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Grzegorczyk's Logics. Part I

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Summary. This article is the second in a series formalizing some results in my joint work with Prof. Joanna Golińska-Pilarek ([9] and [10]) concerning a logic proposed by Prof. Andrzej Grzegorczyk ([11]).

This part presents the syntax and axioms of Grzegorczyk's *Logic of Descriptions* (LD) as originally proposed by him, as well as some theorems not depending on any semantic constructions. There are both some clear similarities and fundamental differences between LD and the non-Fregean logics introduced by Roman Suszko in [15]. In particular, we were inspired by Suszko's semantics for his non-Fregean logic SCI, presented in [16].

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The notation and terminology used in this paper have been introduced in the following articles: [3], [17], [14], [2], [8], [4], [5], [1], [6], [12], [19], [21], [20], [13], [18], and [7].

1. The Construction of Grzegorczyk's LD Language

From now on k, m, n denote elements of $\mathbb{N}$, i, j denote natural numbers, a, b, c denote objects, X, Y, Z denote sets, D, D_1, D_2 denote non empty sets, and p, q, r, s denote finite sequences.

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The functor VAR yielding a finite sequence-membered set is defined by the term

(Def. 1) the set of all $\langle 0, k \rangle$ where k is an element of $\mathbb{N}$.

Note that VAR is non empty and antichain-like.

A variable is an element of VAR. The functors: $'$not$'$, &, and $'{=}'$ yielding finite sequences are defined by terms

(Def. 2) $\langle 1 \rangle$,

(Def. 3) $\langle 2 \rangle$,

(Def. 4) $\langle 3 \rangle$,

respectively. The functor GRZ-ops yielding a non empty, finite sequence-membered set is defined by the term

(Def. 5) $\{'\mathrm{not}', \&, '{=}'\}$.

Let us note that the functor GRZ-ops yields a Polish language. The functor GRZ-symbols yielding a non empty, finite sequence-membered set is defined by the term

(Def. 6) $\mathrm{VAR} \cup \mathrm{GRZ\text{-}ops}$.

The functors: $'$not$'$, &, and $'{=}'$ yield elements of GRZ-symbols. Now we state the proposition:

(1) (i) $'\mathrm{not}' \neq \&$, and

(ii) $'\mathrm{not}' \neq '{=}'$, and

(iii) $\& \neq '{=}'$.

Observe that GRZ-symbols is non trivial and antichain-like.

The functor GRZ-op-arity yielding a function from GRZ-ops into $\mathbb{N}$ is defined by

(Def. 7) $it('\mathrm{not}') = 1$ and $it(\&) = 2$ and $it('{=}') = 2$.

The functor GRZ-arity yielding a Polish arity-function of GRZ-symbols is defined by

(Def. 8) for every a such that $a \in$ GRZ-symbols holds if $a \in$ GRZ-ops, then $it(a) = \mathrm{GRZ\text{-}op\text{-}arity}(a)$ and if $a \notin$ GRZ-ops, then $it(a) = 0$.

Now we state the propositions:

(2) (i) GRZ-arity$('\mathrm{not}') = 1$, and

(ii) GRZ-arity$(\&) = 2$, and

(iii) GRZ-arity$('{=}') = 2$.

(3) The Polish atoms(GRZ-symbols , GRZ-arity) = VAR. The theorem is a consequence of (2).

The functor GRZ-formula-set yielding a Polish language of GRZ-symbols is defined by the term

(Def. 9) $\text{Polish-WFF-set}(\text{GRZ-symbols}, \text{GRZ-arity})$.

A GRZ-formula is a Polish WFF of GRZ-symbols and GRZ-arity. One can verify that there exists a subset of GRZ-formula-set which is non empty.

Let us consider n. The functor x_n yielding a GRZ-formula is defined by the term

(Def. 10) $\langle 0, n\rangle$.

From now on φ, ψ, ϑ, η denote GRZ-formulas.

Let us consider φ. The functor $\neg\varphi$ yielding a GRZ-formula is defined by the term

(Def. 11) $(\text{Polish-unOp}(\text{GRZ-symbols}, \text{GRZ-arity}, '\text{not}'))(\varphi)$.

