]$ from [2, Sch. 2]. $\square$

(11) If $p \in P \frown m$ and $q \in P \frown n$, then $p \frown q \in P \frown (m+n)$. The theorem is a consequence of (10).

(12) If p, $q \in P^*$, then $p \frown q \in P^*$. The theorem is a consequence of (5) and (11).

(13) If $P \subseteq R^*$ and $Q \subseteq R^*$, then $P \frown Q \subseteq R^*$. The theorem is a consequence of (12).

(14) If $Q \subseteq P^*$, then $Q \frown n \subseteq P^*$.

PROOF: Define $\mathcal{X}$[natural number] $\equiv Q \frown \$_1 \subseteq P^*$. $\mathcal{X}[0]$. For every k such that $\mathcal{X}[k]$ holds $\mathcal{X}[k+1]$. For every k, $\mathcal{X}[k]$ from [2, Sch. 2]. $\square$

(15) If $Q \subseteq P^*$, then $Q^* \subseteq P^*$. The theorem is a consequence of (5) and (14).

(16) If $P_1 \subseteq P_2$ and $Q_1 \subseteq Q_2$, then $P_1 \frown Q_1 \subseteq P_2 \frown Q_2$.

(17) If $P \subseteq Q$, then for every n, $P \frown n \subseteq Q \frown n$.

PROOF: Define $\mathcal{S}$[natural number] $\equiv P \frown \$_1 \subseteq Q \frown \$_1$. $P \frown 0 = \{\emptyset\}$. For every n such that $\mathcal{S}[n]$ holds $\mathcal{S}[n+1]$. For every n, $\mathcal{S}[n]$ from [2, Sch. 2]. $\square$

Let us consider S and n. Let us observe that $S \frown n$ is non empty and finite sequence-membered.

2. The Language

In the sequel α denotes a function from P into $\mathbb{N}$ and U, V, W denote subsets of P^*.

Let us consider P, α, and U. The Polish-expression layer(P, α, U) yielding a subset of P^* is defined by

(Def. 6) for every a, $a \in it$ iff $a \in P^*$ and there exists p and there exists q and there exists n such that $a = p \frown q$ and $p \in P$ and $n = \alpha(p)$ and $q \in U \frown n$.

Now we state the proposition:

(18) Suppose $p \in P$ and $n = \alpha(p)$ and $q \in U \frown n$. Then $p \frown q \in$ the Polish-expression layer(P, α, U). The theorem is a consequence of (14), (9), and (12).

Let us consider P and α. The Polish atoms(P, α) yielding a subset of P^* is defined by

(Def. 7) for every a, $a \in it$ iff $a \in P$ and $\alpha(a) = 0$.

The Polish operations(P, α) yielding a subset of P is defined by the term

(Def. 8) $\{t$, where t is an element of $P^* : t \in P$ and $\alpha(t) \neq 0\}$.

Now we state the propositions:

(19) The Polish atoms(P, α) $\subseteq$ the Polish-expression layer(P, α, U). The theorem is a consequence of (4) and (18).

(20) Suppose $U \subseteq V$. Then the Polish-expression layer(P, α, U) $\subseteq$ the Polish-expression layer(P, α, V). The theorem is a consequence of (17).

(21) Suppose $u \in$ the Polish-expression layer(P, α, U). Then there exists p and there exists q such that $p \in P$ and $u = p \frown q$.

Let us consider P and α. The Polish-expression hierarchy(P, α) yielding a function is defined by

(Def. 9) $\operatorname{dom} it = \mathbb{N}$ and $it(0) =$ the Polish atoms(P, α) and for every n, there exists U such that $U = it(n)$ and $it(n + 1) =$ the Polish-expression layer(P, α, U).

Let us consider n. The Polish-expression hierarchy(P, α, n) yielding a subset of P^* is defined by the term

(Def. 10) (the Polish-expression hierarchy(P, α))(n).

Now we state the proposition:

(22) The Polish-expression hierarchy(P, α, 0) = the Polish atoms(P, α).

Let us consider P, α, and n. Now we state the propositions:

(23) The Polish-expression hierarchy(P, α, $n + 1$) = the Polish-expression layer(P, α, the Polish-expression hierarchy(P, α, n)).

