

Grzegorzczuk's Logics. Part I

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Summary. This article is the second in a series formalizing some results in my joint work with Prof. Joanna Golińska-Pilarek ([9] and [10]) concerning a logic proposed by Prof. Andrzej Grzegorzczuk ([11]).

This part presents the syntax and axioms of Grzegorzczuk's *Logic of Descriptions* (LD) as originally proposed by him, as well as some theorems not depending on any semantic constructions. There are both some clear similarities and fundamental differences between LD and the non-Fregean logics introduced by Roman Suszko in [15]. In particular, we were inspired by Suszko's semantics for his non-Fregean logic SCI, presented in [16].

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The notation and terminology used in this paper have been introduced in the following articles: [3], [17], [14], [2], [8], [4], [5], [1], [6], [12], [19], [21], [20], [13], [18], and [7].

1. THE CONSTRUCTION OF GRZEGORCZYK'S LD LANGUAGE

From now on k, m, n denote elements of \mathbb{N} , i, j denote natural numbers, a, b, c denote objects, X, Y, Z denote sets, D, D_1, D_2 denote non empty sets, and p, q, r, s denote finite sequences.

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The functor VAR yielding a finite sequence-membered set is defined by the term

(Def. 1) the set of all $\langle 0, k \rangle$ where k is an element of \mathbb{N} .

Note that VAR is non empty and antichain-like.

A variable is an element of VAR. The functors: 'not', &, and '=' yielding finite sequences are defined by terms

(Def. 2) $\langle 1 \rangle$,

(Def. 3) $\langle 2 \rangle$,

(Def. 4) $\langle 3 \rangle$,

respectively. The functor GRZ-ops yielding a non empty, finite sequence-membered set is defined by the term

(Def. 5) $\{ \text{'not'}, \&, '=' \}$.

Let us note that the functor GRZ-ops yields a Polish language. The functor GRZ-symbols yielding a non empty, finite sequence-membered set is defined by the term

(Def. 6) $\text{VAR} \cup \text{GRZ-ops}$.

The functors: 'not', &, and '=' yield elements of GRZ-symbols. Now we state the proposition:

- (1) (i) 'not' \neq &, and
 (ii) 'not' \neq '=', and
 (iii) & \neq '='.

Observe that GRZ-symbols is non trivial and antichain-like.

The functor GRZ-op-arity yielding a function from GRZ-ops into \mathbb{N} is defined by

(Def. 7) $it(\text{'not'}) = 1$ and $it(\&) = 2$ and $it('=') = 2$.

The functor GRZ-arity yielding a Polish arity-function of GRZ-symbols is defined by

(Def. 8) for every a such that $a \in \text{GRZ-symbols}$ holds if $a \in \text{GRZ-ops}$, then $it(a) = \text{GRZ-op-arity}(a)$ and if $a \notin \text{GRZ-ops}$, then $it(a) = 0$.

Now we state the propositions:

- (2) (i) $\text{GRZ-arity}(\text{'not'}) = 1$, and
 (ii) $\text{GRZ-arity}(\&) = 2$, and
 (iii) $\text{GRZ-arity}('=') = 2$.

(3) The Polish atoms($\text{GRZ-symbols}, \text{GRZ-arity}$) = VAR. The theorem is a consequence of (2).

The functor GRZ-formula-set yielding a Polish language of GRZ-symbols is defined by the term

(Def. 9) Polish-WFF-set(GRZ-symbols, GRZ-arity).

A GRZ-formula is a Polish WFF of GRZ-symbols and GRZ-arity. One can verify that there exists a subset of GRZ-formula-set which is non empty.

Let us consider n . The functor x_n yielding a GRZ-formula is defined by the term

(Def. 10) $\langle 0, n \rangle$.

From now on $\varphi, \psi, \vartheta, \eta$ denote GRZ-formulas.

Let us consider φ . The functor $\neg\varphi$ yielding a GRZ-formula is defined by the term

(Def. 11) (Polish-unOp(GRZ-symbols, GRZ-arity, 'not'))(φ).