Let us consider ψ. The functors: $\varphi \wedge \psi$ and $\varphi{=}\psi$ yielding GRZ-formulas are defined by terms

(Def. 12) $(\text{Polish-binOp}(\text{GRZ-symbols}, \text{GRZ-arity}, \&))(\varphi, \psi)$,

(Def. 13) $(\text{Polish-binOp}(\text{GRZ-symbols}, \text{GRZ-arity}, '{=}'))(\varphi, \psi)$,

respectively. The functors: $\varphi \vee \psi$ and $\varphi \Rightarrow \psi$ yielding GRZ-formulas are defined by terms

(Def. 14) $\neg(\neg\varphi \wedge \neg\psi)$,

(Def. 15) $\varphi{=}(\varphi \wedge \psi)$,

respectively. The functor $\varphi \Leftrightarrow \psi$ yielding a GRZ-formula is defined by the term

(Def. 16) $(\varphi \Rightarrow \psi) \wedge (\psi \Rightarrow \varphi)$.

We say that φ is atomic if and only if

(Def. 17) $\varphi \in$ the Polish atoms(GRZ-symbols , GRZ-arity).

We say that φ is negative if and only if

(Def. 18) $\text{Polish-WFF-head}\,\varphi = '\text{not}'$.

We say that φ is conjunctive if and only if

(Def. 19) $\text{Polish-WFF-head}\,\varphi = \&$.

We say that φ is an equality if and only if

(Def. 20) $\text{Polish-WFF-head}\,\varphi = '{=}'$.

Let us consider φ. Now we state the propositions:

(4) φ is atomic if and only if $\varphi \in \mathrm{VAR}$.

(5) φ is negative if and only if there exists ψ such that $\varphi = \neg\psi$.

Proof: If φ is negative, then there exists ψ such that $\varphi = \neg\psi$ by (2), [12, (80)]. $\square$

(6) φ is conjunctive if and only if there exists ψ and there exists ϑ such that $\varphi = \psi \wedge \vartheta$.
PROOF: If φ is conjunctive, then there exists ψ and there exists ϑ such that $\varphi = \psi \wedge \vartheta$ by (2), [12, (82)]. □

(7) φ is an equality if and only if there exists ψ and there exists ϑ such that $\varphi = \psi{=}\vartheta$.
PROOF: If φ is an equality, then there exists ψ and there exists ϑ such that $\varphi = \psi{=}\vartheta$ by (2), [12, (82)]. □

(8) φ is atomic or negative or conjunctive or an equality. The theorem is a consequence of (3).

Let us observe that every GRZ-formula which is atomic is also non negative and every GRZ-formula which is atomic is also non conjunctive and every GRZ-formula which is atomic is also non equality and every GRZ-formula which is negative is also non conjunctive and every GRZ-formula which is negative is also non equality and every GRZ-formula which is conjunctive is also non equality.

2. Axioms and Rules

The functors: GRZ-axioms and LD-specific axioms yielding non empty subsets of GRZ-formula-set are defined by conditions

(Def. 21) for every a, $a \in$ GRZ-axioms iff there exists φ and there exists ψ and there exists ϑ such that $a = \neg(\varphi \wedge \neg\varphi)$ or $a = (\neg\neg\varphi){=}\varphi$ or $a = \varphi{=}(\varphi \wedge \varphi)$ or $a = (\varphi \wedge \psi){=}(\psi \wedge \varphi)$ or $a = (\varphi \wedge (\psi \wedge \vartheta)){=}((\varphi \wedge \psi) \wedge \vartheta)$ or $a = (\varphi \wedge (\psi \vee \vartheta)){=}(\varphi \wedge \psi \vee \varphi \wedge \vartheta)$ or $a = (\neg(\varphi \wedge \psi)){=}(\neg\varphi \vee \neg\psi)$ or $a = (\varphi{=}\psi){=}(\psi{=}\varphi)$ or $a = (\varphi{=}\psi){=}((\neg\varphi){=}(\neg\psi))$,

(Def. 22) for every a, $a \in$ LD-specific axioms iff there exists φ and there exists ψ and there exists ϑ such that $a = \varphi{=}\psi \Rightarrow (\varphi \wedge \vartheta){=}(\psi \wedge \vartheta)$ or $a = \varphi{=}\psi \Rightarrow (\varphi \vee \vartheta){=}(\psi \vee \vartheta)$ or $a = \varphi{=}\psi \Rightarrow (\varphi{=}\vartheta){=}(\psi{=}\vartheta)$,

respectively. The functor LD-axioms yielding a non empty subset of GRZ-formula-set is defined by the term

(Def. 23) GRZ-axioms $\cup$ LD-specific axioms.