(24) The Polish-expression hierarchy(P, α, n) $\subseteq$ the Polish-expression hierarchy(P, α, $n + 1$).
PROOF: Define $\mathcal{S}$[natural number] $\equiv$ the Polish-expression hierarchy(P, α, $\$_1$) $\subseteq$ the Polish-expression hierarchy(P, α, $\$_1 + 1$). $\mathcal{S}[0]$. For every k such that $\mathcal{S}[k]$ holds $\mathcal{S}[k + 1]$. For every k, $\mathcal{S}[k]$ from [2, Sch. 2]. □

Now we state the proposition:

(25) The Polish-expression hierarchy(P, α, n) $\subseteq$ the Polish-expression hierarchy(P, α, $n + m$).
PROOF: Define $\mathcal{S}$[natural number] $\equiv$ the Polish-expression hierarchy(P, α, n) $\subseteq$ the Polish-expression hierarchy(P, α, $n + \$_1$). For every k such that $\mathcal{S}[k]$ holds $\mathcal{S}[k + 1]$. For every k, $\mathcal{S}[k]$ from [2, Sch. 2]. □

Let us consider P and α. The Polish-expression set(P, α) yielding a subset of P^* is defined by the term

(Def. 11) $\bigcup$the set of all the Polish-expression hierarchy(P, α, n) where n is a natural number.

Now we state the propositions:

(26) The Polish-expression hierarchy(P, α, n) $\subseteq$ the Polish-expression set(P, α).

(27) Suppose $q \in$ (the Polish-expression set(P, α)) $\frown n$. Then there exists m such that $q \in$ (the Polish-expression hierarchy(P, α, m)) $\frown n$.
PROOF: Define $\mathcal{S}$[natural number] $\equiv$ for every q such that $q \in$ (the Polish-expression set(P, α))$\frown \$_1$ there exists m such that $q \in$ (the Polish-expression hierarchy(P, α, m)) $\frown \$_1$. $\mathcal{S}[0]$. For every k such that $\mathcal{S}[k]$ holds $\mathcal{S}[k+1]$. For every k, $\mathcal{S}[k]$ from [2, Sch. 2]. □

(28) Suppose $a \in$ the Polish-expression set(P, α). Then there exists n such that $a \in$ the Polish-expression hierarchy(P, α, $n+1$). The theorem is a consequence of (24).

Let us consider P and α.

A Polish expression of P and α is an element of the Polish-expression set(P, α). Let us consider n and t. Assume $t \in P$. The Polish operation(P, α, n, t) yielding a function from (the Polish-expression set(P, α)) $\frown n$ into P^* is defined by

(Def. 12) for every q such that $q \in \operatorname{dom} it$ holds $it(q) = t \frown q$.

Let us consider X and Y. Let F be a partial function from X to 2^Y. One can check that F is disjoint valued if and only if the condition (Def. 13) is satisfied.

(Def. 13) for every a and b such that a, $b \in \operatorname{dom} F$ and $a \neq b$ holds $F(a)$ misses $F(b)$.

Let X be a set. One can check that there exists a finite sequence of elements of 2^X which is disjoint valued.

Now we state the proposition:

(29) Let us consider a set X, a disjoint valued finite sequence B of elements of 2^X, a, b, and c. If $a \in B(b)$ and $a \in B(c)$, then $b = c$ and $b \in \operatorname{dom} B$.

Let us consider X. Let B be a disjoint valued finite sequence of elements of 2^X. The arity from list B yielding a function from X into $\mathbb{N}$ is defined by

(Def. 14) for every a such that $a \in X$ holds there exists n such that $a \in B(n)$ and $a \in B(it(a))$ or there exists no n such that $a \in B(n)$ and $it(a) = 0$.

Now we state the propositions:

(30) Let us consider a disjoint valued finite sequence B of elements of 2^X, and a. Suppose $a \in X$. Then (the arity from list B)(a) $\neq 0$ if and only if

there exists n such that $a \in B(n)$. The theorem is a consequence of (29).

(31) Let us consider a disjoint valued finite sequence B of elements of 2^X, a, and n. Suppose $a \in B(n)$. Then (the arity from list B)$(a) = n$. The theorem is a consequence of (29).

(32) Suppose $r \in$ the Polish-expression set(P, α). Then there exists n and there exists p and there exists q such that $p \in P$ and $n = \alpha(p)$ and $q \in$ (the Polish-expression set(P, α)) $\frown n$ and $r = p \frown q$. The theorem is a consequence of (28), (23), (26), and (17).

Let us consider P, α, and Q. We say that Q is α-closed if and only if

(Def. 15) for every p, n, and q such that $p \in P$ and $n = \alpha(p)$ and $q \in Q \frown n$ holds $p \frown q \in Q$.

Now we state the propositions:

(33) The Polish-expression set(P, α) is α-closed. The theorem is a consequence of (27), (18), (23), and (26).