Let us consider ψ . The functors: $\varphi \wedge \psi$ and $\varphi = \psi$ yielding GRZ-formulas are defined by terms

(Def. 12) (Polish-binOp(GRZ-symbols, GRZ-arity, &))(φ, ψ),

(Def. 13) (Polish-binOp(GRZ-symbols, GRZ-arity, '='))(φ, ψ),

respectively. The functors: $\varphi \vee \psi$ and $\varphi \Rightarrow \psi$ yielding GRZ-formulas are defined by terms

(Def. 14) $\neg(\neg\varphi \wedge \neg\psi)$,

(Def. 15) $\varphi = (\varphi \wedge \psi)$,

respectively. The functor $\varphi \Leftrightarrow \psi$ yielding a GRZ-formula is defined by the term

(Def. 16) $(\varphi \Rightarrow \psi) \wedge (\psi \Rightarrow \varphi)$.

We say that φ is atomic if and only if

(Def. 17) $\varphi \in$ the Polish atoms(GRZ-symbols, GRZ-arity).

We say that φ is negative if and only if

(Def. 18) Polish-WFF-head $\varphi =$ 'not'.

We say that φ is conjunctive if and only if

(Def. 19) Polish-WFF-head $\varphi =$ &.

We say that φ is an equality if and only if

(Def. 20) Polish-WFF-head $\varphi =$ '='.

Let us consider φ . Now we state the propositions:

(4) φ is atomic if and only if $\varphi \in \text{VAR}$.

(5) φ is negative if and only if there exists ψ such that $\varphi = \neg\psi$.

PROOF: If φ is negative, then there exists ψ such that $\varphi = \neg\psi$ by (2), [12, (80)]. \square

- (6) φ is conjunctive if and only if there exists ψ and there exists ϑ such that $\varphi = \psi \wedge \vartheta$.
 PROOF: If φ is conjunctive, then there exists ψ and there exists ϑ such that $\varphi = \psi \wedge \vartheta$ by (2), [12, (82)]. \square
- (7) φ is an equality if and only if there exists ψ and there exists ϑ such that $\varphi = \psi = \vartheta$.
 PROOF: If φ is an equality, then there exists ψ and there exists ϑ such that $\varphi = \psi = \vartheta$ by (2), [12, (82)]. \square
- (8) φ is atomic or negative or conjunctive or an equality. The theorem is a consequence of (3).

Let us observe that every GRZ-formula which is atomic is also non negative and every GRZ-formula which is atomic is also non conjunctive and every GRZ-formula which is atomic is also non equality and every GRZ-formula which is negative is also non conjunctive and every GRZ-formula which is negative is also non equality and every GRZ-formula which is conjunctive is also non equality.

2. AXIOMS AND RULES

The functors: GRZ-axioms and LD-specific axioms yielding non empty subsets of GRZ-formula-set are defined by conditions

(Def. 21) for every a , $a \in \text{GRZ-axioms}$ iff there exists φ and there exists ψ and there exists ϑ such that $a = \neg(\varphi \wedge \neg\varphi)$ or $a = (\neg\neg\varphi) = \varphi$ or $a = \varphi = (\varphi \wedge \varphi)$ or $a = (\varphi \wedge \psi) = (\psi \wedge \varphi)$ or $a = (\varphi \wedge (\psi \wedge \vartheta)) = ((\varphi \wedge \psi) \wedge \vartheta)$ or $a = (\varphi \wedge (\psi \vee \vartheta)) = (\varphi \wedge \psi \vee \varphi \wedge \vartheta)$ or $a = (\neg(\varphi \wedge \psi)) = (\neg\varphi \vee \neg\psi)$ or $a = (\varphi = \psi) = (\psi = \varphi)$ or $a = (\varphi = \psi) = ((\neg\varphi) = (\neg\psi))$,

(Def. 22) for every a , $a \in \text{LD-specific axioms}$ iff there exists φ and there exists ψ and there exists ϑ such that $a = \varphi = \psi \Rightarrow (\varphi \wedge \vartheta) = (\psi \wedge \vartheta)$ or $a = \varphi = \psi \Rightarrow (\varphi \vee \vartheta) = (\psi \vee \vartheta)$ or $a = \varphi = \psi \Rightarrow (\varphi = \vartheta) = (\psi = \vartheta)$,

respectively. The functor LD-axioms yielding a non empty subset of GRZ-formula-set is defined by the term

(Def. 23) $\text{GRZ-axioms} \cup \text{LD-specific axioms}$.