A GRZ-rule is a relation between $2^{\text{GRZ-formula-set}}$ and GRZ-formula-set. In the sequel R, R_1, R_2 denote GRZ-rules.

Let us consider R_1 and R_2. Note that the functor $R_1 \cup R_2$ yields a GRZ-rule. The functors: GRZ-MP, GRZ-ConjIntro, GRZ-ConjElimL, and GRZ-ConjElimR yielding GRZ-rules are defined by terms

(Def. 24) the set of all $\langle\{\varphi, \varphi{=}\psi\}, \psi\rangle$ where φ is a GRZ-formula, ψ is a GRZ-formula, (Def. 25) the set of all $\langle\{\varphi,\psi\},\ \varphi\wedge\psi\rangle$ where φ is a GRZ-formula, ψ is a GRZ-formula,

(Def. 26) the set of all $\langle\{\varphi\wedge\psi\},\ \varphi\rangle$ where φ is a GRZ-formula, ψ is a GRZ-formula,

(Def. 27) the set of all $\langle\{\varphi\wedge\psi\},\ \psi\rangle$ where φ is a GRZ-formula, ψ is a GRZ-formula,

respectively. The functor GRZ-rules yielding a GRZ-rule is defined by

(Def. 28) for every a, $a \in it$ iff $a \in$ GRZ-MP or $a \in$ GRZ-ConjIntro or $a \in$ GRZ-ConjElimL or $a \in$ GRZ-ConjElimR.

A GRZ-formula sequence is a finite sequence of elements of GRZ-formula-set.

A finite GRZ-formula set is a finite subset of GRZ-formula-set. From now on Γ, Γ_1, Γ_2 denote non empty subsets of GRZ-formula-set, Δ, Δ_1, Δ_2 denote subsets of GRZ-formula-set, P, P_1, P_2 denote GRZ-formula sequences, and Σ, Σ_1, Σ_2 denote finite GRZ-formula sets.

Let us consider Σ_1 and Σ_2. Observe that the functor $\Sigma_1 \cup \Sigma_2$ yields a finite GRZ-formula set. Let us consider Γ, R, P, and n. We say that (P, n) is a correct step w.r.t. Γ, R if and only if

(Def. 29) $P(n) \in \Gamma$ or there exists a finite GRZ-formula set Q such that $\langle Q, P(n)\rangle \in R$ and for every q such that $q \in Q$ there exists k such that $k \in \operatorname{dom} P$ and $k < n$ and $P(k) = q$.

We say that P is (Γ, R)-correct if and only if

(Def. 30) for every k such that $k \in \operatorname{dom} P$ holds (P, k) is a correct step w.r.t. Γ, R.

Let a be an element of Γ. One can verify that the functor $\langle a\rangle$ yields a GRZ-formula sequence. Now we state the proposition:

(9) Let us consider an element a of Γ. Then $\langle a\rangle$ is (Γ, R)-correct.

Let us consider Γ and R. Note that there exists a GRZ-formula sequence which is non empty and (Γ, R)-correct.

Let us consider Σ. We say that Σ is (Γ, R)-correct if and only if

(Def. 31) there exists P such that $\Sigma = \operatorname{rng} P$ and P is (Γ, R)-correct.

Now we state the propositions:

(10) If P is (Γ, R)-correct and $P = P_1 \frown P_2$, then P_1 is (Γ, R)-correct.

(11) If P_1 is (Γ, R)-correct and P_2 is (Γ, R)-correct, then $P_1 \frown P_2$ is (Γ, R)-correct.

(12) If Σ_1 is (Γ, R)-correct and Σ_2 is (Γ, R)-correct, then $\Sigma_1 \cup \Sigma_2$ is (Γ, R)-correct. The theorem is a consequence of (11).

(13) If $\Gamma \subseteq \Gamma_1$ and $R \subseteq R_1$ and P is (Γ, R)-correct, then P is (Γ_1, R_1)-correct.

