

# Polish Notation

Taneli Huuskonen<sup>1</sup>  
Department of Mathematics and Statistics  
University of Helsinki  
Finland

**Summary.** This article is the first in a series formalizing some results in my joint work with Prof. Joanna Golińska-Pilarek ([12] and [13]) concerning a logic proposed by Prof. Andrzej Grzegorzczak ([14]).

We present some *mathematical folklore* about representing formulas in “Polish notation”, that is, with operators of fixed arity prepended to their arguments. This notation, which was published by Jan Łukasiewicz in [15], eliminates the need for parentheses and is generally well suited for rigorous reasoning about syntactic properties of formulas.

MSC: 68R15 03B35

Keywords: Polish notation; syntax; well-formed formula

MML identifier: POLNOT\_1, version: 8.1.04 5.32.1240

The notation and terminology used in this paper have been introduced in the following articles: [5], [1], [4], [11], [7], [8], [3], [9], [16], [19], [17], [18], and [10].

## 1. PRELIMINARIES

From now on  $k, m, n$  denote natural numbers,  $a, b, c, c_1, c_2$  denote objects,  $x, y, z, X, Y, Z$  denote sets,  $D$  denotes a non empty set,  $p, q, r, s, t, u, v$  denote finite sequences,  $P, Q, R, P_1, P_2, Q_1, Q_2, R_1, R_2$  denote finite sequence-membered sets, and  $S, T$  denote non empty, finite sequence-membered sets.

Let  $D$  be a non empty set and  $P, Q$  be subsets of  $D^*$ . The functor  $\frown(D, P, Q)$  yielding a subset of  $D^*$  is defined by the term

---

<sup>1</sup>Work supported by Polish National Science Center (NCN) grant “Logic of language experience” nr 2011/03/B/HS1/04580.

(Def. 1)  $\{p \frown q, \text{ where } p \text{ is a finite sequence of elements of } D, q \text{ is a finite sequence of elements of } D : p \in P \text{ and } q \in Q\}$ .

Let us consider  $P$  and  $Q$ . The functor  $P \frown Q$  yielding a finite sequence-membered set is defined by

(Def. 2) for every  $a, a \in it$  iff there exists  $p$  and there exists  $q$  such that  $a = p \frown q$  and  $p \in P$  and  $q \in Q$ .

Let  $\beta$  be an empty set. One can check that  $\beta \frown P$  is empty and  $P \frown \beta$  is empty.

Let us consider  $S$  and  $T$ . One can check that  $S \frown T$  is non empty.

Now we state the propositions:

- (1) If  $p \frown q = r \frown s$ , then there exists  $t$  such that  $p \frown t = r$  or  $p = r \frown t$ .
- (2)  $(P \frown Q) \frown R = P \frown (Q \frown R)$ .

PROOF: For every  $a, a \in (P \frown Q) \frown R$  iff  $a \in P \frown (Q \frown R)$  by [4, (32)].  $\square$

Note that  $\{\emptyset\}$  is non empty and finite sequence-membered.

- (3) (i)  $P \frown \{\emptyset\} = P$ , and
- (ii)  $\{\emptyset\} \frown P = P$ .

PROOF: For every  $a, a \in P \frown \{\emptyset\}$  iff  $a \in P$  by [4, (34)]. For every  $a, a \in \{\emptyset\} \frown P$  iff  $a \in P$  by [4, (34)].  $\square$

Let us consider  $P$ . The functor  $P \frown \frown$  yielding a function is defined by

(Def. 3)  $\text{dom } it = \mathbb{N}$  and  $it(0) = \{\emptyset\}$  and for every  $n$ , there exists  $Q$  such that  $Q = it(n)$  and  $it(n + 1) = Q \frown P$ .

Let us consider  $n$ . The functor  $P \frown n$  yielding a finite sequence-membered set is defined by the term

(Def. 4)  $(P \frown \frown)(n)$ .

Now we state the proposition:

- (4)  $\emptyset \in P \frown 0$ .

Let us consider  $P$ . Let  $n$  be a zero natural number. Note that  $P \frown n$  is non empty.

Let  $\beta$  be an empty set and  $n$  be a non zero natural number. One can verify that  $\beta \frown n$  is empty.

Let us consider  $P$ . The functor  $P^*$  yielding a non empty, finite sequence-membered set is defined by the term

(Def. 5)  $\bigcup$  the set of all  $P \frown n$  where  $n$  is a natural number.

- (5)  $a \in P^*$  if and only if there exists  $n$  such that  $a \in P \frown n$ .

Let us consider  $P$ .

- (6) (i)  $P \frown 0 = \{\emptyset\}$ , and
- (ii) for every  $n, P \frown (n + 1) = (P \frown n) \frown P$ .

(7)  $P \frown 1 = P$ . The theorem is a consequence of (6) and (3).

(8)  $P \frown n \subseteq P^*$ .

(9) (i)  $\emptyset \in P^*$ , and

(ii)  $P \subseteq P^*$ .

The theorem is a consequence of (4), (5), and (7).

(10)  $P \frown (m + n) = (P \frown m) \frown (P \frown n)$ .

PROOF: Define  $\mathcal{X}[\text{natural number}] \equiv P \frown (m + \$_1) = (P \frown m) \frown (P \frown \$_1)$ .  $\mathcal{X}[0]$ . For every  $k$  such that  $\mathcal{X}[k]$  holds  $\mathcal{X}[k + 1]$ . For every  $k$ ,  $\mathcal{X}[k]$  from [2, Sch. 2].  $\square$

(11) If  $p \in P \frown m$  and  $q \in P \frown n$ , then  $p \frown q \in P \frown (m + n)$ . The theorem is a consequence of (10).

(12) If  $p, q \in P^*$ , then  $p \frown q \in P^*$ . The theorem is a consequence of (5) and (11).

(13) If  $P \subseteq R^*$  and  $Q \subseteq R^*$ , then  $P \frown Q \subseteq R^*$ . The theorem is a consequence of (12).

(14) If  $Q \subseteq P^*$ , then  $Q \frown n \subseteq P^*$ .

PROOF: Define  $\mathcal{X}[\text{natural number}] \equiv Q \frown \$_1 \subseteq P^*$ .  $\mathcal{X}[0]$ . For every  $k$  such that  $\mathcal{X}[k]$  holds  $\mathcal{X}[k + 1]$ . For every  $k$ ,  $\mathcal{X}[k]$  from [2, Sch. 2].  $\square$

(15) If  $Q \subseteq P^*$ , then  $Q^* \subseteq P^*$ . The theorem is a consequence of (5) and (14).