A GRZ-rule is a relation between $2^{\text{GRZ-formula-set}}$ and GRZ-formula-set. In the sequel R , R_1 , R_2 denote GRZ-rules.

Let us consider R_1 and R_2 . Note that the functor $R_1 \cup R_2$ yields a GRZ-rule. The functors: GRZ-MP, GRZ-ConjIntro, GRZ-ConjElimL, and GRZ-ConjElimR yielding GRZ-rules are defined by terms

(Def. 24) the set of all $\{\{\varphi, \varphi = \psi\}, \psi\}$ where φ is a GRZ-formula, ψ is a GRZ-formula,

(Def. 25) the set of all $\langle \{\varphi, \psi\}, \varphi \wedge \psi \rangle$ where φ is a GRZ-formula, ψ is a GRZ-formula,

(Def. 26) the set of all $\langle \{\varphi \wedge \psi\}, \varphi \rangle$ where φ is a GRZ-formula, ψ is a GRZ-formula,

(Def. 27) the set of all $\langle \{\varphi \wedge \psi\}, \psi \rangle$ where φ is a GRZ-formula, ψ is a GRZ-formula, respectively. The functor GRZ-rules yielding a GRZ-rule is defined by

(Def. 28) for every a , $a \in it$ iff $a \in \text{GRZ-MP}$ or $a \in \text{GRZ-ConjIntro}$ or $a \in \text{GRZ-ConjElimL}$ or $a \in \text{GRZ-ConjElimR}$.

A GRZ-formula sequence is a finite sequence of elements of GRZ-formula-set.

A finite GRZ-formula set is a finite subset of GRZ-formula-set. From now on Γ , Γ_1 , Γ_2 denote non empty subsets of GRZ-formula-set, Δ , Δ_1 , Δ_2 denote subsets of GRZ-formula-set, P , P_1 , P_2 denote GRZ-formula sequences, and Σ , Σ_1 , Σ_2 denote finite GRZ-formula sets.

Let us consider Σ_1 and Σ_2 . Observe that the functor $\Sigma_1 \cup \Sigma_2$ yields a finite GRZ-formula set. Let us consider Γ , R , P , and n . We say that (P, n) is a correct step w.r.t. Γ , R if and only if

(Def. 29) $P(n) \in \Gamma$ or there exists a finite GRZ-formula set Q such that $\langle Q, P(n) \rangle \in R$ and for every q such that $q \in Q$ there exists k such that $k \in \text{dom } P$ and $k < n$ and $P(k) = q$.

We say that P is (Γ, R) -correct if and only if

(Def. 30) for every k such that $k \in \text{dom } P$ holds (P, k) is a correct step w.r.t. Γ , R .

Let a be an element of Γ . One can verify that the functor $\langle a \rangle$ yields a GRZ-formula sequence. Now we state the proposition:

(9) Let us consider an element a of Γ . Then $\langle a \rangle$ is (Γ, R) -correct.

Let us consider Γ and R . Note that there exists a GRZ-formula sequence which is non empty and (Γ, R) -correct.

Let us consider Σ . We say that Σ is (Γ, R) -correct if and only if

(Def. 31) there exists P such that $\Sigma = \text{rng } P$ and P is (Γ, R) -correct.

Now we state the propositions:

(10) If P is (Γ, R) -correct and $P = P_1 \wedge P_2$, then P_1 is (Γ, R) -correct.

(11) If P_1 is (Γ, R) -correct and P_2 is (Γ, R) -correct, then $P_1 \wedge P_2$ is (Γ, R) -correct.

(12) If Σ_1 is (Γ, R) -correct and Σ_2 is (Γ, R) -correct, then $\Sigma_1 \cup \Sigma_2$ is (Γ, R) -correct. The theorem is a consequence of (11).

(13) If $\Gamma \subseteq \Gamma_1$ and $R \subseteq R_1$ and P is (Γ, R) -correct, then P is (Γ_1, R_1) -correct.

Let us consider Γ , R , and φ . We say that $\Gamma, R \vdash \varphi$ if and only if

(Def. 32) there exists P such that $\varphi \in \text{rng } P$ and P is (Γ, R) -correct.

Let us consider Δ . We say that $\Gamma, R \vdash \Delta$ if and only if

(Def. 33) for every φ such that $\varphi \in \Delta$ holds $\Gamma, R \vdash \varphi$.