(34) If Q is α-closed, then the Polish atoms(P, α) $\subseteq Q$. The theorem is a consequence of (4).

(35) If Q is α-closed, then the Polish-expression hierarchy(P, α, n) $\subseteq Q$.
PROOF: Define $\mathcal{X}$[natural number] $\equiv$ the Polish-expression hierarchy(P, α, $\$_1$) $\subseteq Q$. $\mathcal{X}[0]$. For every k such that $\mathcal{X}[k]$ holds $\mathcal{X}[k+1]$. For every k, $\mathcal{X}[k]$ from [2, Sch. 2]. $\square$

(36) The Polish atoms(P, α) $\subseteq$ the Polish-expression set(P, α). The theorem is a consequence of (33) and (34).

(37) If Q is α-closed, then the Polish-expression set(P, α) $\subseteq Q$. The theorem is a consequence of (28) and (35).

(38) Suppose $r \in$ the Polish-expression set(P, α). Then there exists n and there exists t and there exists q such that $t \in P$ and $n = \alpha(t)$ and $r =$ (the Polish operation(P, α, n, t))(q). The theorem is a consequence of (28), (23), (26), and (17).

(39) Suppose $p \in P$ and $n = \alpha(p)$ and $q \in$ (the Polish-expression set(P, α)) $\frown n$. Then (the Polish operation(P, α, n, p))$(q) \in$ the Polish-expression set(P, α). The theorem is a consequence of (33).

The scheme *AInd* deals with a finite sequence-membered set $\mathcal{P}$ and a function α from $\mathcal{P}$ into $\mathbb{N}$ and a unary predicate $\mathcal{X}$ and states that

(Sch. 1) For every a such that $a \in$ the Polish-expression set($\mathcal{P}$, α) holds $\mathcal{X}[a]$

provided

- for every p, q, and n such that $p \in \mathcal{P}$ and $n = \alpha(p)$ and $q \in$ (the Polish-expression set($\mathcal{P}$, α)) $\frown n$ holds $\mathcal{X}[p \frown q]$.

3. Parsing

In the sequel k, l, m, n, i, j denote natural numbers, a, b, c, c_1, c_2 denote objects, x, y, z, X, Y, Z denote sets, D, D_1, D_2 denote non empty sets, p, q, r, s, t, u, v denote finite sequences, and P, Q, R denote finite sequence-membered sets.

Let us consider P. We say that P is antichain-like if and only if

(Def. 16) for every p and q such that $p, p \frown q \in P$ holds $q = \emptyset$.

Now we state the propositions:

(40) P is antichain-like if and only if for every p and q such that $p, p \frown q \in P$ holds $p = p \frown q$.

PROOF: If P is antichain-like, then for every p and q such that $p, p \frown q \in P$ holds $p = p \frown q$ by [4, (34)]. $\square$

(41) If $P \subseteq Q$ and Q is antichain-like, then P is antichain-like.

Note that every finite sequence-membered set which is trivial is also antichain-like.

Now we state the proposition:

(42) If $P = \{a\}$, then P is antichain-like.

Note that there exists a non empty, finite sequence-membered set which is antichain-like and every finite sequence-membered set which is empty is also antichain-like.

An antichain is an antichain-like, finite sequence-membered set. In the sequel B, C denote antichains.

Let us consider B. One can verify that every subset of B is antichain-like and finite sequence-membered.

A Polish-language is a non empty antichain. From now on S, T denote Polish-languages.

Let D be a non empty set and ψ be a subset of D^*. Note that ψ is antichain-like if and only if the condition (Def. 17) is satisfied.

(Def. 17) for every finite sequence g of elements of D and for every finite sequence h of elements of D such that $g, g \frown h \in \psi$ holds $h = \varepsilon_D$.

Now we state the proposition:

(43) If $p \frown q = r \frown s$ and $p, r \in B$, then $p = r$ and $q = s$. The theorem is a consequence of (1) and (40).

Let us consider B and C. Note that $B \frown C$ is antichain-like.

Now we state the propositions:

(44) If for every p and q such that $p, q \in P$ holds $\operatorname{dom} p = \operatorname{dom} q$, then P is antichain-like.

Proof: For every p and q such that p, $p \frown q \in P$ holds $p = p \frown q$ by [4, (21)]. $\square$

(45) If for every p such that $p \in P$ holds $\operatorname{dom} p = a$, then P is antichain-like. The theorem is a consequence of (44).

(46) If $\emptyset \in B$, then $B = \{\emptyset\}$.