(16) If  $P_1 \subseteq P_2$  and  $Q_1 \subseteq Q_2$ , then  $P_1 \frown Q_1 \subseteq P_2 \frown Q_2$ .

(17) If  $P \subseteq Q$ , then for every  $n$ ,  $P \frown n \subseteq Q \frown n$ .

PROOF: Define  $\mathcal{S}[\text{natural number}] \equiv P \frown \$_1 \subseteq Q \frown \$_1$ .  $P \frown 0 = \{\emptyset\}$ . For every  $n$  such that  $\mathcal{S}[n]$  holds  $\mathcal{S}[n + 1]$ . For every  $n$ ,  $\mathcal{S}[n]$  from [2, Sch. 2].  $\square$

Let us consider  $S$  and  $n$ . Let us observe that  $S \frown n$  is non empty and finite sequence-membered.

## 2. THE LANGUAGE

In the sequel  $\alpha$  denotes a function from  $P$  into  $\mathbb{N}$  and  $U, V, W$  denote subsets of  $P^*$ .

Let us consider  $P, \alpha$ , and  $U$ . The Polish-expression layer( $P, \alpha, U$ ) yielding a subset of  $P^*$  is defined by

(Def. 6) for every  $a$ ,  $a \in it$  iff  $a \in P^*$  and there exists  $p$  and there exists  $q$  and there exists  $n$  such that  $a = p \frown q$  and  $p \in P$  and  $n = \alpha(p)$  and  $q \in U \frown n$ .

Now we state the proposition:

(18) Suppose  $p \in P$  and  $n = \alpha(p)$  and  $q \in U \frown n$ . Then  $p \frown q \in$  the Polish-expression layer( $P, \alpha, U$ ). The theorem is a consequence of (14), (9), and (12).

Let us consider  $P$  and  $\alpha$ . The Polish atoms( $P, \alpha$ ) yielding a subset of  $P^*$  is defined by

(Def. 7) for every  $a, a \in it$  iff  $a \in P$  and  $\alpha(a) = 0$ .

The Polish operations( $P, \alpha$ ) yielding a subset of  $P$  is defined by the term

(Def. 8)  $\{t, \text{ where } t \text{ is an element of } P^* : t \in P \text{ and } \alpha(t) \neq 0\}$ .

Now we state the propositions:

(19) The Polish atoms( $P, \alpha$ )  $\subseteq$  the Polish-expression layer( $P, \alpha, U$ ). The theorem is a consequence of (4) and (18).

(20) Suppose  $U \subseteq V$ . Then the Polish-expression layer( $P, \alpha, U$ )  $\subseteq$  the Polish-expression layer( $P, \alpha, V$ ). The theorem is a consequence of (17).

(21) Suppose  $u \in$  the Polish-expression layer( $P, \alpha, U$ ). Then there exists  $p$  and there exists  $q$  such that  $p \in P$  and  $u = p \wedge q$ .

Let us consider  $P$  and  $\alpha$ . The Polish-expression hierarchy( $P, \alpha$ ) yielding a function is defined by

(Def. 9)  $\text{dom } it = \mathbb{N}$  and  $it(0) =$  the Polish atoms( $P, \alpha$ ) and for every  $n$ , there exists  $U$  such that  $U = it(n)$  and  $it(n + 1) =$  the Polish-expression layer( $P, \alpha, U$ ).

Let us consider  $n$ . The Polish-expression hierarchy( $P, \alpha, n$ ) yielding a subset of  $P^*$  is defined by the term

(Def. 10) (the Polish-expression hierarchy( $P, \alpha$ ))( $n$ ).

Now we state the proposition:

(22) The Polish-expression hierarchy( $P, \alpha, 0$ ) = the Polish atoms( $P, \alpha$ ).

Let us consider  $P, \alpha$ , and  $n$ . Now we state the propositions:

(23) The Polish-expression hierarchy( $P, \alpha, n + 1$ ) = the Polish-expression layer( $P, \alpha, \text{ the Polish-expression hierarchy}(P, \alpha, n)$ ).

(24) The Polish-expression hierarchy( $P, \alpha, n$ )  $\subseteq$  the Polish-expression hierarchy( $P, \alpha, n + 1$ ).

PROOF: Define  $\mathcal{S}[\text{natural number}] \equiv$  the Polish-expression hierarchy( $P, \alpha, \$1$ )  $\subseteq$  the Polish-expression hierarchy( $P, \alpha, \$1 + 1$ ).  $\mathcal{S}[0]$ . For every  $k$  such that  $\mathcal{S}[k]$  holds  $\mathcal{S}[k + 1]$ . For every  $k, \mathcal{S}[k]$  from [2, Sch. 2].  $\square$

Now we state the proposition:

(25) The Polish-expression hierarchy( $P, \alpha, n$ )  $\subseteq$  the Polish-expression hierarchy( $P, \alpha, n + m$ ).

PROOF: Define  $\mathcal{S}[\text{natural number}] \equiv$  the Polish-expression hierarchy( $P, \alpha, n$ )  $\subseteq$  the Polish-expression hierarchy( $P, \alpha, n + \$1$ ). For every  $k$  such that  $\mathcal{S}[k]$  holds  $\mathcal{S}[k + 1]$ . For every  $k, \mathcal{S}[k]$  from [2, Sch. 2].  $\square$

Let us consider  $P$  and  $\alpha$ . The Polish-expression set( $P, \alpha$ ) yielding a subset of  $P^*$  is defined by the term

(Def. 11)  $\cup$  the set of all the Polish-expression hierarchy( $P, \alpha, n$ ) where  $n$  is a natural number.

Now we state the propositions:

(26) The Polish-expression hierarchy( $P, \alpha, n$ )  $\subseteq$  the Polish-expression set( $P, \alpha$ ).

(27) Suppose  $q \in$  (the Polish-expression set( $P, \alpha$ ))  $\cap n$ . Then there exists  $m$  such that  $q \in$  (the Polish-expression hierarchy( $P, \alpha, m$ ))  $\cap n$ .