Let us consider Γ, R, and φ. We say that $\Gamma, R \vdash \varphi$ if and only if

(Def. 32) there exists P such that $\varphi \in \operatorname{rng} P$ and P is (Γ, R)-correct.

Let us consider Δ. We say that $\Gamma, R \vdash \Delta$ if and only if

(Def. 33) for every φ such that $\varphi \in \Delta$ holds $\Gamma, R \vdash \varphi$.

Let us consider Γ, R, and φ. Now we state the propositions:

(14) $\Gamma, R \vdash \varphi$ if and only if there exists Σ such that $\varphi \in \Sigma$ and Σ is (Γ, R)-correct.

(15) If $\varphi \in \Gamma$, then $\Gamma, R \vdash \varphi$. The theorem is a consequence of (9).

Now we state the propositions:

(16) If $\Gamma, R \vdash \Sigma$, then there exists Σ_1 such that $\Sigma \subseteq \Sigma_1$ and Σ_1 is (Γ, R)-correct.

PROOF: Define $\mathcal{X}[\text{set}] \equiv$ there exists Σ_1 such that $\$_1 \subseteq \Sigma_1$ and Σ_1 is (Γ, R)-correct. $\mathcal{X}[\emptyset]$. For every sets x, Δ such that $x \in \Sigma$ and $\Delta \subseteq \Sigma$ and $\mathcal{X}[\Delta]$ holds $\mathcal{X}[\Delta \cup \{x\}]$. $\mathcal{X}[\Sigma]$ from [8, Sch. 2]. $\square$

(17) If $\Gamma, R \vdash \Sigma$ and $\langle \Sigma, \varphi \rangle \in R$, then $\Gamma, R \vdash \varphi$. The theorem is a consequence of (16).

(18) If $\Gamma, R \vdash \varphi$, then $\varphi \in \Gamma$ or there exists Σ such that $\langle \Sigma, \varphi \rangle \in R$ and $\Gamma, R \vdash \Sigma$.

(19) If $\Gamma \subseteq \Gamma_1$ and $R \subseteq R_1$ and $\Gamma, R \vdash \varphi$, then $\Gamma_1, R_1 \vdash \varphi$.

Let us consider Γ, φ, and ψ. Now we state the propositions:

(20) $\Gamma, \text{GRZ-rules} \vdash \varphi \wedge \psi$ if and only if $\Gamma, \text{GRZ-rules} \vdash \varphi$ and $\Gamma, \text{GRZ-rules} \vdash \psi$. The theorem is a consequence of (17).

(21) Suppose $\Gamma, \text{GRZ-rules} \vdash \varphi$ and $\Gamma, \text{GRZ-rules} \vdash \varphi {=} \psi$. Then $\Gamma, \text{GRZ-rules} \vdash \psi$. The theorem is a consequence of (17).

(22) Suppose $\Gamma, \text{GRZ-rules} \vdash \varphi$ and $\Gamma, \text{GRZ-rules} \vdash \varphi \Rightarrow \psi$. Then $\Gamma, \text{GRZ-rules} \vdash \psi$. The theorem is a consequence of (21) and (20).

(23) If $\Gamma, \text{GRZ-rules} \vdash \varphi \wedge \psi$, then $\Gamma, \text{GRZ-rules} \vdash \psi \wedge \varphi$. The theorem is a consequence of (20).

Let us consider φ. We say that φ is GRZ-axiomatic if and only if

(Def. 34) $\varphi \in$ GRZ-axioms.

We say that φ is GRZ-provable if and only if

(Def. 35) $\text{GRZ-axioms}, \text{GRZ-rules} \vdash \varphi$.

We say that φ is LD-axiomatic if and only if

(Def. 36) $\varphi \in$ LD-axioms.

We say that φ is LD-provable if and only if

(Def. 37) $\text{LD-axioms}, \text{GRZ-rules} \vdash \varphi$.

Observe that $\neg(\varphi \wedge \neg\varphi)$ is GRZ-axiomatic and $(\neg\neg\varphi){=}\varphi$ is GRZ-axiomatic and $\varphi{=}(\varphi \wedge \varphi)$ is GRZ-axiomatic.