Let us consider Γ, R , and φ . Now we state the propositions:

(14) $\Gamma, R \vdash \varphi$ if and only if there exists Σ such that $\varphi \in \Sigma$ and Σ is (Γ, R) -correct.

(15) If $\varphi \in \Gamma$, then $\Gamma, R \vdash \varphi$. The theorem is a consequence of (9).

Now we state the propositions:

(16) If $\Gamma, R \vdash \Sigma$, then there exists Σ_1 such that $\Sigma \subseteq \Sigma_1$ and Σ_1 is (Γ, R) -correct.

PROOF: Define $\mathcal{X}[\text{set}] \equiv$ there exists Σ_1 such that $\text{\$}_1 \subseteq \Sigma_1$ and Σ_1 is (Γ, R) -correct. $\mathcal{X}[\emptyset]$. For every sets x, Δ such that $x \in \Sigma$ and $\Delta \subseteq \Sigma$ and $\mathcal{X}[\Delta]$ holds $\mathcal{X}[\Delta \cup \{x\}]$. $\mathcal{X}[\Sigma]$ from [8, Sch. 2]. \square

(17) If $\Gamma, R \vdash \Sigma$ and $\langle \Sigma, \varphi \rangle \in R$, then $\Gamma, R \vdash \varphi$. The theorem is a consequence of (16).

(18) If $\Gamma, R \vdash \varphi$, then $\varphi \in \Gamma$ or there exists Σ such that $\langle \Sigma, \varphi \rangle \in R$ and $\Gamma, R \vdash \Sigma$.

(19) If $\Gamma \subseteq \Gamma_1$ and $R \subseteq R_1$ and $\Gamma, R \vdash \varphi$, then $\Gamma_1, R_1 \vdash \varphi$.

Let us consider Γ, φ , and ψ . Now we state the propositions:

(20) $\Gamma, \text{GRZ-rules} \vdash \varphi \wedge \psi$ if and only if $\Gamma, \text{GRZ-rules} \vdash \varphi$ and $\Gamma, \text{GRZ-rules} \vdash \psi$. The theorem is a consequence of (17).

(21) Suppose $\Gamma, \text{GRZ-rules} \vdash \varphi$ and $\Gamma, \text{GRZ-rules} \vdash \varphi = \psi$. Then $\Gamma, \text{GRZ-rules} \vdash \psi$. The theorem is a consequence of (17).

(22) Suppose $\Gamma, \text{GRZ-rules} \vdash \varphi$ and $\Gamma, \text{GRZ-rules} \vdash \varphi \Rightarrow \psi$.

Then $\Gamma, \text{GRZ-rules} \vdash \psi$. The theorem is a consequence of (21) and (20).

(23) If $\Gamma, \text{GRZ-rules} \vdash \varphi \wedge \psi$, then $\Gamma, \text{GRZ-rules} \vdash \psi \wedge \varphi$. The theorem is a consequence of (20).

Let us consider φ . We say that φ is GRZ-axiomatic if and only if

(Def. 34) $\varphi \in \text{GRZ-axioms}$.

We say that φ is GRZ-provable if and only if

(Def. 35) $\text{GRZ-axioms}, \text{GRZ-rules} \vdash \varphi$.

We say that φ is LD-axiomatic if and only if

(Def. 36) $\varphi \in \text{LD-axioms}$.

We say that φ is LD-provable if and only if

(Def. 37) $\text{LD-axioms}, \text{GRZ-rules} \vdash \varphi$.

Observe that $\neg(\varphi \wedge \neg\varphi)$ is GRZ-axiomatic and $(\neg\neg\varphi) = \varphi$ is GRZ-axiomatic and $\varphi = (\varphi \wedge \varphi)$ is GRZ-axiomatic.

Let us consider ψ . Observe that $(\varphi \wedge \psi) = (\psi \wedge \varphi)$ is GRZ-axiomatic and $(\neg(\varphi \wedge \psi)) = (\neg\varphi \vee \neg\psi)$ is GRZ-axiomatic and $(\varphi = \psi) = (\psi = \varphi)$ is GRZ-axiomatic and $(\varphi = \psi) = ((\neg\varphi) = (\neg\psi))$ is GRZ-axiomatic.