PROOF: Define  $\mathcal{S}$ [natural number]  $\equiv$  for every  $q$  such that  $q \in$  (the Polish-expression set( $P, \alpha$ ))  $\cap \mathbb{N}_1$  there exists  $m$  such that  $q \in$  (the Polish-expression hierarchy( $P, \alpha, m$ ))  $\cap \mathbb{N}_1$ .  $\mathcal{S}[0]$ . For every  $k$  such that  $\mathcal{S}[k]$  holds  $\mathcal{S}[k+1]$ . For every  $k$ ,  $\mathcal{S}[k]$  from [2, Sch. 2].  $\square$

(28) Suppose  $a \in$  the Polish-expression set( $P, \alpha$ ). Then there exists  $n$  such that  $a \in$  the Polish-expression hierarchy( $P, \alpha, n+1$ ). The theorem is a consequence of (24).

Let us consider  $P$  and  $\alpha$ .

A Polish expression of  $P$  and  $\alpha$  is an element of the Polish-expression set( $P, \alpha$ ). Let us consider  $n$  and  $t$ . Assume  $t \in P$ . The Polish operation( $P, \alpha, n, t$ ) yielding a function from (the Polish-expression set( $P, \alpha$ ))  $\cap n$  into  $P^*$  is defined by

(Def. 12) for every  $q$  such that  $q \in \text{dom } it$  holds  $it(q) = t \cap q$ .

Let us consider  $X$  and  $Y$ . Let  $F$  be a partial function from  $X$  to  $2^Y$ . One can check that  $F$  is disjoint valued if and only if the condition (Def. 13) is satisfied.

(Def. 13) for every  $a$  and  $b$  such that  $a, b \in \text{dom } F$  and  $a \neq b$  holds  $F(a)$  misses  $F(b)$ .

Let  $X$  be a set. One can check that there exists a finite sequence of elements of  $2^X$  which is disjoint valued.

Now we state the proposition:

(29) Let us consider a set  $X$ , a disjoint valued finite sequence  $B$  of elements of  $2^X$ ,  $a, b$ , and  $c$ . If  $a \in B(b)$  and  $a \in B(c)$ , then  $b = c$  and  $b \in \text{dom } B$ .

Let us consider  $X$ . Let  $B$  be a disjoint valued finite sequence of elements of  $2^X$ . The arity from list  $B$  yielding a function from  $X$  into  $\mathbb{N}$  is defined by

(Def. 14) for every  $a$  such that  $a \in X$  holds there exists  $n$  such that  $a \in B(n)$  and  $a \in B(it(a))$  or there exists no  $n$  such that  $a \in B(n)$  and  $it(a) = 0$ .

Now we state the propositions:

(30) Let us consider a disjoint valued finite sequence  $B$  of elements of  $2^X$ , and  $a$ . Suppose  $a \in X$ . Then (the arity from list  $B$ )( $a$ )  $\neq 0$  if and only if

there exists  $n$  such that  $a \in B(n)$ . The theorem is a consequence of (29).

(31) Let us consider a disjoint valued finite sequence  $B$  of elements of  $2^X$ ,  $a$ , and  $n$ . Suppose  $a \in B(n)$ . Then (the arity from list  $B$ )( $a$ ) =  $n$ . The theorem is a consequence of (29).

(32) Suppose  $r \in$  the Polish-expression set( $P, \alpha$ ). Then there exists  $n$  and there exists  $p$  and there exists  $q$  such that  $p \in P$  and  $n = \alpha(p)$  and  $q \in$  (the Polish-expression set( $P, \alpha$ ))  $\hat{\ } n$  and  $r = p \hat{\ } q$ . The theorem is a consequence of (28), (23), (26), and (17).

Let us consider  $P, \alpha$ , and  $Q$ . We say that  $Q$  is  $\alpha$ -closed if and only if

(Def. 15) for every  $p, n$ , and  $q$  such that  $p \in P$  and  $n = \alpha(p)$  and  $q \in Q \hat{\ } n$  holds  $p \hat{\ } q \in Q$ .

Now we state the propositions:

(33) The Polish-expression set( $P, \alpha$ ) is  $\alpha$ -closed. The theorem is a consequence of (27), (18), (23), and (26).

(34) If  $Q$  is  $\alpha$ -closed, then the Polish atoms( $P, \alpha$ )  $\subseteq Q$ . The theorem is a consequence of (4).

(35) If  $Q$  is  $\alpha$ -closed, then the Polish-expression hierarchy( $P, \alpha, n$ )  $\subseteq Q$ .

PROOF: Define  $\mathcal{X}$ [natural number]  $\equiv$  the Polish-expression hierarchy( $P, \alpha, \mathbb{S}_1$ )  $\subseteq Q, \mathcal{X}[0]$ . For every  $k$  such that  $\mathcal{X}[k]$  holds  $\mathcal{X}[k + 1]$ . For every  $k, \mathcal{X}[k]$  from [2, Sch. 2].  $\square$

(36) The Polish atoms( $P, \alpha$ )  $\subseteq$  the Polish-expression set( $P, \alpha$ ). The theorem is a consequence of (33) and (34).

(37) If  $Q$  is  $\alpha$ -closed, then the Polish-expression set( $P, \alpha$ )  $\subseteq Q$ . The theorem is a consequence of (28) and (35).

(38) Suppose  $r \in$  the Polish-expression set( $P, \alpha$ ). Then there exists  $n$  and there exists  $t$  and there exists  $q$  such that  $t \in P$  and  $n = \alpha(t)$  and  $r =$  (the Polish operation( $P, \alpha, n, t$ ))( $q$ ). The theorem is a consequence of (28), (23), (26), and (17).

(39) Suppose  $p \in P$  and  $n = \alpha(p)$  and  $q \in$  (the Polish-expression set( $P, \alpha$ ))  $\hat{\ } n$ . Then (the Polish operation( $P, \alpha, n, p$ ))( $q$ )  $\in$  the Polish-expression set( $P, \alpha$ ). The theorem is a consequence of (33).

The scheme *AInd* deals with a finite sequence-membered set  $\mathcal{P}$  and a function  $\alpha$  from  $\mathcal{P}$  into  $\mathbb{N}$  and a unary predicate  $\mathcal{X}$  and states that

(Sch. 1) For every  $a$  such that  $a \in$  the Polish-expression set( $\mathcal{P}, \alpha$ ) holds  $\mathcal{X}[a]$  provided

- for every  $p, q$ , and  $n$  such that  $p \in \mathcal{P}$  and  $n = \alpha(p)$  and  $q \in$  (the Polish-expression set( $\mathcal{P}, \alpha$ ))  $\hat{\ } n$  holds  $\mathcal{X}[p \hat{\ } q]$ .