Proof: For every a such that $a \in B$ holds $a = \emptyset$ by [4, (34)]. $\square$

Let us consider B and n. Note that $B \frown n$ is antichain-like.

Let us consider T. Let us observe that there exists a subset of T^* which is non empty and antichain-like and $T \frown n$ is non empty.

A Polish-language of T is a non empty, antichain-like subset of T^*.

A Polish arity-function of T is a function from T into $\mathbb{N}$ and is defined by

(Def. 18) there exists a such that $a \in T$ and $it(a) = 0$.

One can verify that every Polish-language of T is non empty, antichain-like, and finite sequence-membered.

In the sequel α denotes a Polish arity-function of T and U, V, W denote Polish-languages of T.

Let us consider T and α. Let t be an element of T. Let us observe that the functor $\alpha(t)$ yields a natural number. Let us consider U. Note that the Polish-expression layer(T, α, U) is defined by

(Def. 19) for every a, $a \in it$ iff there exists an element t of T and there exists an element u of T^* such that $a = t \frown u$ and $u \in U \frown \alpha(t)$.

Let us consider B and p. We say that p is B-headed if and only if

(Def. 20) there exists q and there exists r such that $q \in B$ and $p = q \frown r$.

Let us consider P. We say that P is B-headed if and only if

(Def. 21) for every p such that $p \in P$ holds p is B-headed.

Now we state the propositions:

(47) If p is B-headed and $B \subseteq C$, then p is C-headed.

(48) If P is B-headed and $B \subseteq C$, then P is C-headed.

Let us consider B and P. Observe that $B \frown P$ is B-headed.

Now we state the propositions:

(49) If p is $(B \frown C)$-headed, then p is B-headed.

(50) B is B-headed. The theorem is a consequence of (3).

Let us consider B. Let us observe that there exists a finite sequence-membered set which is B-headed.

Let P be a B-headed, finite sequence-membered set. Let us note that every subset of P is B-headed.

Let us consider S. Let us observe that there exists a finite sequence-membered set which is non empty and S-headed.

Now we state the proposition:

(51) $S \frown (m+n)$ is $(S \frown m)$-headed. The theorem is a consequence of (10).

Let us consider S and p. The functor S-head(p) yielding a finite sequence is defined by

(Def. 22) (i) $it \in S$ and there exists r such that $p = it \frown r$, **if** p is S-headed,

(ii) $it = \emptyset$, **otherwise**.

The functor S-tail(p) yielding a finite sequence is defined by

(Def. 23) $p = (S\text{-head}(p)) \frown it$.

Now we state the propositions:

(52) If $s \in S$, then S-head$(s \frown t) = s$ and S-tail$(s \frown t) = t$.

(53) If $s \in S$, then S-head$(s) = s$ and S-tail$(s) = \emptyset$. The theorem is a consequence of (52).

Let us consider S, T, and u. Now we state the propositions:

(54) If $u \in S \frown T$, then S-head$(u) \in S$ and S-tail$(u) \in T$. The theorem is a consequence of (52).

(55) If $S \subseteq T$ and u is S-headed, then S-head$(u) = T$-head(u) and S-tail$(u) = T$-tail(u). The theorem is a consequence of (52).

Now we state the propositions:

(56) Suppose s is S-headed. Then

(i) $s \frown t$ is S-headed, and

(ii) S-head$(s \frown t) = S$-head(s), and

(iii) S-tail$(s \frown t) = (S\text{-tail}(s)) \frown t$.

The theorem is a consequence of (52).

(57) If $m + 1 \leqslant n$ and $s \in S \frown n$, then s is $(S \frown m)$-headed and $S \frown m$-tail(s) is S-headed. The theorem is a consequence of (51), (10), (54), and (7).

(58) (i) s is $(S \frown 0)$-headed, and

(ii) $S \frown 0$-head$(s) = \emptyset$, and

(iii) $S \frown 0$-tail$(s) = s$.

The theorem is a consequence of (4) and (52).

Let us consider T and α. One can verify that the Polish atoms(T, α) is non empty and antichain-like.

Let us consider U. Let us observe that the Polish-expression layer(T, α, U) is non empty and antichain-like.

One can verify that the Polish-expression layer(T, α, U) yields a Polish-language of T. The Polish operations(T, α) yielding a subset of T is defined by the term

(Def. 24) $\{t$, where t is an element of $T : \alpha(t) \neq 0\}$.

Let us consider n. Let us note that the Polish-expression hierarchy(T, α, n) is antichain-like and non empty.