Let us consider ψ. Observe that $(\varphi \wedge \psi)$=$(\psi \wedge \varphi)$ is GRZ-axiomatic and $(\neg(\varphi \wedge \psi))$=$(\neg\varphi \vee \neg\psi)$ is GRZ-axiomatic and $(\varphi$=$\psi)$=$(\psi$=$\varphi)$ is GRZ-axiomatic and $(\varphi$=$\psi)$=$((\neg\varphi)$=$(\neg\psi))$ is GRZ-axiomatic.

Let us consider ϑ. Observe that $(\varphi \wedge (\psi \wedge \vartheta))$=$((\varphi \wedge \psi) \wedge \vartheta)$ is GRZ-axiomatic and $(\varphi \wedge (\psi \vee \vartheta))$=$(\varphi \wedge \psi \vee \varphi \wedge \vartheta)$ is GRZ-axiomatic and φ=$\psi \Rightarrow (\varphi \wedge \vartheta)$=$(\psi \wedge \vartheta)$ is LD-axiomatic and φ=$\psi \Rightarrow (\varphi \vee \vartheta)$=$(\psi \vee \vartheta)$ is LD-axiomatic and φ=$\psi \Rightarrow (\varphi$=$\vartheta)$=$(\psi$=$\vartheta)$ is LD-axiomatic and every GRZ-formula which is GRZ-axiomatic is also LD-axiomatic and every GRZ-formula which is GRZ-axiomatic is also GRZ-provable and every GRZ-formula which is LD-axiomatic is also LD-provable and every GRZ-formula which is GRZ-provable is also LD-provable and there exists a GRZ-formula which is GRZ-axiomatic, GRZ-provable, LD-axiomatic, and LD-provable.

Now we state the proposition:

(24) Suppose GRZ-axioms $\subseteq \Gamma$ and Γ, GRZ-rules $\vdash \varphi$=ψ.
Then Γ, GRZ-rules $\vdash \psi$=φ. The theorem is a consequence of (15) and (21).

3. Provability

Let us consider φ and ψ. Now we state the propositions:

(25) If φ=ψ is GRZ-provable, then ψ=φ is GRZ-provable.

(26) If φ=ψ is LD-provable, then ψ=φ is LD-provable.

Now we state the propositions:

(27) If φ=ψ is LD-provable and ψ=ϑ is LD-provable, then φ=ϑ is LD-provable.
The theorem is a consequence of (24), (22), and (21).

(28) φ=φ is LD-provable. The theorem is a consequence of (24) and (27).

Let us consider φ and ψ. We say that $\varphi =_{\mathrm{LD}} \psi$ if and only if

(Def. 38) φ=ψ is LD-provable.

One can check that the predicate is reflexive and symmetric.

Now we state the proposition:

(29) If $\varphi =_{\mathrm{LD}} \psi$, then $\neg\varphi =_{\mathrm{LD}} \neg\psi$. The theorem is a consequence of (21).

The scheme *BinReplace* deals with a non empty set $\mathcal{X}$ and a binary functor $\mathcal{F}$ yielding an element of $\mathcal{X}$ and a binary predicate $\mathcal{R}$ and states that

(Sch. 1) For every elements a, b, c, d of $\mathcal{X}$ such that $\mathcal{R}[a, b]$ and $\mathcal{R}[c, d]$ holds $\mathcal{R}[\mathcal{F}(a, c), \mathcal{F}(b, d)]$

provided

- for every elements a, b, c of $\mathcal{X}$ such that $\mathcal{R}[a, b]$ and $\mathcal{R}[b, c]$ holds $\mathcal{R}[a, c]$ and

- for every elements a, b of $\mathcal{X}$, $\mathcal{R}[\mathcal{F}(a,b),\mathcal{F}(b,a)]$ and
- for every elements a, b, c of $\mathcal{X}$ such that $\mathcal{R}[a,b]$ holds $\mathcal{R}[\mathcal{F}(a,c),\mathcal{F}(b,c)]$.

Let us consider φ, ψ, ϑ, and η.

Let us assume that $\varphi =_{\mathrm{LD}} \psi$ and $\vartheta =_{\mathrm{LD}} \eta$. Now we state the propositions:

(30) $\varphi \wedge \vartheta =_{\mathrm{LD}} \psi \wedge \eta$.