## 3. PARSING

In the sequel  $k, l, m, n, i, j$  denote natural numbers,  $a, b, c, c_1, c_2$  denote objects,  $x, y, z, X, Y, Z$  denote sets,  $D, D_1, D_2$  denote non empty sets,  $p, q, r, s, t, u, v$  denote finite sequences, and  $P, Q, R$  denote finite sequence-membered sets.

Let us consider  $P$ . We say that  $P$  is antichain-like if and only if

(Def. 16) for every  $p$  and  $q$  such that  $p, p \wedge q \in P$  holds  $q = \emptyset$ .

Now we state the propositions:

(40)  $P$  is antichain-like if and only if for every  $p$  and  $q$  such that  $p, p \wedge q \in P$  holds  $p = p \wedge q$ .

PROOF: If  $P$  is antichain-like, then for every  $p$  and  $q$  such that  $p, p \wedge q \in P$  holds  $p = p \wedge q$  by [4, (34)].  $\square$

(41) If  $P \subseteq Q$  and  $Q$  is antichain-like, then  $P$  is antichain-like.

Note that every finite sequence-membered set which is trivial is also antichain-like.

Now we state the proposition:

(42) If  $P = \{a\}$ , then  $P$  is antichain-like.

Note that there exists a non empty, finite sequence-membered set which is antichain-like and every finite sequence-membered set which is empty is also antichain-like.

An antichain is an antichain-like, finite sequence-membered set. In the sequel  $B, C$  denote antichains.

Let us consider  $B$ . One can verify that every subset of  $B$  is antichain-like and finite sequence-membered.

A Polish-language is a non empty antichain. From now on  $S, T$  denote Polish-languages.

Let  $D$  be a non empty set and  $\psi$  be a subset of  $D^*$ . Note that  $\psi$  is antichain-like if and only if the condition (Def. 17) is satisfied.

(Def. 17) for every finite sequence  $g$  of elements of  $D$  and for every finite sequence  $h$  of elements of  $D$  such that  $g, g \wedge h \in \psi$  holds  $h = \varepsilon_D$ .

Now we state the proposition:

(43) If  $p \wedge q = r \wedge s$  and  $p, r \in B$ , then  $p = r$  and  $q = s$ . The theorem is a consequence of (1) and (40).

Let us consider  $B$  and  $C$ . Note that  $B \wedge C$  is antichain-like.

Now we state the propositions:

(44) If for every  $p$  and  $q$  such that  $p, q \in P$  holds  $\text{dom } p = \text{dom } q$ , then  $P$  is antichain-like.

PROOF: For every  $p$  and  $q$  such that  $p, p \wedge q \in P$  holds  $p = p \wedge q$  by [4, (21)].  $\square$

(45) If for every  $p$  such that  $p \in P$  holds  $\text{dom } p = a$ , then  $P$  is antichain-like. The theorem is a consequence of (44).

(46) If  $\emptyset \in B$ , then  $B = \{\emptyset\}$ .

PROOF: For every  $a$  such that  $a \in B$  holds  $a = \emptyset$  by [4, (34)].  $\square$

Let us consider  $B$  and  $n$ . Note that  $B \wedge n$  is antichain-like.

Let us consider  $T$ . Let us observe that there exists a subset of  $T^*$  which is non empty and antichain-like and  $T \wedge n$  is non empty.

A Polish-language of  $T$  is a non empty, antichain-like subset of  $T^*$ .

A Polish arity-function of  $T$  is a function from  $T$  into  $\mathbb{N}$  and is defined by

(Def. 18) there exists  $a$  such that  $a \in T$  and  $it(a) = 0$ .

One can verify that every Polish-language of  $T$  is non empty, antichain-like, and finite sequence-membered.

In the sequel  $\alpha$  denotes a Polish arity-function of  $T$  and  $U, V, W$  denote Polish-languages of  $T$ .

Let us consider  $T$  and  $\alpha$ . Let  $t$  be an element of  $T$ . Let us observe that the functor  $\alpha(t)$  yields a natural number. Let us consider  $U$ . Note that the Polish-expression  $\text{layer}(T, \alpha, U)$  is defined by

(Def. 19) for every  $a, a \in it$  iff there exists an element  $t$  of  $T$  and there exists an element  $u$  of  $T^*$  such that  $a = t \wedge u$  and  $u \in U \wedge \alpha(t)$ .

Let us consider  $B$  and  $p$ . We say that  $p$  is  $B$ -headed if and only if

(Def. 20) there exists  $q$  and there exists  $r$  such that  $q \in B$  and  $p = q \wedge r$ .

Let us consider  $P$ . We say that  $P$  is  $B$ -headed if and only if

(Def. 21) for every  $p$  such that  $p \in P$  holds  $p$  is  $B$ -headed.

Now we state the propositions:

(47) If  $p$  is  $B$ -headed and  $B \subseteq C$ , then  $p$  is  $C$ -headed.

(48) If  $P$  is  $B$ -headed and  $B \subseteq C$ , then  $P$  is  $C$ -headed.

Let us consider  $B$  and  $P$ . Observe that  $B \wedge P$  is  $B$ -headed.

Now we state the propositions:

(49) If  $p$  is  $(B \wedge C)$ -headed, then  $p$  is  $B$ -headed.

(50)  $B$  is  $B$ -headed. The theorem is a consequence of (3).

Let us consider  $B$ . Let us observe that there exists a finite sequence-membered set which is  $B$ -headed.

Let  $P$  be a  $B$ -headed, finite sequence-membered set. Let us note that every subset of  $P$  is  $B$ -headed.

Let us consider  $S$ . Let us observe that there exists a finite sequence-membered set which is non empty and  $S$ -headed.



Now we state the proposition:

(51)  $S \frown (m + n)$  is  $(S \frown m)$ -headed. The theorem is a consequence of (10).

Let us consider  $S$  and  $p$ . The functor  $S\text{-head}(p)$  yielding a finite sequence is defined by

- (Def. 22) (i)  $it \in S$  and there exists  $r$  such that  $p = it \frown r$ , **if**  $p$  is  $S$ -headed,  
 (ii)  $it = \emptyset$ , **otherwise**.