One can check that the Polish-expression hierarchy(T, α, n) yields a Polish-language of T. The functor Polish-WFF-set(T, α) yielding a Polish-language of T is defined by the term

(Def. 25) the Polish-expression set(T, α).

A Polish WFF of T and α is an element of Polish-WFF-set(T, α). Let t be an element of T. The Polish operation(T, α, t) yielding a function from Polish-WFF-set$(T, \alpha)^{\frown} \alpha(t)$ into Polish-WFF-set(T, α) is defined by the term

(Def. 26) the Polish operation(T, α, $\alpha(t)$, t).

Assume $\alpha(t) = 1$. The functor Polish-unOp(T, α, t) yielding a unary operation on Polish-WFF-set(T, α) is defined by the term

(Def. 27) the Polish operation(T, α, t).

Assume $\alpha(t) = 2$. The functor Polish-binOp(T, α, t) yielding a binary operation on Polish-WFF-set(T, α) is defined by

(Def. 28) for every u and v such that u, $v \in$ Polish-WFF-set(T, α) holds $it(u, v) =$ (the Polish operation(T, α, t))$(u \frown v)$.

In the sequel φ, ψ denote Polish WFFs of T and α.

Let us consider X and Y. Let F be a partial function from X to 2^Y. We say that F is exhaustive if and only if

(Def. 29) for every a such that $a \in Y$ there exists b such that $b \in \operatorname{dom} F$ and $a \in F(b)$.

Let X be a non empty set. Observe that there exists a finite sequence of elements of 2^X which is non exhaustive and disjoint valued.

Now we state the proposition:

(59) Let us consider a partial function F from X to 2^Y. Then F is not exhaustive if and only if there exists a such that $a \in Y$ and for every b such that $b \in \operatorname{dom} F$ holds $a \notin F(b)$.

Let us consider T. Let B be a non exhaustive, disjoint valued finite sequence of elements of 2^T. The Polish arity from list B yielding a Polish arity-function of T is defined by the term

(Def. 30) the arity from list B.

One can check that there exists an antichain-like, finite sequence-membered set which has non empty elements and there exists a Polish-language which is non trivial and every antichain-like, finite sequence-membered set which is non trivial has also non empty elements.

Let us consider S, n, and m. Let p be an element of $S \frown (n+1+m)$. The functor $\operatorname{decomp}(S,n,m,p)$ yielding an element of S is defined by the term

(Def. 31) $S\text{-head}(S \frown n\text{-tail}(p))$.

Let p be an element of $S \frown n$. The functor $\operatorname{decomp}(S,n,p)$ yielding a finite sequence of elements of S is defined by

(Def. 32) $\operatorname{dom} it = \operatorname{Seg} n$ and for every m such that $m \in \operatorname{Seg} n$ there exists k such that $m = k+1$ and $it(m) = S\text{-head}(S \frown k\text{-tail}(p))$.

Now we state the propositions:

(60) Let us consider an element s of $S \frown n$, and an element t of $T \frown n$. If $S \subseteq T$ and $s = t$, then $\operatorname{decomp}(S,n,s) = \operatorname{decomp}(T,n,t)$.
PROOF: Set $p = \operatorname{decomp}(S,n,s)$. Set $q = \operatorname{decomp}(T,n,t)$. For every a such that $a \in \operatorname{Seg} n$ holds $p(a) = q(a)$ by (17), [4, (1)], (57), (55). □

(61) Let us consider an element q of $S \frown 0$. Then $\operatorname{decomp}(S,0,q) = \emptyset$.

(62) Let us consider an element q of $S \frown n$. Then $\operatorname{len} \operatorname{decomp}(S,n,q) = n$.

(63) Let us consider an element q of $S \frown 1$. Then $\operatorname{decomp}(S,1,q) = \langle q \rangle$. The theorem is a consequence of (7), (58), (53), and (62).

(64) Let us consider elements p, q of S, and an element r of $S \frown 2$. Suppose $r = p \frown q$. Then $\operatorname{decomp}(S,2,r) = \langle p, q \rangle$. The theorem is a consequence of (58), (52), (7), (53), and (62).

(65) Polish-WFF-set(T, α) is T-headed. The theorem is a consequence of (28), (23), and (21).

(66) The Polish-expression hierarchy$(T,\ \alpha,\ n)$ is T-headed. The theorem is a consequence of (26) and (65).

Let us consider T, α, and φ. The functor Polish-WFF-head φ yielding an element of T is defined by the term

(Def. 33) $T\text{-head}(\varphi)$.