Let us consider ϑ . Observe that $(\varphi \wedge (\psi \wedge \vartheta)) = ((\varphi \wedge \psi) \wedge \vartheta)$ is GRZ-axiomatic and $(\varphi \wedge (\psi \vee \vartheta)) = (\varphi \wedge \psi \vee \varphi \wedge \vartheta)$ is GRZ-axiomatic and $\varphi = \psi \Rightarrow (\varphi \wedge \vartheta) = (\psi \wedge \vartheta)$ is LD-axiomatic and $\varphi = \psi \Rightarrow (\varphi \vee \vartheta) = (\psi \vee \vartheta)$ is LD-axiomatic and $\varphi = \psi \Rightarrow (\varphi = \vartheta) = (\psi = \vartheta)$ is LD-axiomatic and every GRZ-formula which is GRZ-axiomatic is also LD-axiomatic and every GRZ-formula which is GRZ-axiomatic is also GRZ-provable and every GRZ-formula which is LD-axiomatic is also LD-provable and every GRZ-formula which is GRZ-provable is also LD-provable and there exists a GRZ-formula which is GRZ-axiomatic, GRZ-provable, LD-axiomatic, and LD-provable.

Now we state the proposition:

(24) Suppose $\text{GRZ-axioms} \subseteq \Gamma$ and $\Gamma, \text{GRZ-rules} \vdash \varphi = \psi$.

Then $\Gamma, \text{GRZ-rules} \vdash \psi = \varphi$. The theorem is a consequence of (15) and (21).

3. PROVABILITY

Let us consider φ and ψ . Now we state the propositions:

(25) If $\varphi = \psi$ is GRZ-provable, then $\psi = \varphi$ is GRZ-provable.

(26) If $\varphi = \psi$ is LD-provable, then $\psi = \varphi$ is LD-provable.

Now we state the propositions:

(27) If $\varphi = \psi$ is LD-provable and $\psi = \vartheta$ is LD-provable, then $\varphi = \vartheta$ is LD-provable.

The theorem is a consequence of (24), (22), and (21).

(28) $\varphi = \varphi$ is LD-provable. The theorem is a consequence of (24) and (27).

Let us consider φ and ψ . We say that $\varphi =_{\text{LD}} \psi$ if and only if

(Def. 38) $\varphi = \psi$ is LD-provable.

One can check that the predicate is reflexive and symmetric.

Now we state the proposition:

(29) If $\varphi =_{\text{LD}} \psi$, then $\neg\varphi =_{\text{LD}} \neg\psi$. The theorem is a consequence of (21).

The scheme *BinReplace* deals with a non empty set \mathcal{X} and a binary functor \mathcal{F} yielding an element of \mathcal{X} and a binary predicate \mathcal{R} and states that

(Sch. 1) For every elements a, b, c, d of \mathcal{X} such that $\mathcal{R}[a, b]$ and $\mathcal{R}[c, d]$ holds $\mathcal{R}[\mathcal{F}(a, c), \mathcal{F}(b, d)]$

provided

- for every elements a, b, c of \mathcal{X} such that $\mathcal{R}[a, b]$ and $\mathcal{R}[b, c]$ holds $\mathcal{R}[a, c]$ and

- for every elements a, b of \mathcal{X} , $\mathcal{R}[\mathcal{F}(a, b), \mathcal{F}(b, a)]$ and
- for every elements a, b, c of \mathcal{X} such that $\mathcal{R}[a, b]$ holds $\mathcal{R}[\mathcal{F}(a, c), \mathcal{F}(b, c)]$.

Let us consider φ, ψ, ϑ , and η .

Let us assume that $\varphi =_{LD} \psi$ and $\vartheta =_{LD} \eta$. Now we state the propositions:

(30) $\varphi \wedge \vartheta =_{LD} \psi \wedge \eta$.

PROOF: Define $\mathcal{F}(\text{GRZ-formula}, \text{GRZ-formula}) = \$_1 \wedge \$_2$. Define $\mathcal{P}[\text{GRZ-formula}, \text{GRZ-formula}] \equiv \$_1 = \$_2$ is LD-provable. For every φ, ψ , and ϑ such that $\mathcal{P}[\varphi, \psi]$ and $\mathcal{P}[\psi, \vartheta]$ holds $\mathcal{P}[\varphi, \vartheta]$. For every φ, ψ, ϑ , and η such that $\mathcal{P}[\varphi, \psi]$ and $\mathcal{P}[\vartheta, \eta]$ holds $\mathcal{P}[\mathcal{F}(\varphi, \vartheta), \mathcal{F}(\psi, \eta)]$ from *BinReplace*. \square

(31) $\varphi =_{LD} \vartheta =_{LD} \psi =_{LD} \eta$.