The functor  $S\text{-tail}(p)$  yielding a finite sequence is defined by

- (Def. 23)  $p = (S\text{-head}(p)) \frown it$ .

Now we state the propositions:

(52) If  $s \in S$ , then  $S\text{-head}(s \frown t) = s$  and  $S\text{-tail}(s \frown t) = t$ .

(53) If  $s \in S$ , then  $S\text{-head}(s) = s$  and  $S\text{-tail}(s) = \emptyset$ . The theorem is a consequence of (52).

Let us consider  $S, T$ , and  $u$ . Now we state the propositions:

(54) If  $u \in S \frown T$ , then  $S\text{-head}(u) \in S$  and  $S\text{-tail}(u) \in T$ . The theorem is a consequence of (52).

(55) If  $S \subseteq T$  and  $u$  is  $S$ -headed, then  $S\text{-head}(u) = T\text{-head}(u)$  and  $S\text{-tail}(u) = T\text{-tail}(u)$ . The theorem is a consequence of (52).

Now we state the propositions:

(56) Suppose  $s$  is  $S$ -headed. Then

- (i)  $s \frown t$  is  $S$ -headed, and  
 (ii)  $S\text{-head}(s \frown t) = S\text{-head}(s)$ , and  
 (iii)  $S\text{-tail}(s \frown t) = (S\text{-tail}(s)) \frown t$ .

The theorem is a consequence of (52).

(57) If  $m + 1 \leq n$  and  $s \in S \frown n$ , then  $s$  is  $(S \frown m)$ -headed and  $S \frown m\text{-tail}(s)$  is  $S$ -headed. The theorem is a consequence of (51), (10), (54), and (7).

(58) (i)  $s$  is  $(S \frown 0)$ -headed, and

- (ii)  $S \frown 0\text{-head}(s) = \emptyset$ , and  
 (iii)  $S \frown 0\text{-tail}(s) = s$ .

The theorem is a consequence of (4) and (52).

Let us consider  $T$  and  $\alpha$ . One can verify that the Polish atoms( $T, \alpha$ ) is non empty and antichain-like.

Let us consider  $U$ . Let us observe that the Polish-expression layer( $T, \alpha, U$ ) is non empty and antichain-like.

One can verify that the Polish-expression layer( $T, \alpha, U$ ) yields a Polish-language of  $T$ . The Polish operations( $T, \alpha$ ) yielding a subset of  $T$  is defined by the term

(Def. 24)  $\{t, \text{ where } t \text{ is an element of } T : \alpha(t) \neq 0\}$ .

Let us consider  $n$ . Let us note that the Polish-expression hierarchy( $T, \alpha, n$ ) is antichain-like and non empty.

One can check that the Polish-expression hierarchy( $T, \alpha, n$ ) yields a Polish-language of  $T$ . The functor Polish-WFF-set( $T, \alpha$ ) yielding a Polish-language of  $T$  is defined by the term

(Def. 25) the Polish-expression set( $T, \alpha$ ).

A Polish WFF of  $T$  and  $\alpha$  is an element of Polish-WFF-set( $T, \alpha$ ). Let  $t$  be an element of  $T$ . The Polish operation( $T, \alpha, t$ ) yielding a function from Polish-WFF-set( $T, \alpha$ )  $\cap$   $\alpha(t)$  into Polish-WFF-set( $T, \alpha$ ) is defined by the term

(Def. 26) the Polish operation( $T, \alpha, \alpha(t), t$ ).

Assume  $\alpha(t) = 1$ . The functor Polish-unOp( $T, \alpha, t$ ) yielding a unary operation on Polish-WFF-set( $T, \alpha$ ) is defined by the term

(Def. 27) the Polish operation( $T, \alpha, t$ ).

Assume  $\alpha(t) = 2$ . The functor Polish-binOp( $T, \alpha, t$ ) yielding a binary operation on Polish-WFF-set( $T, \alpha$ ) is defined by

(Def. 28) for every  $u$  and  $v$  such that  $u, v \in$  Polish-WFF-set( $T, \alpha$ ) holds  $it(u, v) =$  (the Polish operation( $T, \alpha, t$ ))( $u \cap v$ ).

In the sequel  $\varphi, \psi$  denote Polish WFFs of  $T$  and  $\alpha$ .

Let us consider  $X$  and  $Y$ . Let  $F$  be a partial function from  $X$  to  $2^Y$ . We say that  $F$  is exhaustive if and only if

(Def. 29) for every  $a$  such that  $a \in Y$  there exists  $b$  such that  $b \in \text{dom } F$  and  $a \in F(b)$ .

Let  $X$  be a non empty set. Observe that there exists a finite sequence of elements of  $2^X$  which is non exhaustive and disjoint valued.

Now we state the proposition:

(59) Let us consider a partial function  $F$  from  $X$  to  $2^Y$ . Then  $F$  is not exhaustive if and only if there exists  $a$  such that  $a \in Y$  and for every  $b$  such that  $b \in \text{dom } F$  holds  $a \notin F(b)$ .

Let us consider  $T$ . Let  $B$  be a non exhaustive, disjoint valued finite sequence of elements of  $2^T$ . The Polish arity from list  $B$  yielding a Polish arity-function of  $T$  is defined by the term

(Def. 30) the arity from list  $B$ .

One can check that there exists an antichain-like, finite sequence-membered set which has non empty elements and there exists a Polish-language which is non trivial and every antichain-like, finite sequence-membered set which is non trivial has also non empty elements.

Let us consider  $S$ ,  $n$ , and  $m$ . Let  $p$  be an element of  $S \frown (n + 1 + m)$ . The functor  $\text{decomp}(S, n, m, p)$  yielding an element of  $S$  is defined by the term

(Def. 31)  $S\text{-head}(S \frown n\text{-tail}(p))$ .

Let  $p$  be an element of  $S \frown n$ . The functor  $\text{decomp}(S, n, p)$  yielding a finite sequence of elements of  $S$  is defined by

(Def. 32)  $\text{dom } it = \text{Seg } n$  and for every  $m$  such that  $m \in \text{Seg } n$  there exists  $k$  such that  $m = k + 1$  and  $it(m) = S\text{-head}(S \frown k\text{-tail}(p))$ .

Now we state the propositions:

(60) Let us consider an element  $s$  of  $S \frown n$ , and an element  $t$  of  $T \frown n$ . If  $S \subseteq T$  and  $s = t$ , then  $\text{decomp}(S, n, s) = \text{decomp}(T, n, t)$ .