Let us consider n. Let φ be an element of the Polish-expression hierarchy$(T,\ \alpha,\ n)$. The functor Polish-WFF-head φ yielding an element of T is defined by the term

(Def. 34) $T\text{-head}(\varphi)$.

Let us consider φ. The Polish arity φ yielding a natural number is defined by the term

(Def. 35) $\alpha(\text{Polish-WFF-head}\,\varphi)$.

Let us consider n. Let φ be an element of the Polish-expression hierarchy$(T,\ \alpha,\ n)$. The Polish arity φ yielding a natural number is defined by the term

(Def. 36) $\alpha(\text{Polish-WFF-head}\,\varphi)$.

Now we state the propositions:

(67) T-tail$(\varphi) \in$ Polish-WFF-set$(T, \alpha) ^\frown$ (the Polish arity φ). The theorem is a consequence of (32) and (52).

(68) Let us consider an element φ of the Polish-expression hierarchy(T, α, $n+1$). Then T-tail$(\varphi) \in$ (the Polish-expression hierarchy(T, α, n)) $^\frown$ (the Polish arity φ). The theorem is a consequence of (23) and (52).

Let us consider T, α, and φ. The functor (T, α)-tail φ yielding an element of Polish-WFF-set$(T, \alpha) ^\frown$ (the Polish arity φ) is defined by the term

(Def. 37) T-tail(φ).

Now we state the proposition:

(69) If T-head$(\varphi) \in$ the Polish atoms(T, α), then $\varphi = T$-head(φ). The theorem is a consequence of (67) and (6).

Let us consider T, α, and n. Let φ be an element of the Polish-expression hierarchy$(T, \alpha, n+1)$. The functor (T, α)-tail φ yielding an element of (the Polish-expression hierarchy(T, α, n)) $^\frown$ (the Polish arity φ) is defined by the term

(Def. 38) T-tail(φ).

Let us consider φ. The functor Polish-WFF-args φ yielding a finite sequence of elements of Polish-WFF-set(T, α) is defined by the term

(Def. 39) decomp(Polish-WFF-set(T, α), the Polish arity φ, (T, α)-tail φ).

Let us consider n. Let φ be an element of the Polish-expression hierarchy(T, α, $n+1$). The functor Polish-WFF-args φ yielding a finite sequence of elements of the Polish-expression hierarchy(T, α, n) is defined by the term

(Def. 40) decomp(the Polish-expression hierarchy(T, α, n), the Polish arity φ, (T, α)-tail φ).

Now we state the propositions:

(70) Let us consider an element t of T, and u.
Suppose $u \in$ Polish-WFF-set$(T, \alpha) ^\frown \alpha(t)$.
Then T-tail((the Polish operation(T, α, t))(u)) $= u$. The theorem is a consequence of (52).

(71) Suppose $\varphi \in$ the Polish-expression hierarchy(T, α, $n+1$).
Then rng Polish-WFF-args $\varphi \subseteq$ the Polish-expression hierarchy(T, α, n). The theorem is a consequence of (60) and (26).

(72) Let us consider a finite sequence p, a function f from Y into D, and a function g from Z into D. Suppose rng $p \subseteq Y$ and rng $p \subseteq Z$ and for every a such that $a \in$ rng p holds $f(a) = g(a)$. Then $f \cdot p = g \cdot p$.
PROOF: Reconsider $p_1 = p$ as a finite sequence of elements of Y. Reconsider $q = f \cdot p_1$ as a finite sequence. Reconsider $p_2 = p$ as a finite sequence of elements of Z. Reconsider $r = g \cdot p_2$ as a finite sequence. $q = r$ by [6, (33)], [4, (1)], [7, (13), (3)]. $\square$

Let us consider T, α, and D. The Polish recursion-domain(α, D) yielding a subset of $T \times D^*$ is defined by the term

(Def. 41) $\{\langle t, p\rangle$, where t is an element of T, p is a finite sequence of elements of $D : \operatorname{len} p = \alpha(t)\}$.

A Polish recursion-function of α and D is a function from the Polish recursion-domain(α, D) into D. From now on f denotes a Polish recursion-function of α and D and γ, γ_1, γ_2 denote functions from Polish-WFF-set(T, α) into D.

Let us consider T, α, D, f, and γ. We say that γ is f-recursive if and only if

(Def. 42) for every φ, $\gamma(\varphi) = f(\langle T\text{-head}(\varphi), \gamma \cdot \text{Polish-WFF-args}\, \varphi\rangle)$.

Now we state the proposition:

(73) If γ_1 is f-recursive and γ_2 is f-recursive, then $\gamma_1 = \gamma_2$. The theorem is a consequence of (36), (17), (33), (52), (60), (72), and (37).