PROOF: Define $\mathcal{F}(\text{GRZ-formula}, \text{GRZ-formula}) = \$_1 = \$_2$. Define $\mathcal{P}[\text{GRZ-formula}, \text{GRZ-formula}] \equiv \$_1 = \$_2$ is LD-provable. For every φ, ψ , and ϑ such that $\mathcal{P}[\varphi, \psi]$ and $\mathcal{P}[\psi, \vartheta]$ holds $\mathcal{P}[\varphi, \vartheta]$. For every φ, ψ, ϑ , and η such that $\mathcal{P}[\varphi, \psi]$ and $\mathcal{P}[\vartheta, \eta]$ holds $\mathcal{P}[\mathcal{F}(\varphi, \vartheta), \mathcal{F}(\psi, \eta)]$ from *BinReplace*. \square

The functor LD-IdR yielding an equivalence relation of GRZ-formula-set is defined by

(Def. 39) for every φ and ψ , $\langle \varphi, \psi \rangle \in it$ iff $\varphi =_{LD} \psi$.

Note that there exists a family of subsets of GRZ-formula-set which is non empty.

The functor LD-IdClasses yielding a non empty family of subsets of GRZ-formula-set is defined by the term

(Def. 40) Classes LD-IdR.

An LD-identity class is an element of LD-IdClasses. Let us consider φ . The functor LD-IdClassOf φ yielding an LD-identity class is defined by the term

(Def. 41) $[\varphi]_{LD-IdR}$.

Now we state the proposition:

(32) $\varphi =_{LD} \psi$ if and only if LD-IdClassOf $\varphi =$ LD-IdClassOf ψ .

PROOF: If $\varphi =_{LD} \psi$, then LD-IdClassOf $\varphi =$ LD-IdClassOf ψ by [14, (18), (23)]. \square

The scheme *UnOpCongr* deals with a non empty set \mathcal{X} and a unary functor \mathcal{F} yielding an element of \mathcal{X} and an equivalence relation \mathcal{E} of \mathcal{X} and states that

(Sch. 2) There exists a unary operation f on Classes \mathcal{E} such that for every element

$$x \text{ of } \mathcal{X}, f([x]_{\mathcal{E}}) = [\mathcal{F}(x)]_{\mathcal{E}}$$

provided

- for every elements x, y of \mathcal{X} such that $\langle x, y \rangle \in \mathcal{E}$ holds $\langle \mathcal{F}(x), \mathcal{F}(y) \rangle \in \mathcal{E}$.

The scheme *BinOpCongr* deals with a non empty set \mathcal{X} and a binary functor \mathcal{F} yielding an element of \mathcal{X} and an equivalence relation \mathcal{E} of \mathcal{X} and states that

(Sch. 3) There exists a binary operation f on Classes \mathcal{E} such that for every elements x, y of \mathcal{X} , $f([x]_{\mathcal{E}}, [y]_{\mathcal{E}}) = [\mathcal{F}(x, y)]_{\mathcal{E}}$

provided

- for every elements x_1, x_2, y_1, y_2 of \mathcal{X} such that $\langle x_1, x_2 \rangle, \langle y_1, y_2 \rangle \in \mathcal{E}$ holds $\langle \mathcal{F}(x_1, y_1), \mathcal{F}(x_2, y_2) \rangle \in \mathcal{E}$.

From now on x, y, z denote LD-identity classes.

Now we state the proposition:

(33) There exists φ such that $x = \text{LD-IdClassOf } \varphi$.

Let us consider x . We say that x is LD-provable if and only if

(Def. 42) there exists φ such that $x = \text{LD-IdClassOf } \varphi$ and φ is LD-provable.

The functor $\neg x$ yielding an LD-identity class is defined by

(Def. 43) there exists φ such that $x = \text{LD-IdClassOf } \varphi$ and $it = \text{LD-IdClassOf } \neg\varphi$.