PROOF: Set  $p = \text{decomp}(S, n, s)$ . Set  $q = \text{decomp}(T, n, t)$ . For every  $a$  such that  $a \in \text{Seg } n$  holds  $p(a) = q(a)$  by (17), [4, (1)], (57), (55).  $\square$

(61) Let us consider an element  $q$  of  $S \frown 0$ . Then  $\text{decomp}(S, 0, q) = \emptyset$ .

(62) Let us consider an element  $q$  of  $S \frown n$ . Then  $\text{len } \text{decomp}(S, n, q) = n$ .

(63) Let us consider an element  $q$  of  $S \frown 1$ . Then  $\text{decomp}(S, 1, q) = \langle q \rangle$ . The theorem is a consequence of (7), (58), (53), and (62).

(64) Let us consider elements  $p, q$  of  $S$ , and an element  $r$  of  $S \frown 2$ . Suppose  $r = p \frown q$ . Then  $\text{decomp}(S, 2, r) = \langle p, q \rangle$ . The theorem is a consequence of (58), (52), (7), (53), and (62).

(65) Polish-WFF-set( $T, \alpha$ ) is  $T$ -headed. The theorem is a consequence of (28), (23), and (21).

(66) The Polish-expression hierarchy( $T, \alpha, n$ ) is  $T$ -headed. The theorem is a consequence of (26) and (65).

Let us consider  $T, \alpha$ , and  $\varphi$ . The functor Polish-WFF-head  $\varphi$  yielding an element of  $T$  is defined by the term

(Def. 33)  $T\text{-head}(\varphi)$ .

Let us consider  $n$ . Let  $\varphi$  be an element of the Polish-expression hierarchy( $T, \alpha, n$ ). The functor Polish-WFF-head  $\varphi$  yielding an element of  $T$  is defined by the term

(Def. 34)  $T\text{-head}(\varphi)$ .

Let us consider  $\varphi$ . The Polish arity  $\varphi$  yielding a natural number is defined by the term

(Def. 35)  $\alpha(\text{Polish-WFF-head } \varphi)$ .

Let us consider  $n$ . Let  $\varphi$  be an element of the Polish-expression hierarchy( $T, \alpha, n$ ). The Polish arity  $\varphi$  yielding a natural number is defined by the term

(Def. 36)  $\alpha(\text{Polish-WFF-head } \varphi)$ .

Now we state the propositions:

(67)  $T\text{-tail}(\varphi) \in \text{Polish-WFF-set}(T, \alpha) \frown (\text{the Polish arity } \varphi)$ . The theorem is a consequence of (32) and (52).

(68) Let us consider an element  $\varphi$  of the Polish-expression hierarchy( $T, \alpha, n + 1$ ). Then  $T\text{-tail}(\varphi) \in (\text{the Polish-expression hierarchy}(T, \alpha, n)) \frown (\text{the Polish arity } \varphi)$ . The theorem is a consequence of (23) and (52).

Let us consider  $T, \alpha$ , and  $\varphi$ . The functor  $(T, \alpha)\text{-tail } \varphi$  yielding an element of  $\text{Polish-WFF-set}(T, \alpha) \frown (\text{the Polish arity } \varphi)$  is defined by the term

(Def. 37)  $T\text{-tail}(\varphi)$ .

Now we state the proposition:

(69) If  $T\text{-head}(\varphi) \in \text{the Polish atoms}(T, \alpha)$ , then  $\varphi = T\text{-head}(\varphi)$ . The theorem is a consequence of (67) and (6).

Let us consider  $T, \alpha$ , and  $n$ . Let  $\varphi$  be an element of the Polish-expression hierarchy( $T, \alpha, n+1$ ). The functor  $(T, \alpha)\text{-tail } \varphi$  yielding an element of  $(\text{the Polish-expression hierarchy}(T, \alpha, n)) \frown (\text{the Polish arity } \varphi)$  is defined by the term

(Def. 38)  $T\text{-tail}(\varphi)$ .

Let us consider  $\varphi$ . The functor  $\text{Polish-WFF-args } \varphi$  yielding a finite sequence of elements of  $\text{Polish-WFF-set}(T, \alpha)$  is defined by the term

(Def. 39)  $\text{decomp}(\text{Polish-WFF-set}(T, \alpha), \text{the Polish arity } \varphi, (T, \alpha)\text{-tail } \varphi)$ .

Let us consider  $n$ . Let  $\varphi$  be an element of the Polish-expression hierarchy( $T, \alpha, n + 1$ ). The functor  $\text{Polish-WFF-args } \varphi$  yielding a finite sequence of elements of the Polish-expression hierarchy( $T, \alpha, n$ ) is defined by the term

(Def. 40)  $\text{decomp}(\text{the Polish-expression hierarchy}(T, \alpha, n), \text{the Polish arity } \varphi, (T, \alpha)\text{-tail } \varphi)$ .

Now we state the propositions:

(70) Let us consider an element  $t$  of  $T$ , and  $u$ .

Suppose  $u \in \text{Polish-WFF-set}(T, \alpha) \frown \alpha(t)$ .

Then  $T\text{-tail}((\text{the Polish operation}(T, \alpha, t))(u)) = u$ . The theorem is a consequence of (52).

(71) Suppose  $\varphi \in \text{the Polish-expression hierarchy}(T, \alpha, n + 1)$ .

Then  $\text{rng Polish-WFF-args } \varphi \subseteq \text{the Polish-expression hierarchy}(T, \alpha, n)$ . The theorem is a consequence of (60) and (26).

(72) Let us consider a finite sequence  $p$ , a function  $f$  from  $Y$  into  $D$ , and a function  $g$  from  $Z$  into  $D$ . Suppose  $\text{rng } p \subseteq Y$  and  $\text{rng } p \subseteq Z$  and for every  $a$  such that  $a \in \text{rng } p$  holds  $f(a) = g(a)$ . Then  $f \cdot p = g \cdot p$ .