From now on L denotes a non trivial Polish-language, β denotes a Polish arity-function of L, g denotes a Polish recursion-function of β and D, J, J_1 denote subsets of Polish-WFF-set(L, β), H denotes a function from J into D, H_1 denotes a function from J_1 into D.

Let us consider L, β, D, g, J, and H. We say that H is g-recursive if and only if

(Def. 43) for every Polish WFF φ of L and β such that $\varphi \in J$ and rng Polish-WFF-args $\varphi \subseteq J$ holds
$H(\varphi) = g(\langle L\text{-head}(\varphi), H \cdot \text{Polish-WFF-args}\, \varphi\rangle)$.

Now we state the propositions:

(74) There exists J and there exists H such that $J =$ the Polish-expression hierarchy(L, β, n) and H is g-recursive.
PROOF: Define $\mathcal{X}$[natural number] $\equiv$ there exists J and there exists H such that $J =$ the Polish-expression hierarchy(L, β, $\$_1$) and H is g-recursive. For every n, $\mathcal{X}[n]$ from [2, Sch. 2]. $\square$

(75) There exists a function γ from Polish-WFF-set(L, β) into D such that γ is g-recursive.
PROOF: Set $W =$ Polish-WFF-set(L, β). Define $\mathcal{X}$[object, object] $\equiv$ there exists n and there exists J_1 and there exists H_1 such that $J_1 =$ the Polish-expression hierarchy(L, β, n) and H_1 is g-recursive and $\$_1 \in J_1$ and $\$_2 = H_1(\$_1)$. For every a such that $a \in W$ there exists b such that $b \in D$ and $\mathcal{X}[a, b]$ by (28), (74), [8, (5)]. Consider γ being a function from W into D such that for every a such that $a \in W$ holds $\mathcal{X}[a, \gamma(a)]$ from [8, Sch. 1]. $\square$

(76) Let us consider an element t of L. Then the Polish operation(L, β, t) is one-to-one.

PROOF: Set $f =$ the Polish operation(L, β, t). For every a and b such that $a, b \in \operatorname{dom} f$ and $f(a) = f(b)$ holds $a = b$ by [4, (33)]. $\square$

(77) Let us consider elements t, u of L. Suppose rng(the Polish operation(L, β, t)) meets rng(the Polish operation(L, β, u)). Then $t = u$. The theorem is a consequence of (43).

(78) Let us consider an element t of L, and a. Suppose $a \in$ dom(the Polish operation(L, β, t)). Then there exists p such that

(i) $p =$ (the Polish operation(L, β, t))(a), and

(ii) L-head$(p) = t$.

The theorem is a consequence of (52).

Let us consider L, β, an element t of L, and a Polish WFF φ of L and β. Now we state the proposition:

(79) Polish-WFF-head $\varphi = t$ if and only if there exists an element u of Polish-WFF-set$(L, \beta)^{\frown}\beta(t)$ such that $\varphi =$ (the Polish operation(L, β, t))(u). The theorem is a consequence of (52).

Let us assume that $\beta(t) = 1$. Now we state the propositions:

(80) If Polish-WFF-head $\varphi = t$, then there exists a Polish WFF ψ of L and β such that $\varphi =$ (Polish-unOp(L, β, t))(ψ). The theorem is a consequence of (79) and (7).

(81) (i) Polish-WFF-head((Polish-unOp(L, β, t))(φ)) $= t$, and

(ii) Polish-WFF-args((Polish-unOp(L, β, t))(φ)) $= \langle\varphi\rangle$.

The theorem is a consequence of (7), (79), (70), and (63).

Now we state the proposition:

(82) Suppose $\beta(t) = 2$. Then suppose Polish-WFF-head $\varphi = t$. Then there exist Polish WFFs ψ, H of L and β such that $\varphi =$ (Polish-binOp(L, β, t)) (ψ, H). The theorem is a consequence of (79), (6), and (7).

Now we state the propositions:

(83) Let us consider an element t of L. Suppose $\beta(t) = 2$. Let us consider Polish WFFs φ, ψ of L and β. Then

(i) Polish-WFF-head(Polish-binOp(L, β, t))$(\varphi, \psi) = t$, and

(ii) Polish-WFF-args(Polish-binOp(L, β, t))$(\varphi, \psi) = \langle\varphi, \psi\rangle$.

The theorem is a consequence of (7), (11), (79), (64), and (70).

(84) Let us consider a Polish WFF φ of L and β. Then $\varphi \in$ the Polish atoms(L, β) if and only if the Polish arity $\varphi = 0$. The theorem is a consequence of (53), (67), and (6).