PROOF: Define $\mathcal{F}(\text{GRZ-formula}, \text{GRZ-formula}) = \$_1 \wedge \$_2$. Define $\mathcal{P}[\text{GRZ-formula}, \text{GRZ-formula}] \equiv \$_1{=}\$_2$ is LD-provable. For every φ, ψ, and ϑ such that $\mathcal{P}[\varphi,\psi]$ and $\mathcal{P}[\psi,\vartheta]$ holds $\mathcal{P}[\varphi,\vartheta]$. For every φ, ψ, ϑ, and η such that $\mathcal{P}[\varphi,\psi]$ and $\mathcal{P}[\vartheta,\eta]$ holds $\mathcal{P}[\mathcal{F}(\varphi,\vartheta),\mathcal{F}(\psi,\eta)]$ from *BinReplace*. □

(31) $\varphi{=}\vartheta =_{\mathrm{LD}} \psi{=}\eta$.

PROOF: Define $\mathcal{F}(\text{GRZ-formula}, \text{GRZ-formula}) = \$_1{=}\$_2$. Define $\mathcal{P}[\text{GRZ-formula}, \text{GRZ-formula}] \equiv \$_1{=}\$_2$ is LD-provable. For every φ, ψ, and ϑ such that $\mathcal{P}[\varphi,\psi]$ and $\mathcal{P}[\psi,\vartheta]$ holds $\mathcal{P}[\varphi,\vartheta]$. For every φ, ψ, ϑ, and η such that $\mathcal{P}[\varphi,\psi]$ and $\mathcal{P}[\vartheta,\eta]$ holds $\mathcal{P}[\mathcal{F}(\varphi,\vartheta),\mathcal{F}(\psi,\eta)]$ from *BinReplace*. □

The functor LD-IdR yielding an equivalence relation of GRZ-formula-set is defined by

(Def. 39) for every φ and ψ, $\langle \varphi, \psi \rangle \in it$ iff $\varphi =_{\mathrm{LD}} \psi$.

Note that there exists a family of subsets of GRZ-formula-set which is non empty.

The functor LD-IdClasses yielding a non empty family of subsets of GRZ-formula-set is defined by the term

(Def. 40) Classes LD-IdR.

An LD-identity class is an element of LD-IdClasses. Let us consider φ. The functor LD-IdClassOf φ yielding an LD-identity class is defined by the term

(Def. 41) $[\varphi]_{\text{LD-IdR}}$.

Now we state the proposition:

(32) $\varphi =_{\mathrm{LD}} \psi$ if and only if LD-IdClassOf φ = LD-IdClassOf ψ.

PROOF: If $\varphi =_{\mathrm{LD}} \psi$, then LD-IdClassOf φ = LD-IdClassOf ψ by [14, (18), (23)]. □

The scheme *UnOpCongr* deals with a non empty set $\mathcal{X}$ and a unary functor $\mathcal{F}$ yielding an element of $\mathcal{X}$ and an equivalence relation $\mathcal{E}$ of $\mathcal{X}$ and states that

(Sch. 2) There exists a unary operation f on Classes $\mathcal{E}$ such that for every element x of $\mathcal{X}$, $f([x]_{\mathcal{E}}) = [\mathcal{F}(x)]_{\mathcal{E}}$

provided

- for every elements x, y of $\mathcal{X}$ such that $\langle x, y\rangle \in \mathcal{E}$ holds $\langle \mathcal{F}(x), \mathcal{F}(y)\rangle \in \mathcal{E}$.

The scheme *BinOpCongr* deals with a non empty set $\mathcal{X}$ and a binary functor $\mathcal{F}$ yielding an element of $\mathcal{X}$ and an equivalence relation $\mathcal{E}$ of $\mathcal{X}$ and states that

(Sch. 3) There exists a binary operation f on Classes $\mathcal{E}$ such that for every elements x, y of $\mathcal{X}$, $f([x]_{\mathcal{E}}, [y]_{\mathcal{E}}) = [\mathcal{F}(x, y)]_{\mathcal{E}}$

provided

- for every elements x_1, x_2, y_1, y_2 of $\mathcal{X}$ such that $\langle x_1, x_2\rangle$, $\langle y_1, y_2\rangle \in \mathcal{E}$ holds $\langle \mathcal{F}(x_1, y_1), \mathcal{F}(x_2, y_2)\rangle \in \mathcal{E}$.

From now on x, y, z denote LD-identity classes.