PROOF: Reconsider  $p_1 = p$  as a finite sequence of elements of  $Y$ . Reconsider  $q = f \cdot p_1$  as a finite sequence. Reconsider  $p_2 = p$  as a finite sequence of elements of  $Z$ . Reconsider  $r = g \cdot p_2$  as a finite sequence.  $q = r$  by [6, (33)], [4, (1)], [7, (13), (3)].  $\square$

Let us consider  $T$ ,  $\alpha$ , and  $D$ . The Polish recursion-domain( $\alpha$ ,  $D$ ) yielding a subset of  $T \times D^*$  is defined by the term

(Def. 41)  $\{\langle t, p \rangle, \text{ where } t \text{ is an element of } T, p \text{ is a finite sequence of elements of } D : \text{len } p = \alpha(t)\}$ .

A Polish recursion-function of  $\alpha$  and  $D$  is a function from the Polish recursion-domain( $\alpha$ ,  $D$ ) into  $D$ . From now on  $f$  denotes a Polish recursion-function of  $\alpha$  and  $D$  and  $\gamma$ ,  $\gamma_1$ ,  $\gamma_2$  denote functions from Polish-WFF-set( $T$ ,  $\alpha$ ) into  $D$ .

Let us consider  $T$ ,  $\alpha$ ,  $D$ ,  $f$ , and  $\gamma$ . We say that  $\gamma$  is  $f$ -recursive if and only if

(Def. 42) for every  $\varphi$ ,  $\gamma(\varphi) = f(\langle T\text{-head}(\varphi), \gamma \cdot \text{Polish-WFF-args } \varphi \rangle)$ .

Now we state the proposition:

(73) If  $\gamma_1$  is  $f$ -recursive and  $\gamma_2$  is  $f$ -recursive, then  $\gamma_1 = \gamma_2$ . The theorem is a consequence of (36), (17), (33), (52), (60), (72), and (37).

From now on  $L$  denotes a non trivial Polish-language,  $\beta$  denotes a Polish arity-function of  $L$ ,  $g$  denotes a Polish recursion-function of  $\beta$  and  $D$ ,  $J$ ,  $J_1$  denote subsets of Polish-WFF-set( $L$ ,  $\beta$ ),  $H$  denotes a function from  $J$  into  $D$ ,  $H_1$  denotes a function from  $J_1$  into  $D$ .

Let us consider  $L$ ,  $\beta$ ,  $D$ ,  $g$ ,  $J$ , and  $H$ . We say that  $H$  is  $g$ -recursive if and only if

(Def. 43) for every Polish WFF  $\varphi$  of  $L$  and  $\beta$  such that  $\varphi \in J$  and  $\text{rng Polish-WFF-args } \varphi \subseteq J$  holds  
 $H(\varphi) = g(\langle L\text{-head}(\varphi), H \cdot \text{Polish-WFF-args } \varphi \rangle)$ .

Now we state the propositions:

(74) There exists  $J$  and there exists  $H$  such that  $J =$  the Polish-expression hierarchy( $L$ ,  $\beta$ ,  $n$ ) and  $H$  is  $g$ -recursive.

PROOF: Define  $\mathcal{X}[\text{natural number}] \equiv$  there exists  $J$  and there exists  $H$  such that  $J =$  the Polish-expression hierarchy( $L$ ,  $\beta$ ,  $\$1$ ) and  $H$  is  $g$ -recursive. For every  $n$ ,  $\mathcal{X}[n]$  from [2, Sch. 2].  $\square$

(75) There exists a function  $\gamma$  from Polish-WFF-set( $L$ ,  $\beta$ ) into  $D$  such that  $\gamma$  is  $g$ -recursive.

PROOF: Set  $W = \text{Polish-WFF-set}(L, \beta)$ . Define  $\mathcal{X}[\text{object, object}] \equiv$  there exists  $n$  and there exists  $J_1$  and there exists  $H_1$  such that  $J_1 =$  the Polish-expression hierarchy( $L$ ,  $\beta$ ,  $n$ ) and  $H_1$  is  $g$ -recursive and  $\$1 \in J_1$  and  $\$2 = H_1(\$1)$ . For every  $a$  such that  $a \in W$  there exists  $b$  such that  $b \in D$  and  $\mathcal{X}[a, b]$  by (28), (74), [8, (5)]. Consider  $\gamma$  being a function from  $W$  into  $D$  such that for every  $a$  such that  $a \in W$  holds  $\mathcal{X}[a, \gamma(a)]$  from [8, Sch. 1].  $\square$

(76) Let us consider an element  $t$  of  $L$ . Then the Polish operation( $L$ ,  $\beta$ ,  $t$ ) is one-to-one.

PROOF: Set  $f =$  the Polish operation( $L, \beta, t$ ). For every  $a$  and  $b$  such that  $a, b \in \text{dom } f$  and  $f(a) = f(b)$  holds  $a = b$  by [4, (33)].  $\square$

(77) Let us consider elements  $t, u$  of  $L$ . Suppose  $\text{rng}(\text{the Polish operation}(L, \beta, t))$  meets  $\text{rng}(\text{the Polish operation}(L, \beta, u))$ . Then  $t = u$ . The theorem is a consequence of (43).

(78) Let us consider an element  $t$  of  $L$ , and  $a$ . Suppose  $a \in \text{dom}(\text{the Polish operation}(L, \beta, t))$ . Then there exists  $p$  such that

- (i)  $p = (\text{the Polish operation}(L, \beta, t))(a)$ , and
- (ii)  $L\text{-head}(p) = t$ .

The theorem is a consequence of (52).

Let us consider  $L, \beta$ , an element  $t$  of  $L$ , and a Polish WFF  $\varphi$  of  $L$  and  $\beta$ . Now we state the proposition:

(79) Polish-WFF-head  $\varphi = t$  if and only if there exists an element  $u$  of Polish-WFF-set( $L, \beta$ )  $\cap \beta(t)$  such that  $\varphi = (\text{the Polish operation}(L, \beta, t))(u)$ . The theorem is a consequence of (52).

Let us assume that  $\beta(t) = 1$ . Now we state the propositions:

(80) If Polish-WFF-head  $\varphi = t$ , then there exists a Polish WFF  $\psi$  of  $L$  and  $\beta$  such that  $\varphi = (\text{Polish-unOp}(L, \beta, t))(\psi)$ . The theorem is a consequence of (79) and (7).

(81) (i) Polish-WFF-head( $(\text{Polish-unOp}(L, \beta, t))(\varphi)$ ) =  $t$ , and  
(ii) Polish-WFF-args( $(\text{Polish-unOp}(L, \beta, t))(\varphi)$ ) =  $\langle \varphi \rangle$ .

The theorem is a consequence of (7), (79), (70), and (63).