(85) Let us consider a function γ from Polish-WFF-set(L,β) into D, an element t of L, and a Polish WFF φ of L and β. Suppose γ is g-recursive and $\beta(t)=1$. Then $\gamma((\text{Polish-unOp}(L,\beta,t))(\varphi)) = g(t,\langle\gamma(\varphi)\rangle)$. The theorem is a consequence of (81).

Let us consider S. Let p be a finite sequence of elements of S. The functor Flat(p) yielding an element of $S \frown \operatorname{len} p$ is defined by

(Def. 44) $\operatorname{decomp}(S,\operatorname{len} p, it) = p$.

Let us consider L and β.

A substitution of L and β is a partial function from the Polish atoms(L,β) to Polish-WFF-set(L,β). Let s be a substitution of L and β. The functor Subst s yielding a Polish recursion-function of β and Polish-WFF-set(L,β) is defined by

(Def. 45) for every element t of L and for every finite sequence p of elements of Polish-WFF-set(L,β) such that $\operatorname{len} p = \beta(t)$ holds if $t \in \operatorname{dom} s$, then $it(t,p) = s(t)$ and if $t \notin \operatorname{dom} s$, then $it(t,p) = t \frown \operatorname{Flat}(p)$.

Let φ be a Polish WFF of L and β. The functor $s[\varphi]$ yielding a Polish WFF of L and β is defined by

(Def. 46) there exists a function H from Polish-WFF-set(L,β) into Polish-WFF-set(L,β) such that H is (Subst s)-recursive and $it = H(\varphi)$.

Now we state the proposition:

(86) Let us consider a substitution s of L and β, and a Polish WFF φ of L and β. If $s=\emptyset$, then $s[\varphi]=\varphi$.
PROOF: Set $W=$ Polish-WFF-set(L,β). Set $g=$ Subst s. Set $\gamma = \mathrm{id}_W$. γ is g-recursive by (62), [6, (32), (33)], [7, (3), (17), (13)]. □

References

[1] Grzegorz Bancerek. Cardinal numbers. *Formalized Mathematics*, 1(**2**):377–382, 1990.
[2] Grzegorz Bancerek. The fundamental properties of natural numbers. *Formalized Mathematics*, 1(**1**):41–46, 1990.
[3] Grzegorz Bancerek. The ordinal numbers. *Formalized Mathematics*, 1(**1**):91–96, 1990.
[4] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Formalized Mathematics*, 1(**1**):107–114, 1990.
[5] Czesław Byliński. Binary operations. *Formalized Mathematics*, 1(**1**):175–180, 1990.
[6] Czesław Byliński. Finite sequences and tuples of elements of a non-empty sets. *Formalized Mathematics*, 1(**3**):529–536, 1990.
[7] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**): 55–65, 1990.
[8] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.
[9] Czesław Byliński. Partial functions. *Formalized Mathematics*, 1(**2**):357–367, 1990.
[10] Czesław Byliński. Some basic properties of sets. *Formalized Mathematics*, 1(**1**):47–53, 1990.

[11] Agata Darmochwał. Finite sets. *Formalized Mathematics*, 1(**1**):165–167, 1990.

[12] Joanna Golińska-Pilarek and Taneli Huuskonen. Logic of descriptions. A new approach to the foundations of mathematics and science. *Studies in Logic, Grammar and Rhetoric*, 40(27), 2012.

[13] Joanna Golińska-Pilarek and Taneli Huuskonen. Grzegorczyk's non-Fregean logics. In Rafał Urbaniak and Gillman Payette, editors, *Applications of Formal Philosophy: The Road Less Travelled*, Logic, Reasoning and Argumentation. Springer, 2015.

[14] Andrzej Grzegorczyk. Filozofia logiki i formalna LOGIKA NIESYMPLIFIKACYJNA. *Zagadnienia Naukoznawstwa*, XLVII(4), 2012. In Polish.

[15] Jan Łukasiewicz. Uwagi o aksjomacie Nicoda i 'dedukcji uogólniającej'. In *Księga pamiątkowa Polskiego Towarzystwa Filozoficznego*, Lwów, 1931. In Polish.

[16] Andrzej Nędzusiak. Probability. *Formalized Mathematics*, 1(**4**):745–749, 1990.

[17] Beata Padlewska. Families of sets. *Formalized Mathematics*, 1(**1**):147–152, 1990.

[18] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(**1**):67–71, 1990.

[19] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1 (**1**):73–83, 1990.

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