One can verify that the functor is involutive. Let us consider y . The functor $x \wedge y$ yielding an LD-identity class is defined by

(Def. 44) there exists φ and there exists ψ such that $x = \text{LD-IdClassOf } \varphi$ and $y = \text{LD-IdClassOf } \psi$ and $it = \text{LD-IdClassOf } (\varphi \wedge \psi)$.

Note that the functor is commutative and idempotent. The functor $x=y$ yielding an LD-identity class is defined by

(Def. 45) there exists φ and there exists ψ such that $x = \text{LD-IdClassOf } \varphi$ and $y = \text{LD-IdClassOf } \psi$ and $it = \text{LD-IdClassOf } \varphi=\psi$.

One can check that the functor is commutative.

The functor $x \vee y$ yielding an LD-identity class is defined by the term

(Def. 46) $\neg(\neg x \wedge \neg y)$.

Let us observe that the functor is commutative and idempotent. The functor $x \Rightarrow y$ yielding an LD-identity class is defined by the term

(Def. 47) $x=(x \wedge y)$.

Let φ be an LD-provable GRZ-formula. Let us observe that $\text{LD-IdClassOf } \varphi$ is LD-provable.

Now we state the proposition:

(34) If $\text{LD-IdClassOf } \varphi$ is LD-provable, then φ is LD-provable. The theorem is a consequence of (32) and (21).

Let us consider x and y . Now we state the propositions:

(35) $x \wedge y$ is LD-provable if and only if x is LD-provable and y is LD-provable. The theorem is a consequence of (34) and (20).

(36) $x=y$ is LD-provable if and only if $x = y$. The theorem is a consequence of (34) and (32).

Now we state the proposition:

$$(37) \quad \text{LD-IdClassOf } \neg\varphi = \neg \text{LD-IdClassOf } \varphi.$$

Let us consider φ and ψ . Now we state the propositions:

$$(38) \quad \text{LD-IdClassOf}(\varphi \wedge \psi) = \text{LD-IdClassOf } \varphi \wedge \text{LD-IdClassOf } \psi.$$

$$(39) \quad \text{LD-IdClassOf } \varphi = \psi = (\text{LD-IdClassOf } \varphi) = (\text{LD-IdClassOf } \psi).$$

$$(40) \quad \text{LD-IdClassOf}(\varphi \vee \psi) = \text{LD-IdClassOf } \varphi \vee \text{LD-IdClassOf } \psi.$$

$$(41) \quad \text{LD-IdClassOf}(\varphi \Rightarrow \psi) = \text{LD-IdClassOf } \varphi \Rightarrow \text{LD-IdClassOf } \psi.$$

Now we state the propositions:

$$(42) \quad (x \wedge y) \wedge z = x \wedge (y \wedge z). \text{ The theorem is a consequence of (33) and (32).}$$

$$(43) \quad x \Rightarrow y \text{ is LD-provable if and only if } x = x \wedge y.$$

$$(44) \quad \text{If } x \Rightarrow y \text{ is LD-provable and } y \Rightarrow z \text{ is LD-provable, then } x \Rightarrow z \text{ is LD-provable. The theorem is a consequence of (36) and (42).}$$

$$(45) \quad \text{If } \varphi \Rightarrow \psi \text{ is LD-provable and } \psi \Rightarrow \vartheta \text{ is LD-provable, then } \varphi \Rightarrow \vartheta \text{ is LD-provable. The theorem is a consequence of (41), (34), and (44).}$$

Let us consider x , y , and z . Now we state the propositions:

$$(46) \quad x \vee (y \vee z) = (x \vee y) \vee z.$$

$$(47) \quad x \wedge (y \vee z) = x \wedge y \vee x \wedge z. \text{ The theorem is a consequence of (33), (32), and (40).}$$

$$(48) \quad x \vee y \wedge z = (x \vee y) \wedge (x \vee z). \text{ The theorem is a consequence of (47).}$$

Let us consider x and y . Now we state the propositions:

$$(49) \quad x \Rightarrow y \text{ is LD-provable and } y \Rightarrow x \text{ is LD-provable if and only if } x = y. \text{ The theorem is a consequence of (36).}$$

$$(50) \quad \text{If } x \text{ is LD-provable, then } x \vee y \text{ is LD-provable. The theorem is a consequence of (33), (35), (47), and (48).}$$

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