Now we state the proposition:

(82) Suppose  $\beta(t) = 2$ . Then suppose Polish-WFF-head  $\varphi = t$ . Then there exist Polish WFFs  $\psi, H$  of  $L$  and  $\beta$  such that  $\varphi = (\text{Polish-binOp}(L, \beta, t))(\psi, H)$ . The theorem is a consequence of (79), (6), and (7).

Now we state the propositions:

(83) Let us consider an element  $t$  of  $L$ . Suppose  $\beta(t) = 2$ . Let us consider Polish WFFs  $\varphi, \psi$  of  $L$  and  $\beta$ . Then

- (i) Polish-WFF-head( $\text{Polish-binOp}(L, \beta, t))(\varphi, \psi) = t$ , and
- (ii) Polish-WFF-args( $\text{Polish-binOp}(L, \beta, t))(\varphi, \psi) = \langle \varphi, \psi \rangle$ .

The theorem is a consequence of (7), (11), (79), (64), and (70).

(84) Let us consider a Polish WFF  $\varphi$  of  $L$  and  $\beta$ . Then  $\varphi \in$  the Polish atoms( $L, \beta$ ) if and only if the Polish arity  $\varphi = 0$ . The theorem is a consequence of (53), (67), and (6).

(85) Let us consider a function  $\gamma$  from Polish-WFF-set( $L, \beta$ ) into  $D$ , an element  $t$  of  $L$ , and a Polish WFF  $\varphi$  of  $L$  and  $\beta$ . Suppose  $\gamma$  is  $g$ -recursive and  $\beta(t) = 1$ . Then  $\gamma((\text{Polish-unOp}(L, \beta, t))(\varphi)) = g(t, \langle \gamma(\varphi) \rangle)$ . The theorem is a consequence of (81).

Let us consider  $S$ . Let  $p$  be a finite sequence of elements of  $S$ . The functor Flat( $p$ ) yielding an element of  $S \wedge \text{len } p$  is defined by

(Def. 44)  $\text{decomp}(S, \text{len } p, it) = p$ .

Let us consider  $L$  and  $\beta$ .

A substitution of  $L$  and  $\beta$  is a partial function from the Polish atoms( $L, \beta$ ) to Polish-WFF-set( $L, \beta$ ). Let  $s$  be a substitution of  $L$  and  $\beta$ . The functor Subst  $s$  yielding a Polish recursion-function of  $\beta$  and Polish-WFF-set( $L, \beta$ ) is defined by

(Def. 45) for every element  $t$  of  $L$  and for every finite sequence  $p$  of elements of Polish-WFF-set( $L, \beta$ ) such that  $\text{len } p = \beta(t)$  holds if  $t \in \text{dom } s$ , then  $it(t, p) = s(t)$  and if  $t \notin \text{dom } s$ , then  $it(t, p) = t \wedge \text{Flat}(p)$ .

Let  $\varphi$  be a Polish WFF of  $L$  and  $\beta$ . The functor  $s[\varphi]$  yielding a Polish WFF of  $L$  and  $\beta$  is defined by

(Def. 46) there exists a function  $H$  from Polish-WFF-set( $L, \beta$ ) into Polish-WFF-set( $L, \beta$ ) such that  $H$  is (Subst  $s$ )-recursive and  $it = H(\varphi)$ .

Now we state the proposition:

(86) Let us consider a substitution  $s$  of  $L$  and  $\beta$ , and a Polish WFF  $\varphi$  of  $L$  and  $\beta$ . If  $s = \emptyset$ , then  $s[\varphi] = \varphi$ .

PROOF: Set  $W = \text{Polish-WFF-set}(L, \beta)$ . Set  $g = \text{Subst } s$ . Set  $\gamma = \text{id}_W$ .  $\gamma$  is  $g$ -recursive by (62), [6, (32), (33)], [7, (3), (17), (13)].  $\square$

## REFERENCES

- [1] Grzegorz Bancerek. Cardinal numbers. *Formalized Mathematics*, 1(2):377–382, 1990.
- [2] Grzegorz Bancerek. The fundamental properties of natural numbers. *Formalized Mathematics*, 1(1):41–46, 1990.
- [3] Grzegorz Bancerek. The ordinal numbers. *Formalized Mathematics*, 1(1):91–96, 1990.
- [4] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Formalized Mathematics*, 1(1):107–114, 1990.
- [5] Czesław Byliński. Binary operations. *Formalized Mathematics*, 1(1):175–180, 1990.
- [6] Czesław Byliński. Finite sequences and tuples of elements of a non-empty sets. *Formalized Mathematics*, 1(3):529–536, 1990.
- [7] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(1):55–65, 1990.
- [8] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(1):153–164, 1990.
- [9] Czesław Byliński. Partial functions. *Formalized Mathematics*, 1(2):357–367, 1990.
- [10] Czesław Byliński. Some basic properties of sets. *Formalized Mathematics*, 1(1):47–53, 1990.

- [11] Agata Darmochwał. Finite sets. *Formalized Mathematics*, 1(1):165–167, 1990.
- [12] Joanna Golińska-Pilarek and Taneli Huuskonen. Logic of descriptions. A new approach to the foundations of mathematics and science. *Studies in Logic, Grammar and Rhetoric*, 40(27), 2012.
- [13] Joanna Golińska-Pilarek and Taneli Huuskonen. Grzegorzczak's non-Fregean logics. In Rafał Urbaniak and Gillman Payette, editors, *Applications of Formal Philosophy: The Road Less Travelled*, Logic, Reasoning and Argumentation. Springer, 2015.
- [14] Andrzej Grzegorzczak. Filozofia logiki i formalna LOGIKA NIESYMPLIKACYJNA. *Zagadnienia Naukoznawstwa*, XLVII(4), 2012. In Polish.
- [15] Jan Łukasiewicz. Uwagi o aksjomacie Nicoda i 'dedukcji uogólniającej'. In *Księga pamiątkowa Polskiego Towarzystwa Filozoficznego*, Lwów, 1931. In Polish.
- [16] Andrzej Nędzusiak. Probability. *Formalized Mathematics*, 1(4):745–749, 1990.
- [17] Beata Padlewska. Families of sets. *Formalized Mathematics*, 1(1):147–152, 1990.
- [18] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(1):67–71, 1990.
- [19] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1(1):73–83, 1990.

*Received April 30, 2015*

---