Now we state the proposition:

(33) There exists φ such that $x =$ LD-IdClassOf φ.

Let us consider x. We say that x is LD-provable if and only if

(Def. 42) there exists φ such that $x =$ LD-IdClassOf φ and φ is LD-provable.

The functor $\neg x$ yielding an LD-identity class is defined by

(Def. 43) there exists φ such that $x =$ LD-IdClassOf φ and $it =$ LD-IdClassOf $\neg\varphi$.

One can verify that the functor is involutive. Let us consider y. The functor $x \wedge y$ yielding an LD-identity class is defined by

(Def. 44) there exists φ and there exists ψ such that $x =$ LD-IdClassOf φ and $y =$ LD-IdClassOf ψ and $it =$ LD-IdClassOf$(\varphi \wedge \psi)$.

Note that the functor is commutative and idempotent. The functor $x{=}y$ yielding an LD-identity class is defined by

(Def. 45) there exists φ and there exists ψ such that $x =$ LD-IdClassOf φ and $y =$ LD-IdClassOf ψ and $it =$ LD-IdClassOf $\varphi{=}\psi$.

One can check that the functor is commutative.

The functor $x \vee y$ yielding an LD-identity class is defined by the term

(Def. 46) $\neg(\neg x \wedge \neg y)$.

Let us observe that the functor is commutative and idempotent. The functor $x \Rightarrow y$ yielding an LD-identity class is defined by the term

(Def. 47) $x{=}(x \wedge y)$.

Let φ be an LD-provable GRZ-formula. Let us observe that LD-IdClassOf φ is LD-provable.

Now we state the proposition:

(34) If LD-IdClassOf φ is LD-provable, then φ is LD-provable. The theorem is a consequence of (32) and (21).

Let us consider x and y. Now we state the propositions:

(35) $x \wedge y$ is LD-provable if and only if x is LD-provable and y is LD-provable. The theorem is a consequence of (34) and (20).

(36) $x{=}y$ is LD-provable if and only if $x = y$. The theorem is a consequence of (34) and (32).

Now we state the proposition:

(37) LD-IdClassOf $\neg\varphi = \neg$ LD-IdClassOf φ.

Let us consider φ and ψ. Now we state the propositions:

(38) LD-IdClassOf$(\varphi \wedge \psi) =$ LD-IdClassOf $\varphi \wedge$ LD-IdClassOf ψ.

(39) LD-IdClassOf $\varphi{=}\psi =$ (LD-IdClassOf φ)=(LD-IdClassOf ψ).

(40) LD-IdClassOf$(\varphi \vee \psi) =$ LD-IdClassOf $\varphi \vee$ LD-IdClassOf ψ.

(41) LD-IdClassOf$(\varphi \Rightarrow \psi) =$ LD-IdClassOf $\varphi \Rightarrow$ LD-IdClassOf ψ.

Now we state the propositions:

(42) $(x \wedge y) \wedge z = x \wedge (y \wedge z)$. The theorem is a consequence of (33) and (32).

(43) $x \Rightarrow y$ is LD-provable if and only if $x = x \wedge y$.

(44) If $x \Rightarrow y$ is LD-provable and $y \Rightarrow z$ is LD-provable, then $x \Rightarrow z$ is LD-provable. The theorem is a consequence of (36) and (42).

(45) If $\varphi \Rightarrow \psi$ is LD-provable and $\psi \Rightarrow \vartheta$ is LD-provable, then $\varphi \Rightarrow \vartheta$ is LD-provable. The theorem is a consequence of (41), (34), and (44).

Let us consider x, y, and z. Now we state the propositions:

(46) $x \vee (y \vee z) = (x \vee y) \vee z$.

(47) $x \wedge (y \vee z) = x \wedge y \vee x \wedge z$. The theorem is a consequence of (33), (32), and (40).

(48) $x \vee y \wedge z = (x \vee y) \wedge (x \vee z)$. The theorem is a consequence of (47).

Let us consider x and y. Now we state the propositions:

(49) $x \Rightarrow y$ is LD-provable and $y \Rightarrow x$ is LD-provable if and only if $x = y$. The theorem is a consequence of (36).

(50) If x is LD-provable, then $x \vee y$ is LD-provable. The theorem is a consequence of (33), (35), (47), and (48).

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